

FIXED POINTS, EIGENVALUES AND SURJECTIVITY

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ABSTRACT. We prove that a countably condensing operator defined on a closed wedge in a Banach space has a fixed point if it is strictly quasibounded, by using an index theory for such operators. From this, the existence of eigenvalues and surjectivity are deduced.

0. Introduction

Let $f : K \rightarrow K$ be a continuous operator defined on a closed wedge K in a Banach space. The basic problems of finding a solution of nonlinear operator equation are as follows:

- (1) Does f have a fixed point ?
- (2) Does f have an eigenvalue ?
- (3) Is a perturbation of the identity I by f surjective ?

As for problem (3), it was shown in [12] that if a condensing operator f is strictly quasibounded, then $I - f$ is surjective. In [9], various conditions on f which ensure the surjectivity were studied. There are several ways of approach: The one is to use a fixed point theorem or an invariance of domain theorem, where the latter case can be found in [3, 10]. The other is a direct method to apply the index theory for condensing operators developed by Nussbaum [8]. It can be easily checked that problems (1) and (3) are equivalent.

It is natural to investigate the above problems for a large class of countably condensing operators, roughly speaking, condensing only on countable subsets of the space; see [7, 11]. In Section 1, we introduce a fixed point index theory for countably condensing operators due to Väth [11]. Using the index theory, we prove in Section 2 that a countably condensing operator f has a fixed point if it is strictly quasibounded. In Sections 3 and 4, it is shown that under suitable quasiboundedness condition the existence of eigenvalues and surjectivity are deduced from the fixed point theorem; more precisely, the problems (1), (2) and (3) are all equivalent. For the surjectivity, we present another direct proof which is based on the index theory stated in Section 1. In Section 5, we will

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observe problems (1) and (2) in a new aspect, which is motivated by recent works of Isac and Németh [4, 5].

1. An index theory

In this section, we introduce a concept of countably condensing maps and a fixed point index theory for such maps due to Väth [11] on which our topological approach is based.

In what follows, K will always be a closed wedge in a Banach space E . Here a wedge means that $ax + by \in K$ whenever $a, b \in [0, \infty)$ and $x, y \in K$.

Given a nonempty subset Ω of K , the closure, the boundary, and the convex hull of Ω in K are denoted by $\bar{\Omega}$, $\partial\Omega$, and $\text{co}\Omega$, respectively.

A function $\gamma : \{M \subseteq K : M \text{ is bounded}\} \rightarrow [0, \infty)$ is said to be a *measure of noncompactness* on K if it satisfies the following properties:

- (1) $\gamma(\bar{M}) = \gamma(M)$;
- (2) $\gamma(\text{co } M) = \gamma(M)$;
- (3) $\gamma(M) = 0$ if and only if M is relatively compact;
- (4) $\gamma(M \cup N) = \max\{\gamma(M), \gamma(N)\}$;
- (5) $\gamma(M + N) \leq \gamma(M) + \gamma(N)$; and
- (6) $\gamma(\alpha M) = \alpha\gamma(M)$ for all $\alpha \geq 0$.

In this case, (4) implies that $\gamma(N) \leq \gamma(M)$ if $N \subseteq M$. Note that the Kuratowski or the Hausdorff measure of noncompactness has the above properties; see [1].

Let Ω be a nonempty subset of K and γ a measure of noncompactness on K , and $k > 0$. A continuous map $f : \Omega \rightarrow K$ is said to be *countably k - γ -condensing* if $\gamma(f(C)) < k\gamma(C)$ for each countable bounded set $C \subseteq \Omega$ with $\gamma(C) > 0$. If $k = 1$, we call f *countably γ -condensing*. In particular, if $\gamma(f(B)) < k\gamma(B)$ for all bounded sets $B \subseteq \Omega$ with $\gamma(B) > 0$, then f is called *k - γ -condensing*. If $k = 1$, we call f *γ -condensing*.

A continuous homotopy $h : [0, 1] \times \Omega \rightarrow K$ is said to be *countably γ -condensing* if $\gamma(h([0, 1] \times C)) < \gamma(C)$ for each countable bounded set $C \subseteq \Omega$ with $\gamma(C) > 0$.

An example of a countably γ -condensing map is given in [7], where γ is the Kuratowski or the Hausdorff measure of noncompactness; see the proof of Theorem 3.1 in [7].

Let Ω be a nonempty open set in K . If $f : \bar{\Omega} \rightarrow K$ is a countably γ -condensing map that has no fixed points on $\partial\Omega$, one may define the fixed point index of f on Ω as an integer, denoted by $\text{ind}_K(f, \Omega)$. For details of this definition, we refer to [11]. The fixed point index for γ -condensing maps, where γ is the Kuratowski measure of noncompactness, has been developed by Nussbaum [8]. In case when f is a compact map, the above indices agree with the Leray-Schauder degree; see [2, 8, 11].

The fixed point index has the following basic properties; see [11, Theorem 1.3] and [11, Corollary 2.1].

Lemma 1. *Let Ω be a nonempty bounded open set in K and $f : \overline{\Omega} \rightarrow K$ a countably γ -condensing map such that f has no fixed points on $\partial\Omega$. Then the following statements hold:*

- (1) (Existence) *If $\text{ind}_K(f, \Omega) \neq 0$, then f has a fixed point in Ω .*
- (2) (Normalization) *If $f \equiv 0$ and $0 \in \Omega$, then $\text{ind}_K(f, \Omega) = 1$.*
- (3) (Homotopy invariance) *If $h : [0, 1] \times \overline{\Omega} \rightarrow K$ is a countably γ -condensing homotopy such that $h(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial\Omega$, then*

$$\text{ind}_K(h(0, \cdot), \Omega) = \text{ind}_K(h(1, \cdot), \Omega).$$

2. Fixed points

In this section, we give a fixed point theorem for countably condensing maps, where the method is to use the index theory stated in Section 1. This is a starting point for establishing relations with eigenvalues and surjectivity.

Theorem 1. *Let $(E, \|\cdot\|)$ be a Banach space and K a closed wedge in E . Suppose that $f : K \rightarrow K$ is a countably γ -condensing map such that it is strictly quasibounded, that is,*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|f(x)\|}{\|x\|} < 1.$$

Then f has a fixed point in K .

Proof. It follows from the definition of ℓ that there is a real number $R > 0$ such that

$$\|f(x)\| \leq \ell\|x\| \quad \text{for all } x \in K \text{ with } \|x\| \geq R.$$

Let $B = \{x \in E : \|x\| < R\}$ be an open ball with center at 0 and radius R and set $B_K := B \cap K$. Then 0 belongs to B_K and $\partial B_K = \{x \in K : \|x\| = R\}$. Consider a homotopy $h : [0, 1] \times \overline{B}_K \rightarrow K$ defined by

$$h(t, x) := tf(x) \quad \text{for } (t, x) \in [0, 1] \times \overline{B}_K.$$

Then h is countably γ -condensing on $[0, 1] \times \overline{B}_K$: For each countable set $C \subseteq \overline{B}_K$ with $\gamma(C) > 0$, since $h([0, 1] \times C) \subseteq \text{co}[f(C) \cup \{0\}]$ and f is countably γ -condensing, we have

$$\gamma(h([0, 1] \times C)) \leq \gamma(\text{co}[f(C) \cup \{0\}]) = \gamma(f(C)) < \gamma(C).$$

We now claim that $h(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial B_K$. Indeed, if $h(t, x) = x$ for some $(t, x) \in [0, 1] \times \partial B_K$, then

$$R = \|x\| = \|h(t, x)\| \leq \|f(x)\| \leq \ell\|x\| = \ell R$$

and hence $(1 - \ell)R \leq 0$, which contradicts the fact that $0 \leq \ell < 1$ and $R > 0$. Therefore, the homotopy invariance of the fixed point index implies that

$$\text{ind}_K(h(1, \cdot), B_K) = \text{ind}_K(h(0, \cdot), B_K).$$

Since $h(0, \cdot) \equiv 0$ and $0 \in B_K$, we obtain from Lemma 1 that $\text{ind}_K(h(0, \cdot), B_K) = 1$ and so $\text{ind}_K(h(1, \cdot), B_K) = 1$. Again by Lemma 1, we conclude that f has a fixed point in B_K . This completes the proof. \square

The following result is an immediate consequence of Theorem 1 if we take $K = E$.

Corollary 1. *Let $f : E \rightarrow E$ be a countably γ -condensing map on a Banach space E . If f is strictly quasibounded, then f has a fixed point in E .*

3. Eigenvalues

In this section, the existence of eigenvalues for countably condensing maps is proved, by using a fixed point theorem in the previous section.

Theorem 2. *Let K be a closed wedge in a Banach space E . Suppose that $f : K \rightarrow K$ is a countably γ -condensing map such that $f(0) \neq 0$ and*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|f(x)\|}{\|x\|} < \infty.$$

If $\lambda > \ell$ and $\lambda \geq 1$, then λ is an eigenvalue of f .

Proof. Let $\lambda > \ell$ and $\lambda \geq 1$. Consider a map $g : K \rightarrow K$ defined by

$$g(x) := \frac{1}{\lambda} f(x) \quad \text{for } x \in K.$$

Then g is countably γ -condensing: For each countable bounded set $C \subseteq K$ with $\gamma(C) > 0$, we obtain from $\lambda \geq 1$ that

$$\gamma(g(C)) = \frac{1}{\lambda} \gamma(f(C)) \leq \gamma(f(C)) < \gamma(C)$$

because f is countably γ -condensing. From $\lambda > \ell$ it follows that

$$\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|g(x)\|}{\|x\|} = \frac{1}{\lambda} \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|f(x)\|}{\|x\|} = \frac{\ell}{\lambda} < 1.$$

Hence Theorem 1 implies that g has a fixed point x_0 in K . Because of $f(0) \neq 0$, we have that $f(x_0) = \lambda x_0$ and $x_0 \neq 0$. Thus, λ is an eigenvalue of f . \square

Corollary 2. *Let $f : E \rightarrow E$ be a countably γ -condensing map on a Banach space E such that $f(0) \neq 0$ and*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in E}} \frac{\|f(x)\|}{\|x\|} < \infty.$$

Then λ is an eigenvalue of f for every $\lambda > \ell$ with $\lambda \geq 1$.

Now we consider eigenvalues of countably k -condensing maps when $0 < k < 1$.

Theorem 3. *Let K be a closed wedge in a Banach space E . Let $f : K \rightarrow K$ be a countably k - γ -condensing map with $0 < k < 1$ such that $f(0) \neq 0$ and*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|f(x)\|}{\|x\|} < \infty.$$

If $\lambda > \ell$ and $\lambda \geq k$, then λ is an eigenvalue of f .

Proof. Fix $\lambda > \ell$ such that $\lambda \geq k$. Consider a map $g : K \rightarrow K$ defined by

$$g(x) := \frac{1}{k}f(x) \quad \text{for } x \in K.$$

Then g is countably γ -condensing: For each countable bounded set $C \subseteq K$ with $\gamma(C) > 0$, we have

$$\gamma(g(C)) = \frac{1}{k}\gamma(f(C)) < \gamma(C)$$

because f is countably k - γ -condensing. It follows from $f(0) \neq 0$ that $g(0) \neq 0$ and

$$\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|g(x)\|}{\|x\|} = \frac{\ell}{k} < \infty.$$

Theorem 2 tells us that λ' is an eigenvalue of g for every $\lambda' > \ell/k$ with $\lambda' \geq 1$. Taking $\lambda' = \lambda/k$, there exists an $x \in K$ with $x \neq 0$ such that $g(x) = (\lambda/k)x$ or $f(x) = \lambda x$. This completes the proof. \square

Corollary 3. *Let $f : E \rightarrow E$ be a countably k - γ -condensing map on a Banach space E with $0 < k < 1$ such that $f(0) \neq 0$ and*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in E}} \frac{\|f(x)\|}{\|x\|} < \infty.$$

Then λ is an eigenvalue of f for every $\lambda > \ell$ with $\lambda \geq k$.

Remark 1. Theorems 1, 2 and 3 are all equivalent. From the proof of Theorems 2 and 3, respectively, it is clear that Theorem 1 implies Theorem 2 which implies Theorem 3. It remains to consider that Theorem 3 implies Theorem 1. Indeed, let f and $\ell < 1$ be as in Theorem 1. Fix $c \in (0, 1)$. The map $g : K \rightarrow K$ defined by $g(x) = cf(x)$ for $x \in K$ is countably c - γ -condensing and $\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \|g(x)\|/\|x\| = c\ell$. If $g(0) = 0$, then $f(0) = 0$, that is, $0 \in K$ is a fixed point of f . Now suppose that $g(0) \neq 0$. In view of Theorem 3, λ is an eigenvalue of g for every $\lambda > c\ell$ with $\lambda \geq c$. Because of $\ell < 1$, we may take $\lambda = c$. Then there exists an $x \in K \setminus \{0\}$ such that $g(x) = cx$ or $f(x) = x$. Thus, Theorem 1 holds.

4. Surjectivity

In this section, we show that under certain conditions a countably condensing perturbation of the identity is surjective, by using a fixed point theorem in Section 2. We present another direct proof which is based on the index theory stated in Section 1.

Theorem 4. *Let K be a closed wedge in a Banach space E . Suppose that $f : K \rightarrow K$ is a countably γ -condensing map such that $(I - f)(K) \subseteq K$ and*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|f(x)\|}{\|x\|} < 1.$$

Then $(I - f)|_K : K \rightarrow K$ is surjective.

Proof. For each fixed $y \in K$, consider a map $f_y : K \rightarrow K$ defined by $f_y(x) := f(x) + y$ for $x \in K$. Then f_y is clearly countably γ -condensing and we have

$$\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|f_y(x)\|}{\|x\|} \leq \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|f(x)\|}{\|x\|} + \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|y\|}{\|x\|} < 1.$$

Applying Theorem 1, the map f_y has a fixed point x in K , that is, $f(x) + y = x$. Thus, $(I - f)|_K$ is surjective. \square

Another Proof. Fix $y \in K$. By the definition of ℓ , there is a real number $R_0 > 0$ such that

$$\|f(x)\| \leq \ell \|x\| \quad \text{for all } x \in K \text{ with } \|x\| \geq R_0.$$

Choose a real number $R \geq R_0$ such that $(1 - \ell)R > \|y\|$. Let $B := \{x \in E : \|x\| < R\}$ and $B_K := B \cap K$. Consider a homotopy $h : [0, 1] \times \overline{B}_K \rightarrow K$ defined by

$$h(t, x) := ty + tf(x) \quad \text{for } (t, x) \in [0, 1] \times \overline{B}_K.$$

Then h is countably γ -condensing on $[0, 1] \times \overline{B}_K$: For each countable set $C \subseteq \overline{B}_K$ with $\gamma(C) > 0$, since $h([0, 1] \times C) \subseteq \text{co}[(\{y\} + f(C)) \cup \{0\}]$ and f is countably γ -condensing, we have

$$\gamma(h([0, 1] \times C)) \leq \gamma(\text{co}[(\{y\} + f(C)) \cup \{0\}]) \leq \gamma(\{y\}) + \gamma(f(C)) < \gamma(C).$$

We now claim that $h(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial B_K$. Indeed, if $h(t, x) = x$ for some $(t, x) \in [0, 1] \times \partial B_K$, then

$$R = \|x\| = \|h(t, x)\| \leq \|y\| + \|f(x)\| \leq \|y\| + \ell \|x\| = \|y\| + \ell R$$

and hence $(1 - \ell)R \leq \|y\|$, which contradicts our choice of R . Lemma 1 implies that

$$\text{ind}_K(h(1, \cdot), B_K) = \text{ind}_K(h(0, \cdot), B_K) = 1.$$

Therefore, there exists an $x \in K$ such that $x = y + f(x)$. We conclude that $(I - f)|_K$ is surjective. \square

The following corollary is a generalization of the corresponding results for compact or condensing maps; see [12, 13].

Corollary 4. *Let $f : E \rightarrow E$ be a countably γ -condensing map on a Banach space E . If f is strictly quasibounded, then $I - f$ is surjective.*

Remark 2. Theorem 1 is equivalent to Theorem 4 whenever $(I - f)(K) \subseteq K$. As we have seen in the proof of Theorem 4, Theorem 1 implies Theorem 4 and Theorem 1 is an immediate consequence of Theorem 4 because 0 belongs to K .

Furthermore, we give the surjectivity of countably k -condensing maps when $0 < k < 1$.

Theorem 5. *Let K be a closed wedge in a Banach space E . Suppose that $f : K \rightarrow K$ is a countably k - γ -condensing map with $0 < k < 1$ such that*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|f(x)\|}{\|x\|} < \infty.$$

If $\lambda > \ell$ and $\lambda \geq k$, then $(\lambda I - f)|_K : K \rightarrow K$ is surjective whenever $(\lambda I - f)(K) \subseteq K$.

Proof. Fix $\lambda > \ell$ such that $\lambda \geq k$. It suffices to show that $(I - (1/\lambda)f)|_K : K \rightarrow K$ is surjective because K is a wedge in E . Consider a map $g : K \rightarrow K$ defined by

$$g(x) := \frac{1}{\lambda}f(x) \quad \text{for } x \in K.$$

Then g is countably γ -condensing because for each countable bounded set $C \subseteq K$ with $\gamma(C) > 0$ we have

$$\gamma(g(C)) = \frac{1}{\lambda}\gamma(f(C)) < \frac{k}{\lambda}\gamma(C) \leq \gamma(C).$$

It follows from $\lambda > \ell$ that

$$\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|g(x)\|}{\|x\|} = \frac{\ell}{\lambda} < 1.$$

By Theorem 4, the map $(I - g)|_K$ is surjective. This completes the proof. \square

Corollary 5. *Let $f : E \rightarrow E$ be a countably k - γ -condensing map on a Banach space E with $0 < k < 1$ such that*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in E}} \frac{\|f(x)\|}{\|x\|} < \infty.$$

Then $\lambda I - f$ is surjective for every $\lambda > \ell$ with $\lambda \geq k$.

Remark 3. Theorems 1–5 are all equivalent. We know by Remarks 1 and 2 that Theorems 1–4 are all equivalent. It is obvious that Theorem 4 implies Theorem 5. Theorem 3 is deduced from Theorem 5 whenever $(\lambda I - f)(K) \subseteq K$ because $0 \in K$ and $f(0) \neq 0$ imply that there is an $x \in K$ such that $f(x) = \lambda x$ and $x \neq 0$.

Remark 4. Corollaries 1–5 are all equivalent. As in Remarks 1–3, it can be proved by similar arguments if we take $K = E$.

5. Further observation

The goal of this section is to observe the previous results in Sections 2 and 3 in a more general setting. It is motivated by recent works of Isac and Németh [4, 5].

We extend Theorem 6.1 of [4] for α -condensing maps to a class of countably γ -condensing maps, where α is the Kuratowski measure of noncompactness.

Theorem A. *Let K be a closed wedge in a Banach space E . Suppose that $f : K \rightarrow K$ is a countably γ -condensing map such that*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{B(f(x), x)}{B(x, x)} < 1,$$

where $B : E \times E \rightarrow \mathbb{R}$ is a map which satisfies the following conditions:

- (b₁) $B(\lambda x, y) = \lambda B(x, y)$ for all $\lambda > 0$ and all $x, y \in E$.
- (b₂) $B(x, x) > 0$ for all $x \in E$ with $x \neq 0$.

Then f has a fixed point in K .

Proof. Consider the map $h : [0, 1] \times K \rightarrow K$ defined by

$$h(t, x) := tf(x) \quad \text{for } (t, x) \in [0, 1] \times K.$$

We first claim that there is a real number $R > 0$ such that $h(t, x) \neq x$ for all $x \in K$ with $\|x\| = R$ and all $t \in [0, 1]$. We assume the contrary, then for each integer $n > 0$ there exist an $x_n \in K$ and a $t_n \in [0, 1]$ such that $\|x_n\| = n$ and $h(t_n, x_n) = x_n$. It follows that $t_n f(x_n) = x_n$ and $t_n \neq 0$. Thus we have by (b₁) and (b₂)

$$\frac{B(f(x_n), x_n)}{B(x_n, x_n)} = \frac{1}{t_n} \quad \text{for all } n \in \mathbb{N}.$$

Since $t_n \in [0, 1]$ for all $n \in \mathbb{N}$, the sequence $\{t_n\}_{n \in \mathbb{N}}$ has a subsequence $\{t_{n_k}\}_{k \in \mathbb{N}}$ which converges to some point t_0 in $[0, 1]$. Hence we have

$$\lim_{k \rightarrow \infty} \frac{B(f(x_{n_k}), x_{n_k})}{B(x_{n_k}, x_{n_k})} \in [1, \infty].$$

From $\lim_{k \rightarrow \infty} \|x_{n_k}\| = \infty$ it follows that

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{B(f(x), x)}{B(x, x)} \geq 1$$

which is a contradiction. Now let $R > 0$ be a fixed real number that has the above property. Set $B := \{x \in E : \|x\| < R\}$ and $B_K := B \cap K$. As in the proof of Theorem 1, since h is countably γ -condensing on $[0, 1] \times \overline{B}_K$ and since $h(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial B_K$, Lemma 1 implies that

$$\text{ind}_K(h(1, \cdot), B_K) = \text{ind}_K(h(0, \cdot), B_K) = 1.$$

Therefore, f has a fixed point in K . □

The following result is a countably condensing version of Theorem 4.4 in [5].

Theorem B. *Let K be a closed wedge in a Banach space E . Suppose that $f : K \rightarrow K$ is a countably γ -condensing map such that $f(0) \neq 0$ and*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{B(f(x), x)}{B(x, x)} < \infty,$$

where $B : E \times E \rightarrow \mathbb{R}$ is as in Theorem A. If $\lambda > \ell$ and $\lambda \geq 1$, then λ is an eigenvalue of f .

Proof. Let $\lambda > \ell$ and $\lambda \geq 1$. The map $g : K \rightarrow K$ defined by

$$g(x) := \frac{1}{\lambda} f(x) \quad \text{for } x \in K$$

is obviously countably γ -condensing. It follows from (b_1) and $\lambda > \ell$ that

$$\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{B(g(x), x)}{B(x, x)} = \frac{\ell}{\lambda} < 1.$$

Theorem A implies that g has a fixed point x_0 in K . From $f(0) \neq 0$ we obtain that $f(x_0) = \lambda x_0$ and $x_0 \neq 0$. This completes the proof. \square

Finally, we give the following result on eigenvalues for countably k -condensing maps.

Theorem C. *Let K be a closed wedge in a Banach space E . Let $f : K \rightarrow K$ be a countably k - γ -condensing map with $0 < k < 1$ such that $f(0) \neq 0$ and*

$$\ell = \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{B(f(x), x)}{B(x, x)} < \infty,$$

where $B : E \times E \rightarrow \mathbb{R}$ is a map which satisfies conditions (b_1) and (b_2) . Then λ is an eigenvalue of f for every $\lambda > \ell$ with $\lambda \geq k$.

Proof. Fix $\lambda > \ell$ such that $\lambda \geq k$. Let $g : K \rightarrow K$ be a map defined by

$$g(x) := \frac{1}{k} f(x) \quad \text{for } x \in K.$$

Then g is countably γ -condensing, $g(0) \neq 0$ and

$$\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{B(g(x), x)}{B(x, x)} = \frac{\ell}{k} < \infty,$$

Theorem B states that λ' is an eigenvalue of g for every $\lambda' > \ell/k$ and $\lambda' \geq 1$. Taking $\lambda' = \lambda/k$, there exists an $x \in K$ with $x \neq 0$ such that $f(x) = \lambda x$. This completes the proof. \square

Remark 5. Theorems A, B and C are all equivalent. We only have to check that Theorem C implies Theorem A. Indeed, let f and $\ell < 1$ be as in Theorem A. Fix $c \in (0, 1)$. The map $g : K \rightarrow K$ given by $g(x) = cf(x)$ for $x \in K$ is countably c - γ -condensing and $\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} B(g(x), x)/B(x, x) = c\ell$. If $g(0) = 0$, then $0 \in K$ is trivially a fixed point of f . Now suppose that $g(0) \neq 0$. Theorem C

says that λ is an eigenvalue of g for every $\lambda > c\ell$ with $\lambda \geq c$. Taking $\lambda = c$, there exists an $x \in K \setminus \{0\}$ such that $f(x) = x$. Thus, Theorem A holds.

Let E be a real vector space. Then $[\cdot, \cdot] : E \times E \rightarrow \mathbb{R}$ is said to be a *semi-inner-product* on E if it satisfies the following conditions:

- (s₁) $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in E$.
- (s₂) $[\lambda x, y] = \lambda[x, y]$ for all $\lambda \in \mathbb{R}$ and all $x, y \in E$.
- (s₃) $[x, x] > 0$ for all $x \in E$ with $x \neq 0$.
- (s₄) $|[x, y]|^2 \leq [x, x][y, y]$ for all $x, y \in E$.

It is known in [6] that for every normed space $(E, \|\cdot\|)$ one can construct at least one semi-inner-product on E consistent with the norm in the sense $[x, x] = \|x\|^2$. In this case, we say that the semi-inner-product is *compatible* with the norm $\|\cdot\|$.

Remark 6. If $[\cdot, \cdot]$ is a semi-inner-product on a real Banach space E that is compatible with the norm $\|\cdot\|$, then Theorem A implies Theorem 1. To see this, let f and $\ell < 1$ be as in Theorem 1 and let $B : E \times E \rightarrow \mathbb{R}$ be defined by

$$B(x, y) := [x, y] \quad \text{for } (x, y) \in E \times E.$$

Since $[x, x] = \|x\|^2$ and $[f(x), x] \leq \|f(x)\| \|x\|$ by (s₄) for all $x \in E$, it follows that

$$\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{[f(x), x]}{[x, x]} \leq \limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\|f(x)\| \|x\|}{\|x\|^2}.$$

Applying Theorem A, the map f has a fixed point in K ; thus Theorem 1 holds. Moreover, a similar argument shows that Theorem B implies Theorem 2 and Theorem C implies Theorem 3, respectively.

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