J. Korean Math. Soc. **45** (2008), No. 1, pp. 205–219

MAXIMUM SUBSPACES RELATED TO A-CONTRACTIONS AND QUASINORMAL OPERATORS

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Abstract. It is shown that if $A \geq 0$ and T are two bounded linear operators on a complex Hilbert space $\mathcal{H}$ satisfying the inequality $T^*AT \leq A$ and the condition $AT = A^{1/2}TA^{1/2}$, then there exists the maximum reducing subspace for A and $A^{1/2}T$ on which the equality $T^*AT = A$ is satisfied. We concretely express this subspace in two ways, and as applications, we derive certain decompositions for quasinormal contractions. Also, some facts concerning the quasi-isometries are obtained.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, $I = I_{\mathcal{H}}$ being the identity operator in $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$ we denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range and the kernel of T, respectively.

Recall that T is a *quasinormal operator* if T and T^*T commute, where T^* is the adjoint operator of T.

Throughout in this paper $A \in \mathcal{B}(\mathcal{H})$ is a non zero positive fixed operator. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an *A-contraction* on $\mathcal{H}$ if it satisfies the following operator inequality

$$T^*AT \leq A. \tag{1.1}$$

According to [1], this means that A is a *lower T-Toeplitz* operator. If the equality in (1.1) occurs, then T is called an *A-isometry* on $\mathcal{H}$. In the terminology of [1], [2] the fact that T is an A-isometry means that A is a *T-Toeplitz* operator.

We say that T is an *A-weighted contraction* on $\mathcal{H}$ if $T^*T \leq A$, and we call T an *A-weighted isometry* on $\mathcal{H}$ if the equality $T^*T = A$ holds. Clearly, T is an A-weighted contraction (A-weighted isometry) if and only if there is a contraction (respectively, an isometry) V from $\overline{\mathcal{R}(A)}$ into $\overline{\mathcal{R}(T)}$ such that $T = VA^{1/2}$, where $A^{1/2}$ is the square root of A. In this case, V is uniquely determined by A and T.

Received June 21, 2006.

2000 *Mathematics Subject Classification.* Primary 47A15, 47A63; Secondary 47B20.

Key words and phrases. A-contraction, A-isometry, A-weighted isometry, quasinormal operator, quasi-isometry.

It immediately follows from (1.1) that T is an A-contraction (A-isometry) if and only if $A^{1/2}T$ is an A-weighted contraction (A-weighted isometry), on $\mathcal{H}$. If T is an A-contraction, we denote by $\widehat{T}$ the (unique) contraction on $\overline{\mathcal{R}(A)}$ satisfying

$$\widehat{T}A^{1/2}h = A^{1/2}Th \quad (h \in \mathcal{H}). \tag{1.2}$$

Clearly, $\widehat{T}$ is an isometry if and only if T is an A-isometry.

For an A-contraction T and any integer $n \geq 0$ we denote $\mathcal{N}_n = \mathcal{N}(A - T^{*(n+1)}AT^{n+1})$. It was shown in Proposition 2.1 [11] that the subspace

$$\mathcal{N}_\infty = \bigcap_{n=0}^{\infty} \mathcal{N}_n \tag{1.3}$$

is invariant for T, and if $\mathcal{N}_\infty$ reduces A then $\mathcal{N}_\infty$ is the maximum invariant subspace for A and T on which T is an A-isometry. For instance, if either the range $\mathcal{R}(A)$ is closed, or T is a *regular* A-contraction, which means that $AT = A^{1/2}TA^{1/2}$, then $\mathcal{N}_\infty$ reduces A. In these cases $\mathcal{N}_\infty$ is also invariant for $A^{1/2}T$ and furthermore, if T is a regular A-contraction then $\mathcal{N}_\infty$ is the maximum subspace into $\mathcal{N}_0$ which is invariant for A and $A^{1/2}T$ (Proposition 2.3 [11]), and obviously, $A^{1/2}T$ is an A-weighted isometry on $\mathcal{N}_\infty$.

This last meaning of $\mathcal{N}_\infty$ suggests us to investigate the existence of the maximum reducing subspace for A and $A^{1/2}T$ on which $A^{1/2}T$ is an A-weighted isometry. We find such a subspace in Section 2, which will be denoted $\mathcal{M}_\infty$, and we show that $\mathcal{M}_\infty$ can be concretely obtained in two ways, using the subspace $\mathcal{N}_\infty$ and the minimal isometric dilation of the contraction $\widehat{T}$, respectively.

Recall ([4]) that the *minimal isometric dilation* of a contraction S on $\mathcal{H}$ is an isometry V on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ with the property

$$P_{\mathcal{H}}V = SP_{\mathcal{H}}, \tag{1.4}$$

$P_{\mathcal{H}}$ being the orthogonal projection onto $\mathcal{H}$, such that

$$\mathcal{K} = \bigvee_{n \geq 0} V^n\mathcal{H}. \tag{1.5}$$

Remark that the property (1.4) means that V is a *lifting* for S.

The space $\mathcal{M}_\infty$ has interesting meanings in the context of quasinormal operators, where different natural A-contractions appear. We consider this context in Section 3, and we obtain an orthogonal decomposition of $\mathcal{H}$ relative to a quasinormal contraction T, where all reducing subspaces can be intrinsicly expressed in terms of T and of the partial isometry from the polar decomposition of T. It is a complete description of the decompositions from [8, 9] concerning the quasinormal contractions. Also, we derive some consequences involving the *quasi-isometries*, that is the operators $T \in \mathcal{B}(\mathcal{H})$ which are T^*T-isometries (see [6], [11]).

2. Maximum A-weighted isometric part

Let T be an A-contraction on $\mathcal{H}$. Using a standard argument which involves Zorn's lemma, one can obtains the existence of the maximum subspace $\mathcal{M}_0$ of $\mathcal{H}$ which reduces A and T such that $T|_{\mathcal{M}_0}$ is a $A|_{\mathcal{M}_0}$-isometry (see [12]). In the case when $\mathcal{H}$ is separable, $\mathcal{M}_0$ can be also obtained from Theorem 1 [5], which in particular gives that any $S \in \mathcal{B}(\mathcal{H})$ has a maximum subspace which reduces A and S on which S is an A-contraction, or an A-isometry, respectively.

In our setting one has $\mathcal{M}_0 \subset \mathcal{N}_\infty$ and $\mathcal{M}_0$ reduces $A^{1/2}T$ to an A-weighted isometry, but $\mathcal{M}_0$ is not the maximum subspace having this property relative to $A^{1/2}T$, in general (see [12]). In the regular case it is possible to get more invariant (or reducing) subspaces in $\mathcal{N}_\infty$ for A and $A^{1/2}T$ on which $A^{1/2}T$ is an A-weighted isometry (see [9-11]), but the maximum such subspace between $\mathcal{M}_0$ and $\mathcal{N}_\infty$ is now obtained in the following.

Theorem 2.1. *Let T be a regular A-contraction on $\mathcal{H}$. Then the maximum subspace which reduces A and $A^{1/2}T$ on which $A^{1/2}T$ is an A-weighted isometry is*

$$\mathcal{M}_\infty = \mathcal{H} \ominus \bigvee_{n\geq 0} A^{1/2}T^n\mathcal{N}_\infty^\perp. \tag{2.1}$$

Moreover, $A^{1/2}T$ is a quasinormal operator on $\mathcal{M}_\infty$, and $\mathcal{M}_\infty$ is an invariant subspace for T such that T is an A-isometry on $\mathcal{M}_\infty$. In particular, if $A^{1/2}T$ is a quasinormal operator on $\mathcal{H}$ then $\mathcal{M}_\infty = \mathcal{N}_0$.

Proof. Let $\mathcal{M}_\infty$ be the subspace defined in (2.1). Since T is a regular A-contraction, $\mathcal{N}_\infty^\perp$ is invariant for A and we have for $n \geq 0$,

$$AA^{1/2}T^n\mathcal{N}_\infty^\perp = A^{1/2}T^nA\mathcal{N}_\infty^\perp \subset A^{1/2}T^n\mathcal{N}_\infty^\perp \subset \mathcal{M}_\infty^\perp,$$

and also

$$A^{1/2}TA^{1/2}T^n\mathcal{N}_\infty^\perp = AT^{n+1}\mathcal{N}_\infty^\perp = A^{1/2}T^{n+1}A^{1/2}\mathcal{N}_\infty^\perp \subset \mathcal{M}_\infty^\perp.$$

It follows that $\mathcal{M}_\infty^\perp$ is an invariant subspace for the operators A and $A^{1/2}T$. Next, since $\mathcal{N}_\infty$ is invariant for $A^{1/2}T$, we have firstly $T^*A^{1/2}\mathcal{N}_\infty^\perp \subset \mathcal{N}_\infty^\perp$. On the other hand, for $n \geq 1$ we obtain

$$\begin{aligned} T^*A^{1/2}T^n\mathcal{N}_\infty^\perp &= T^*A^{1/2}T(T^{n-1}\mathcal{N}_\infty^\perp) \\ &\subset (T^*A^{1/2}T - A^{1/2})T^{n-1}\mathcal{N}_\infty^\perp + A^{1/2}T^{n-1}\mathcal{N}_\infty^\perp \\ &\subset \mathcal{N}_\infty^\perp + \mathcal{M}_\infty^\perp \subset \mathcal{M}_\infty^\perp. \end{aligned}$$

Here we used the fact that $\mathcal{N}_0 = \mathcal{N}(A^{1/2} - T^*A^{1/2}T)$ (by Theorem 2.6 [11]), which gives that $\mathcal{R}(A^{1/2} - T^*A^{1/2}T) \subset \mathcal{N}_0^\perp \subset \mathcal{N}_\infty^\perp$. Thus, we infer that $\mathcal{M}_\infty^\perp$ is invariant for T^* and, because $\mathcal{M}_\infty$ is reducing for $A^{1/2}$, $\mathcal{M}_\infty^\perp$ is also invariant for $T^*A^{1/2}$. Hence $\mathcal{M}_\infty$ is reducing for A and $A^{1/2}T$, and also $\mathcal{M}_\infty$ is invariant for T.

Now, since $\mathcal{M}_\infty \subset \mathcal{N}_\infty \subset \mathcal{N}_0$, we have

$$(A^{1/2}T|_{\mathcal{M}_\infty})^*(A^{1/2}T|_{\mathcal{M}_\infty}) = (T^*AT)|_{\mathcal{M}_\infty} = A|_{\mathcal{M}_\infty},$$

which just means that $A^{1/2}T$ is an A-weighted isometry on $\mathcal{M}_\infty$. As $\mathcal{M}_\infty$ is invariant for A and T, the above relation also gives that T is an A-isometry on $\mathcal{M}_\infty$, and having in view the fact that T is a regular A-contraction on $\mathcal{H}$, and particularly on $\mathcal{M}_\infty$, it follows from Proposition 2.3 [11] that $A^{1/2}T$ is quasinormal on $\mathcal{M}_\infty$.

It remains to prove that $\mathcal{M}_\infty$ is the maximum subspace reducing A and $A^{1/2}T$ on which $A^{1/2}T$ is an A-weighted isometry. Let $\mathcal{M} \subset \mathcal{H}$ be another subspace having these properties. Firstly, for any $h \in \mathcal{M}$ we have $T^*ATh = Ah$. Since T is also a regular $A^{1/4}$-contraction (by Theorem 2.6 [11]), the previous relation implies $A^{3/4}T^*A^{1/4}Th = Ah$ and later $T^*A^{1/2}Th = A^{1/4}T^*A^{1/4}Th = A^{1/2}h$, because $A^{1/4}$ is injective on his range. Then using the fact that $\mathcal{M}$ is invariant for $A^{1/2}T$, we obtain

$$T^{*2}AT^2h = T^*(T^*A^{1/2}T)A^{1/2}Th = T^*A^{1/2}A^{1/2}Th = Ah,$$

and by induction we infer that $T^{*n}AT^nh = Ah$, for $n \geq 1$ and $h \in \mathcal{M}$. So $\mathcal{M} \subset \mathcal{N}(A - T^{*n}AT^n)$ for $n \geq 1$, hence $\mathcal{M} \subset \mathcal{N}_\infty$ (by (1.3)). To prove $\mathcal{M} \subset \mathcal{M}_\infty$ we show that $T^{*m}A^{1/2}\mathcal{M} \subset \mathcal{M}$ for $m \geq 2$.

Let $\{p_n(A)\}$ be an approximation polynomial for $A^{1/2}$ with $p_n(0) = 0$ (as in [7], p. 261). If $p_n(A) = \sum_{j\geq 1} c_jA^j$ (a finite sum, c_j being positive scalars), then for $h \in \mathcal{M}$ we have $T^*A^{1/2}h \in \mathcal{M}$ and also

$$T^{*2}A^{1/2}h = \lim_n \sum_{j\geq 1} c_jT^*A^{1/2}T^*(A^{1/2})^{2j-1}h \in \mathcal{M},$$

because in each term $2j - 1 \geq 1$. Using this fact, we obtain

$$T^{*3}A^{1/2}h = \lim_n \sum_{j\geq 1} c_jT^*A^{1/2}T^{*2}A^{1/2}(A^{1/2})^{2(j-1)}h \in \mathcal{M},$$

and by induction we get that $T^{*m}A^{1/2}h \in \mathcal{M}$ for any $m \geq 1$. Thus, for $m \geq 1$ we have $T^{*m}A^{1/2}\mathcal{M} \subset \mathcal{M} \subset \mathcal{N}_\infty$, whence it follows that $\mathcal{M}$ is orthogonal to $A^{1/2}T^m\mathcal{N}_\infty^\perp$. Hence $\mathcal{M}$ is orthogonal to $\mathcal{M}_\infty^\perp$, that is $\mathcal{M} \subset \mathcal{M}_\infty$.

Finally, we suppose that $A^{1/2}T$ is quasinormal on $\mathcal{H}$. Then by Corollary 2.7 [11] one has $\mathcal{N}_0 = \mathcal{N}_\infty$ and this subspace reduces A and $A^{1/2}T$. Clearly, $A^{1/2}T$ will be an A-weighted isometry on $\mathcal{N}_0$ and consequently, by the maximality of $\mathcal{M}_\infty$ one obtains $\mathcal{M}_\infty = \mathcal{N}_0$. The proof is finished. □

Another description of the subspace $\mathcal{M}_\infty$ is given by the following.

Proposition 2.2. *If T is a regular A-contraction on $\mathcal{H}$ then*

$$\mathcal{M}_\infty = \{h \in \mathcal{H} : V^n\widehat{T}^{*m}A^{j/2}h \in \overline{\mathcal{R}(A)},\ \ n,m \geq 0,\ \ j \geq 1\}, \tag{2.2}$$

where V is the minimal isometric dilation of the contraction $\widehat{T}$ defined by (1.2).

Proof. We know from Theorem 2.5 [12] that the subspace $\widetilde{\mathcal{M}_0}$ defined by the right side in (2.2) reduces A and $A^{1/2}T$, $\widetilde{\mathcal{M}_0}$ is invariant for T and T is an A-isometry on $\widetilde{\mathcal{M}_0}$. Thus, $A^{1/2}T$ is an A-weighted isometry on $\widetilde{\mathcal{M}_0}$, and from Theorem 2.1 we have $\widetilde{\mathcal{M}_0} \subset \mathcal{M}_\infty$.

To prove the converse inclusion, let $h \in \mathcal{M}_\infty$ be arbitrary. Since $A^{1/2}T$ is an A-weighted isometry on $\mathcal{M}_\infty$, there is an isometry S from $\mathcal{R}_\infty := \overline{A^{1/2}\mathcal{M}_\infty}$ into $\overline{A^{1/2}T\mathcal{M}_\infty}$ such that $SA^{1/2}h = A^{1/2}Th$. Then $\widehat{T}A^{1/2}h = A^{1/2}Th = JSA^{1/2}h$, where J is the natural injection of $\overline{A^{1/2}T\mathcal{M}_\infty}$ into $\mathcal{R}_\infty$. Therefore $\mathcal{R}_\infty$ is invariant for $\widehat{T}$ and $\widehat{T}|_{\mathcal{R}_\infty} = JS$ is an isometry on $\mathcal{R}_\infty$. In fact, $\mathcal{R}_\infty$ even reduces $\widehat{T}$ because for $h' \in \mathcal{N}_\infty^\perp$ and $n \geq 0$ one has

$$\langle \widehat{T}^* A^{1/2}h, A^{1/2}T^n h' \rangle = \langle h, AT^{n+1}h' \rangle = \langle h, A^{1/2}T^{n+1}A^{1/2}h' \rangle = 0,$$

having in view that $\mathcal{N}_\infty$ reduces A, and T is a regular A-contraction. This shows that $\widehat{T}^*A^{1/2}\mathcal{M}_\infty$ is orthogonal to $A^{1/2}T^n\mathcal{N}_\infty^\perp$ for $n \geq 0$, therefore $\widehat{T}^*A^{1/2}\mathcal{M}_\infty \subset \mathcal{M}_\infty$. Thus one obtains that

$$\widehat{T}^*\mathcal{R}_\infty \subset \mathcal{M}_\infty \cap \overline{\mathcal{R}(A)} = \mathcal{R}_\infty.$$

To see the previous equality, we first remark that $\mathcal{R}_\infty \subset \mathcal{M}_\infty \cap \overline{\mathcal{R}(A)}$. Next let $k \in \mathcal{M}_\infty \cap \overline{\mathcal{R}(A)}$ and we write $k = k_0 + k_1$ with $k_0 \in \mathcal{R}_\infty$ and $k_1 \in \overline{\mathcal{R}(A)} \ominus \mathcal{R}_\infty$. Then $k_1 = k - k_0 \in \mathcal{M}_\infty$ and k_1 is orthogonal on $\mathcal{R}_\infty$ and so to $A\mathcal{M}_\infty$. Hence k_1 is orthogonal to Ak_1, that is $A^{1/2}k_1 = 0$. This means $k_1 \in \mathcal{N}(A)$, therefore $k_1 = 0$ because we also have $k_1 \in \overline{\mathcal{R}(A)}$. Thus $k = k_0 \in \mathcal{R}_\infty$, which gives the required equality. Consequently, $\mathcal{R}_\infty$ reduces $\widehat{T}$.

Next, if V is the minimal isometric dilation of $\widehat{T}$ then, since V is a lifting for $\widehat{T}$, V will be also a lifting for the isometry $\widehat{T}|_{\mathcal{R}_\infty}$, hence V is an extension for $\widehat{T}|_{\mathcal{R}_\infty}$ (see [4]). Thus, for $h \in \mathcal{M}_\infty$, $n, m \geq 0$ and $j \geq 1$ we obtain

$$V^n\widehat{T}^{*m}A^{j/2}h = \widehat{T}^n\widehat{T}^{*m}A^{j/2}h \in \overline{\mathcal{R}(A)}$$

because $A^{j/2}h \in \mathcal{R}_\infty$. Consequently, $\mathcal{M}_\infty \subset \widetilde{\mathcal{M}_0}$ what ends the proof. □

Corollary 2.3. *Let T be a regular A-contraction on $\mathcal{H}$. Then $\mathcal{H}$ admits a unique orthogonal decomposition of the form*

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \tag{2.3}$$

where both subspaces reduce A and $A^{1/2}T$, such that $A^{1/2}T$ is an A-weighted isometry on $\mathcal{H}_0$ and $A^{1/2}T$ is a completely non A-weighted isometry on $\mathcal{H}_1$. Furthermore, one has $\mathcal{H}_0 = \mathcal{M}_\infty$ and

$$\mathcal{H}_1 = \bigvee_{n,j\geq 0} A^{1/2}T^n\overline{\mathcal{R}(A - T^{*j}AT^j)}. \tag{2.4}$$

Proof. This follows from Theorem 2.1, and (2.4) is obtained from (1.3) and (2.1). □

Having in view this decomposition, we call $\mathcal{M}_\infty$ *the maximum A-weighted isometric part* of $\mathcal{H}$ relative to the A-contraction T.

Remark 2.4. From the corresponding maximality properties of the subspaces $\mathcal{M}_\infty$ and $\mathcal{N}_\infty$ we have immediately the inclusions

$$\mathcal{M}_0 \subset \mathcal{M}_\infty \subset \mathcal{N}_\infty.$$

In addition, one has $\mathcal{M}_0 = \mathcal{M}_\infty$ if and only if $\mathcal{M}_\infty$ is invariant for T^*, and $\mathcal{M}_\infty = \mathcal{N}_\infty$ if and only if $\mathcal{N}_\infty$ is invariant for $T^*A^{1/2}$.

Proposition 2.5. *Let T be an A-contraction on $\mathcal{H}$ such that $AT = TA$. Then the maximum subspace which reduces A and T on which T is an A-isometry is*

$$\mathcal{M}_0 = \mathcal{H} \ominus \bigvee_{n,j\geq 0} T^n(I - T^{*j}T^j)\overline{\mathcal{R}(A)}. \tag{2.5}$$

Proof. Since A and T commute, the A-contraction T is regular. Then from (2.3) and (2.4) we infer that the corresponding A-weighted isometric part of $\mathcal{H}$ is the subspace

$$\mathcal{M}_\infty = \mathcal{H} \ominus \bigvee_{n,j\geq 0} T^n A^{1/2}\overline{\mathcal{R}(A - T^{*j}AT^j)}.$$

This shows that $\mathcal{M}_\infty$ is invariant for T^* and by Remark 2.4 we have $\mathcal{M}_\infty = \mathcal{M}_0$, this being the maximum subspace which reduces T to an A-isometry.

To prove the formula (2.5), we firstly remark that for $j \geq 1$,

$$\overline{\mathcal{R}(A - T^{*j}AT^j)} = \overline{(I - T^{*j}T^j)\overline{A^{1/2}\mathcal{H}}} = \overline{(I - T^{*j}T^j)A^{1/2}\mathcal{H}}.$$

Thus for $n, j \geq 0$ one has

$$[T^n A^{1/2}\overline{\mathcal{R}(A - T^{*j}AT^j)}]^\perp = [\overline{T^n(I - T^{*j}T^j)A\mathcal{H}}]^\perp$$
$$= [T^n(I - T^{*j}T^j)A\mathcal{H}]^\perp = [T^n(I - T^{*j}T^j)\overline{A\mathcal{H}}]^\perp,$$

whence it follows that

$$[\bigvee_{n,j\geq 0} T^n A^{1/2}\overline{\mathcal{R}(A - T^{*j}AT^j)}]^\perp = [\bigvee_{n,j\geq 0} T^n(I - T^{*j}T^j)\overline{\mathcal{R}(A)}]^\perp.$$

Hence the subspace $\mathcal{M}_0 = \mathcal{M}_\infty$ has the form (2.5). □

We remark that the above proposition completes the Proposition 2.8 [12]. In general we cannot obtain $\mathcal{M}_\infty = \mathcal{N}_\infty$, but it is possible to have this equality in certain cases, as we see below. First we recover a usual decomposition of a contraction.

Corollary 2.6. *If T is a contraction on $\mathcal{H}$ and V is the minimal isometric dilation of T, then the maximum subspace which reduces T to an isometry is*

$$\begin{aligned} \mathcal{H}_i &= \mathcal{H} \ominus \bigvee_{n,j\geq 0} T^n(I - T^{*j}T^j)\mathcal{H} \\ &= \{h \in \mathcal{H} : V^n T^{*m} h \in \mathcal{H},\ \ n, m \geq 0\}. \end{aligned} \tag{2.6}$$

Hence, $\mathcal{H}$ admits a unique orthogonal decomposition of the form

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \oplus \mathcal{H}_c, \tag{2.7}$$

where the three subspaces reduce T, such that T is unitary on $\mathcal{H}_u$, T is a shift on $\mathcal{H}_s$ and T is completely nonisometric on $\mathcal{H}_c$.

Proof. One applies Proposition 2.5 and Proposition 2.2 in the case $A = I$. Also, the decomposition (2.7) is obtained by combining in this case the decomposition (2.3) with the Wold decomposition of $\mathcal{H}_i$ in a unitary part and a shift part. □

The following corollary completes Corollary 2.9 [12].

Corollary 2.7. *Let T be a regular A-contraction on $\mathcal{H}$. Then the maximum subspace $\widehat{\mathcal{M}_0}$ which reduces $\widehat{T}$ to an $A|_{\overline{\mathcal{R}(A)}}$-isometry coincides with the maximum subspace $\widehat{\mathcal{H}_0}$ which reduces $\widehat{T}$ to an isometry. In fact, one has*

$$\widehat{\mathcal{H}_0} = \widehat{\mathcal{M}_0} = \mathcal{M}_\infty \cap \overline{\mathcal{R}(A)} = \overline{A^{1/2}\mathcal{M}_\infty}, \tag{2.8}$$

and

$$\mathcal{M}_\infty = \widehat{\mathcal{H}_0} \oplus \mathcal{N}(A). \tag{2.9}$$

Proof. Since $\widehat{T}$ and $A|_{\overline{\mathcal{R}(A)}}$ commute, the relations (2.5) and (2.6) give

$$\begin{aligned}\widehat{\mathcal{M}_0} &= \overline{\mathcal{R}(A)} \ominus \bigvee_{n,j\geq 0} \widehat{T}^n(I - \widehat{T}^{*j}T^j)\overline{A\overline{\mathcal{R}(A)}} \\ &= \overline{\mathcal{R}(A)} \ominus \bigvee_{n,j\geq 0} \widehat{T}^n(I - \widehat{T}^{*j}\widehat{T}^j)\overline{\mathcal{R}(A)} = \widehat{\mathcal{H}_0}.\end{aligned}$$

The second equality in (2.8) is quoted in [12], and the last equality in (2.8) follows from the proof of Proposition 2.2. Finally (2.9) is derived from (2.8) because $\mathcal{M}_\infty$ contains $\mathcal{N}(A)$. □

Notice that if A is injective in this corollary, then

$$\widehat{\mathcal{H}_0} = \widehat{\mathcal{M}_0} = \mathcal{M}_\infty.$$

Remark 2.8. Suppose that T is a regular A-contraction such that $T^2 = 0$. Then one infers that $\mathcal{N}_\infty \subset \mathcal{N}(A)$, hence

$$\mathcal{N}(A) = \mathcal{N} = \mathcal{M}_0 = \mathcal{M}_\infty = \mathcal{N}_\infty,$$

where $\mathcal{N} = \mathcal{N}(A - AT)$. But in general, $\mathcal{N}_\infty \subsetneq \mathcal{N}_0$ (as in Example 4.3 [9]), and in this case, $A^{1/2}T$ is not quasinormal on $\mathcal{H}$ (by Theorem 2.1).

Remark 2.9. Assume that T is a regular A-contraction with $T^2 = T$. Then $A^{1/2}T$ is quasinormal on $\mathcal{H}$ because

$$A^{1/2}TT^*AT = A^{1/2}TAT = ATA^{1/2}T = T^*AA^{1/2}T^2 = T^*ATA^{1/2}T,$$

where we used the fact that $AT = T^*A$ (see [3]). Then both Theorem 2.1 and the fact that $T^*AT = AT$ imply in this case

$$\mathcal{N} = \mathcal{M}_\infty = \mathcal{N}_\infty = \mathcal{N}_0.$$

Furthermore, for $\widehat{T}$ as in Corollary 2.7 we have

$$\widehat{\mathcal{M}_0} = \mathcal{N} \cap \overline{\mathcal{R}(A)} = \mathcal{N}(I - \widehat{T}),$$

hence $\widehat{T}|_{\widehat{\mathcal{M}_0}} = I_{\widehat{\mathcal{M}_0}}$. But $\widehat{T}^2 = \widehat{T}$ (as $T^2 = T$) and so $\widehat{T} = 0$ on $\widehat{\mathcal{M}_0}^\perp = \overline{\mathcal{R}(I - \widehat{T})}$. So, $\widehat{T}$ is the orthogonal projection onto $\mathcal{N}(I - \widehat{T})$. Moreover, when A is injective we have $T = \widehat{T}$. Indeed, since in this case $\widehat{T}^*A^{1/2} = A^{1/2}\widehat{T}^* = T^*A^{1/2}$ and $\widehat{T}^{*2} = \widehat{T}^*$ on $\mathcal{H} = \overline{\mathcal{R}(A)}$, we obtain

$$T^*(I - \widehat{T}^*)A^{1/2}\mathcal{H} = T^*A^{1/2}(I - \widehat{T}^*)\mathcal{H} = A^{1/2}\widehat{T}^*(I - \widehat{T}^*)\mathcal{H} = \{0\},$$

hence $T^*|_{\overline{\mathcal{R}(I - \widehat{T}^*)}} = 0$. On the other hand, for $k = A^{1/2}h$ where $h \in \mathcal{N} = \mathcal{N}(I - \widehat{T})$, we have $T^*k = A^{1/2}\widehat{T}^*h = A^{1/2}h = k$. As $\mathcal{N}(I - \widehat{T}) = \overline{A^{1/2}\mathcal{N}}$ (see [10]), we deduce that T^* is the identity on $\mathcal{N}(I - \widehat{T})$. Thus, T^* is the orthogonal projection onto $\mathcal{N}(I - \widehat{T})$, and consequently, $T = T^* = \widehat{T}$ on $\mathcal{H} = \overline{\mathcal{R}(A)}$.

Finally we remark that if A is invertible and T is an A-contraction with $T^2 = T$, then T is an orthogonal projection if and only if the A-contraction T is regular, or equivalently $AT = TA$.

3. Applications to quasinormal operators and quasi-isometries

We derive from the above results some facts concerning the quasinormal operators.

Proposition 3.1. *Let T be a quasinormal contraction on $\mathcal{H}$. The following statements hold:*

(i) $\mathcal{N}(I - T^*T)$ is the maximum subspace which reduces T to an isometry.

(ii) $\mathcal{H}_q = \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T)$ is the maximum subspace which reduces T to a quasi-isometry, or equivalently, to a partial isometry. Also, $\mathcal{H}_q$ is the maximum T^*T-weighted isometric part of $\mathcal{H}$ relative to the T^*T-contraction T. In addition, T is normal on $\mathcal{H}_q$ if and only if $|T|T$ is normal on $\mathcal{H}_q$.

Proof. Assertion (i) follows applying Theorem 2.1 to the quasinormal I-contraction T, which gives in this case

$$\mathcal{M}_0 = \mathcal{M}_\infty = \mathcal{N}_0 = \mathcal{N}(I - T^*T).$$

For (ii), we apply Theorem 2.1 to the regular T^*T-contraction T, having in view that $|T|T$ is quasinormal. In this case one has

$$\begin{aligned}\mathcal{M}_\infty &= \mathcal{N}_0 = \mathcal{N}(T^*T - T^{*2}T^2) \\ &= \mathcal{N}(T^*T - (T^*T)^2) = \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T),\end{aligned}$$

the orthogonal decomposition being obvious. This subspace, denoted $\mathcal{H}_q$, is the maximum T^*T-weighted isometric part of $\mathcal{H}$ relative to T. On the other hand, since T^* and T^*T commute, $\mathcal{H}_q = \mathcal{M}_\infty$ is also invariant for T^*, and by Remark 2.4 we have $\mathcal{H}_q = \mathcal{M}_0$. So, $\mathcal{H}_q$ is the maximum subspace which reduces T to a T^*T-isometry, that is to a quasi-isometry, or equivalently, to a partial isometry (see [6], [11]).

The last assertion in (*ii*) follows from Theorem 2.9 [6], since T is a quasinormal quasi-isometry on $\mathcal{H}_q$. □

Proposition 3.2. *Let T be a quasinormal contraction on $\mathcal{H}$. The following statements hold:*

(*i*) *$\mathcal{H}_* = \mathcal{H} \ominus \bigvee_{n,j\geq 0} T^{*n}(I - T^jT^{*j})\overline{\mathcal{R}(T^*)}$ is the maximum subspace which reduces T to a normal quasi-isometry, or equivalently, on which T^* is a T^*T-isometry. Furthermore, $\mathcal{H}_*$ is the maximum T^*T-weighted isometric part of $\mathcal{H}$ relative to the T^*T-contraction T^*.*

(*ii*) *$\mathcal{H}$ admits the orthogonal decomposition*

$$\mathcal{H} = \bigcap_{n\geq 0} T^n\mathcal{N}(I - TT^*) \oplus \bigvee_{n,j\geq 0} T^{*n}(I - T^jT^{*j})\overline{\mathcal{R}(T^*)} \oplus \mathcal{N}(T), \tag{3.1}$$

where the first subspace reduce T to a unitary operator, and the second subspace contains no nonzero subspace which reduces T to a normal quasi-isometry.

Proof. Since T is a quasinormal contraction one has $TT^* \leq T^*T \leq I$, whence $TT^*TT^* \leq TT^* \leq T^*T$. Hence T^* is a T^*T-contraction, which is regular because T^* commutes with T^*T. In this case, the maximum T^*T-weighted isometric part relative to T^* is just the subspace $\mathcal{H}_*$ from (*i*), having in view (2.5) and the fact that $\overline{\mathcal{R}(T^*T)} = \overline{\mathcal{R}(T^*)}$. Now the form of $\mathcal{H}_*$ immediately gives that $\mathcal{H}_*$ is invariant for T. But by Theorem 2.1, $\mathcal{H}_*$ is also invariant for T^*, and T^* is a T^*T-isometry on $\mathcal{H}_*$. Consequently, by Remark 2.4 we have that $\mathcal{H}_*$ is the maximum subspace which reduces T, on which T^* is a T^*T-isometry. On the other hand, we have

$$\mathcal{H}_* \subset \mathcal{N}(T^*T - TT^*TT^*) \subset \mathcal{N}(T^*T - TT^*) \cap \mathcal{N}(T^*T - T^{*2}T^2),$$

the second inclusion being proved in Theorem 3.4 [11]. So $\mathcal{H}_*$ reduces T to a normal quasi-isometry, and furthermore, it is the maximum subspace with this property. Indeed, if $\mathcal{M} \subset \mathcal{H}$ is another subspace reducing T to a normal quasi-isometry, then T^* will be a T^*T-isometry on $\mathcal{M}$, hence $\mathcal{M} \subset \mathcal{H}_*$ by the above remark. Hence, $\mathcal{H}_*$ is the maximum subspace which reduces T to a normal quasi-isometry, and all assertions from (*i*) are proved.

Next, by Corollary 3.5 [11] we have

$$\mathcal{H}_* = \bigcap_{n=0}^{\infty} T^n\mathcal{N}(I - TT^*) \oplus \mathcal{N}(T),$$

which leads to the decomposition (3.1). The required properties of the subspaces from (3.1) are obtained from the above remarks on $\mathcal{H}_*$ and from Theorem 3.1 [11]. □

In the sequel we denote as usually $|T| = (T^*T)^{1/2}$.

Corollary 3.3. *Let T be an injective quasinormal operator on $\mathcal{H}$, and $T = W|T|$ be the polar decomposition of T. Then the maximum subspace $\mathcal{H}_u$ which reduces W to a unitary operator is the maximum T^*T-weighted isometric part of $\mathcal{H}$ relative to the T^*T-contraction W^*. Moreover, $\mathcal{H}_u$ reduces T to a normal operator, and $\mathcal{H} \ominus \mathcal{H}_u$ reduces T to a pure quasinormal operator.*

Proof. Since T is quasinormal injective, W is an isometry which commutes with $|T|$. Then the decomposition (3.1) with W instead of T gives $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$, where

$$\mathcal{H}_u = \bigcap_{n \geq 1} W^n \mathcal{H}$$

is the unitary part of $\mathcal{H}$ relative to W. But (by Proposition 3.2) $\mathcal{H}_u$ is just the maximum T^*T-weighted isometric subspace of $\mathcal{H}$ relative to W^* as a T^*T-contraction. Then (by Theorem 2.1) $\mathcal{H}_u$ reduces T^*T, hence $\mathcal{H}_u$ reduces $T = W|T|$ and clearly, T is normal (W being unitary) on $\mathcal{H}_u$. Finally, $\mathcal{H}_s = \mathcal{H} \ominus \mathcal{H}_u$ reduces W to a shift (W being an isometry), and $\mathcal{H}_s$ reduces T. But $\mathcal{H}_s$ contains no non zero subspace which reduces $|T|W^* = T^*$ to a T^*T-weighted isometry, that is with property $TT^* = T^*T$. This means that $\mathcal{H}_s$ reduces T to a pure quasinormal operator. □

As an application one obtains the following result.

Corollary 3.4. *Let T be a quasinormal operator on $\mathcal{H}$ and $T = W|T|$ be the polar decomposition of T. Then the maximum subspace which reduces T to a normal operator is*

$$\mathcal{H}_n = \mathcal{H}_u \oplus \mathcal{N}(T), \tag{3.2}$$

where $\mathcal{H}_u$ is the unitary part of $\mathcal{H}$ relative to W.

Moreover, $\mathcal{H}_n$ is the maximum subspace which reduces W to a normal partial isometry, and W is a shift on $\mathcal{H} \ominus \mathcal{H}_n$.

Proof. Since $\mathcal{N}(T)$ reduces T, one can defines the operator $T_0 = T|_{\overline{\mathcal{R}(T^*)}}$ in $\mathcal{B}(\overline{\mathcal{R}(T^*)})$, and T_0 is an injective quasinormal operator. Then the polar decomposition of T_0 is $T_0 = W_0|T_0|$, where $W_0 = W|_{\overline{\mathcal{R}(T^*)}}$. Clearly, $\overline{\mathcal{R}(T^*)}$ reduces W because W and $|T|$ commute.

Now, if we consider the decomposition $\overline{\mathcal{R}(T^*)} = \mathcal{H}_u \oplus \mathcal{H}_s$ in the unitary part $\mathcal{H}_u$ and the completely non-unitary part $\mathcal{H}_s$ relative to W_0, then (by Corollary 3.3) $\mathcal{H}_u$ reduces T_0 to a normal operator, while $\mathcal{H}_s$ reduces T_0 to a pure quasinormal. Clearly, one has

$$\mathcal{H}_u = \bigcap_{n=1}^{\infty} W_0^n \overline{\mathcal{R}(T^*)} = \bigcap_{n=1}^{\infty} W^n \mathcal{H} \subset W\mathcal{H} \subset \overline{\mathcal{R}(T)},$$

hence $\mathcal{H}_u$ and $\mathcal{N}(T^*)$ are orthogonal. As $\mathcal{N}(T) \subset \mathcal{N}(T^*)$, T being quasinormal, it follows that $\mathcal{H}_u$ and $\mathcal{N}(T)$ are orthogonal. Thus $\mathcal{H}_n = \mathcal{H}_u \oplus \mathcal{N}(T)$ is the maximum subspace which reduces T to a unitary operator, because $\mathcal{H} \ominus \mathcal{H}_n = \overline{\mathcal{R}(T^*)} \ominus \mathcal{H}_u$ reduces T to a pure operator. But W is unitary on $\mathcal{H}_u$, and so $\mathcal{H}_n$ reduces W to a normal operator.

Finally, since $\overline{\mathcal{R}(T^*)}$ reduces W, we infer that $W|_{\mathcal{H} \ominus \mathcal{H}_n} = W_0|_{\overline{\mathcal{R}(T^*)} \ominus \mathcal{H}_u}$ is a shift, and in particular W is pure on $\mathcal{H} \ominus \mathcal{H}_n$ (the unitary part of W being $\mathcal{H}_u$). Consequently, $\mathcal{H}_n$ is the maximum subspace on which W is a normal operator. □

Now we obtain, as application, an orthogonal decomposition of $\mathcal{H}$ induced by a quasinormal contraction T, where all reducing subspaces can be completely described in terms of T and W.

Theorem 3.5. *Let T be a quasinormal contraction on $\mathcal{H}$ with the polar decomposition $T = W|T|$. Then $\mathcal{H}$ has the orthogonal decomposition*

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4, \tag{3.3}$$

where

$$\mathcal{H}_0 = \bigcap_{n=0}^{\infty} T^n \mathcal{N}(I - TT^*), \tag{3.4}$$

$$\mathcal{H}_1 = \bigoplus_{n=0}^{\infty} T^n [\mathcal{N}(I - T^*T) \cap \mathcal{N}(T^*)],$$

$$\mathcal{H}_2 = \mathcal{N}(T - |T|) \ominus \mathcal{N}(I - T) = [\mathcal{N}(I - W) \ominus \mathcal{N}(I - T)] \oplus \mathcal{N}(T),$$

$$\mathcal{H}_3 = \Big(\bigcap_{n=1}^{\infty} W^n \mathcal{H} \ominus \mathcal{H}_0\Big) \ominus [\mathcal{N}(I - W) \ominus \mathcal{N}(I - T)],$$

$$\mathcal{H}_4 = \bigoplus_{n=0}^{\infty} W^n [\mathcal{N}(T^*) \ominus \mathcal{N}(T)] \ominus \mathcal{H}_1.$$

Furthermore, all subspaces in (3.3) *reduce T such that $T|_{\mathcal{H}_0}$ is unitary, $T|_{\mathcal{H}_1}$ is a shift, $T|_{\mathcal{H}_2}$ is positive completely nonisometric quasinormal, $T|_{\mathcal{H}_3}$ is normal completely nonpositive and completely nonisometric quasinormal, and $T|_{\mathcal{H}_4}$ is a completely nonisometric pure quasinormal contraction. Also, one has*

$$\mathcal{N}(T - |T|) = \mathcal{N}(I - W) \oplus \mathcal{N}(T), \tag{3.5}$$

$$\mathcal{N}(|T| - T|T|) = \mathcal{N}(I - T) \oplus \mathcal{N}(T).$$

Proof. Using the notations from the previous proof we have

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \oplus \mathcal{N}(T),$$

where $\mathcal{H}_u = \bigcap_{n=1}^{\infty} W^n \mathcal{H}$ and $\mathcal{H}_s = \bigoplus_{n=0}^{\infty} W_0^n \mathcal{N}(W_0^*)$. Let $\mathcal{H}_0$ be the maximum unitary part for the contraction T. As T and W commute, $\mathcal{H}_0$ will reduces W, and $T = W$ on $\mathcal{H}_0$, T being unitary on $\mathcal{H}_0$. So W is unitary on $\mathcal{H}_0$, hence $\mathcal{H}_0 \subset \mathcal{H}_u$, and the above form of $\mathcal{H}_0$ is given in [8] (see also [11, 12]). We also remark that $\mathcal{H}_u \ominus \mathcal{H}_0 \subset \mathcal{H}_n \ominus \mathcal{H}_0$, therefore T is a completely nonisometric normal contraction.

Now, we know from Theorem 3.1 [9] that the maximum subspace which reduces T to a positive operator is $\mathcal{N}_T = \mathcal{N}(T - |T|)$. We even get the decomposition (3.5) for $\mathcal{N}_T$. Indeed, W being a $|T|^2$-contraction which commutes with $|T|$, we have $\widehat{W} = W|_{\mathcal{N}(T)^\perp}$. Also, $\mathcal{N}(I - \widehat{W}) = \mathcal{N}(I - W)$ because $\mathcal{N}(W) = \mathcal{N}(T)$ and

$$\mathcal{N}(I - W) = \mathcal{N}(I - W^*) \subset \mathcal{N}(T)^\perp.$$

Thus, for the regular $|T|^2$-contraction W we obtain

$$\mathcal{N}_T = \mathcal{N}(|T| - |T|W) = \mathcal{N}(I - \widehat{W}) \oplus \mathcal{N}(|T|) = \mathcal{N}(I - W) \oplus \mathcal{N}(T).$$

Clearly, $\mathcal{N}(I - W)$ reduces T to a positive contraction, and since T is unitary on $\mathcal{N}(I - T)$ and $\mathcal{N}(I - T) = \mathcal{N}(I - T^*) \subset \mathcal{N}(T)^\perp$, we have $T = W$ on $\mathcal{N}(I - T)$. Hence $\mathcal{N}(I - T) \subset \mathcal{N}(I - W)$. Then the operator $T|_{\mathcal{N}_T \ominus \mathcal{N}(I-T)}$ being positive, it is completely nonisometric, or equivalently, a completely non unitary contraction. Hence T has the required properties on the subspace

$$\mathcal{H}_2 := \mathcal{N}_T \ominus \mathcal{N}(I - T) = [\mathcal{N}(I - W) \ominus \mathcal{N}(I - T)] \oplus \mathcal{N}(T),$$

and we have, in addition,

$$\mathcal{H}_2 \subset (\mathcal{H}_u \ominus \mathcal{H}_0) \oplus \mathcal{N}(T) = \mathcal{H}_n \ominus \mathcal{H}_0.$$

Next, it is clear that the subspace

$$\mathcal{H}_3 := (\mathcal{H}_n \ominus \mathcal{H}_0) \ominus [\mathcal{N}_T \ominus \mathcal{N}(I - T)] = (\mathcal{H}_u \ominus \mathcal{H}_0) \ominus [\mathcal{N}(I - W) \ominus \mathcal{N}(I - T)]$$

reduces T to a normal, completely nonisometric and completely nonpositive contraction. Clearly, $\mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 = \mathcal{H}_n$.

It remains to analyse the subspace $\mathcal{H}_s$. Recall that $W_0 = W|_{\mathcal{N}(T)^\perp}$, therefore $\mathcal{N}(W_0^*) = \mathcal{N}(T^*) \ominus \mathcal{N}(T)$, and also

$$\mathcal{H}_s = \bigoplus_{n=0}^{\infty} W^n[\mathcal{N}(T^*) \ominus \mathcal{N}(T)].$$

It is immediate that the subspace

$$\mathcal{H}_1 := \mathcal{H}_s \cap \mathcal{N}(I - T^*T)$$

reduces T to a completely non unitary isometry, hence $T|_{\mathcal{H}_1}$ is a shift. Thus, we have

$$\mathcal{H}_1 = \bigoplus_{n=0}^{\infty} T^n \mathcal{N}(T^*|_{\mathcal{H}_1}),$$

and it is easy to see that

$$\mathcal{N}(T^*|_{\mathcal{H}_1}) = \mathcal{N}(I - T^*T) \cap [\mathcal{N}(T^*) \ominus \mathcal{N}(T)] = \mathcal{N}(I - T^*T) \cap \mathcal{N}(T^*).$$

Also, we remark that $\mathcal{H}_1$ is the maximum subspace of $\mathcal{H}_s$ which reduces T to an isometry, because $\mathcal{N}(I - T^*T)$ has the same property in $\mathcal{H}$ (by Proposition 3.1). This also gives that the subspace $\mathcal{H}_4 := \mathcal{H}_s \ominus \mathcal{H}_1$ reduces T to a completely nonisometric pure quasinormal contraction.

Finally, the second decomposition from (3.5) can be proved in a similar way as before, having in view that T is a regular $|T|^2$-contraction and $\widehat{T} = T|_{\mathcal{N}(T)^\perp}$. The proof is finished. □

Corollary 3.6. *Let T be a quasinormal contraction on $\mathcal{H}$ with the polar decomposition $T = W|T|$. One has:*

(i) If T is completely non unitary, then $\mathcal{H}$ has the decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4, \tag{3.6}$$

where $\mathcal{H}_1$ and $\mathcal{H}_4$ are as in (3.4), *and*

$$\mathcal{H}_2 = \mathcal{N}(T - |T|), \quad \mathcal{H}_3 = \bigcap_{n=1}^{\infty} W^n\mathcal{H} \ominus \mathcal{N}(I - W).$$

(ii) If W is completely non unitary, then T is completely non unitary and the decomposition (3.6) *one reduces to*

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{N}(T) \oplus \{0\} \oplus \mathcal{H}_4. \tag{3.7}$$

Corollary 3.7. *Let T be a quasinormal partial isometry on $\mathcal{H}$. Then T is a quasi-isometry and one has the orthogonal decomposition*

$$\mathcal{H} = \bigcap_{n=0}^{\infty} T^n\mathcal{N}(I - T^*T) \oplus \bigoplus_{n=0}^{\infty} T^n[\mathcal{N}(T^*) \ominus \mathcal{N}(T)] \oplus \mathcal{N}(T). \tag{3.8}$$

Proof. Clearly, $T = W$ in Theorem 3.5, therefore $\mathcal{H}_2 = \mathcal{N}(T)$ and $\mathcal{H}_3 = \{0\}$. Also, $T = W$ is a shift on $\mathcal{H}_s = \mathcal{H} \ominus \mathcal{H}_n$ by Corollary 3.4, hence $\mathcal{H}_1 = \mathcal{H}_s$ and $\mathcal{H}_4 = \{0\}$. Therefore, we infer from (3.3)

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T),$$

which means by Proposition 3.1 (*ii*) that T is a quasi-isometry. Also, this implies that

$$\mathcal{H}_0 = \bigcap_{n=0}^{\infty} T^n\mathcal{H} = \bigcap_{n=0}^{\infty} T^n\mathcal{N}(I - T^*T).$$

Thus we obtain $T = V \oplus 0$ where $V = T|_{\mathcal{N}(I-T^*T)}$ is an isometry. □

Remark from Corollary 2.3 [6] that a quasi-isometry is quasinormal if and only if it is a partial isometry. Now we can obtain the following.

Corollary 3.8. *Let $T \neq 0$ be a quasi-isometry with $|T|T$ a quasinormal operator on $\mathcal{H}$. Then the following assertions are equivalent:*

(i) $||T|| = 1$;

(ii) T is partial isometry;

(iii) T is quasinormal;

(iv) T is hyponormal.

Furthermore, if $|T|T$ is normal then these assertions are also equivalent to each of the following two assertions:

(*v*) *T is normal*;
(*vi*) $\mathcal{N}(T) \subset \mathcal{N}(T^*)$.

Proof. First we suppose $||T|| = 1$. Then $S = |T|T$ is a contraction and $S^*S = T^*T^*TT = T^*T$ because T is a quasi-isometry. Also, we have

$$S^{*2}S^2 = S^*T^*TS = T^*|T|T^*T|T| = T^{*2}TT^*T^2,$$

whence one infers on one hand

$$S^{*2}S^2 - S^*S = T^{*2}TT^*T^2 - T^*T = (T^*T^2 - T)^*(T^*T^2 - T) \geq 0.$$

On the other hand, as $S^*S \leq I$ we have $S^{*2}S^2 - S^*S \leq 0$, hence

$$S^{*2}S^2 = S^*S,$$

or equivalently, $T^*T^2 - T = 0$. This means $T = T^*T^2$, that is $T = |S|S$. Since by hypothesis S is quasinormal, hence S and $|S|$ commute, it follows that T is also quasinormal, or equivalently, T is a partial isometry. Thus, we have that $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$.

Now if we assume that T is hyponormal, then for $h \in \mathcal{H}$ we have

$$||T^*Th|| \leq ||T^2h|| = ||Th||,$$

whence $||T||^2 \leq ||T||$ and so $||T|| \leq 1$. Since we also have $||T|| \geq 1$ (T being a quasi-isometry), it follows that $||T|| = 1$. Hence (iv) implies (i).

Next we suppose that $S = |T|T$ is a normal operator. If $||T|| = 1$ then as above we get that $T = |S|S$ is normal, and in this case $\mathcal{N}(T) = \mathcal{N}(T^*)$. Hence we have $(i) \Rightarrow (v) \Rightarrow (vi)$. The implication $(vi) \Rightarrow (v)$ is even Theorem 2.9 [6], and trivially (v) implies (iv). Consequently, all assertions $(i) - (vi)$ are mutually equivalent, if $|T|T$ is a normal operator. □

Remark 3.9. From the previous proof we infer that for any quasi-isometry T with $||T|| = 1$ one has

$$T = T^*T^2,$$

this fact being also quoted by S. M. Patel in [6]. Concerning the question from Remark 2.1 [6], namely if the condition (vi) for a quasi-isometry T assures that T is normal, we can see a simple example which shows that this fact need not holds unless the assumption that $|T|T$ is normal. So, we consider the operator T on $\mathcal{H} \oplus \mathcal{H}$ given by

$$T = \begin{pmatrix} V & 0 \\ 0 & Q \end{pmatrix},$$

where V is an isometry and Q is an orthogonal projection on $\mathcal{H}$. Then $T = |T|T$ is not normal, but T is a quasi-isometry and

$$\mathcal{N}(T) = \{0\} \oplus \mathcal{N}(Q) \subset \mathcal{N}(V^*) \oplus \mathcal{N}(Q) = \mathcal{N}(T^*).$$

So, the answer to Patel's question is negative. In fact, we have the following.

Corollary 3.10. *A quasi-isometry T is normal if and only if $|T|T$ is normal and $\mathcal{N}(T) \subset \mathcal{N}(T^*)$.*

As we remarked, a quasi-isometry T with $|T|T$ normal is a normal partial isometry (by Theorem 2.9 [6]). So, the normal quasi-isometries, or equivalently, the normal partial isometries, play a similar role in the general context of A-contractions like the unitary operators for contractions. We will refer to this fact in a subsequent paper.

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