J. Korean Math. Soc. **45** (2008), No. 1, pp. 121–136

ON CERTAIN NEW NONLINEAR RETARDED INTEGRAL INEQUALITIES FOR FUNCTIONS IN TWO VARIABLES AND THEIR APPLICATIONS

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ABSTRACT. Some new explicit bounds on the solutions to a class of new nonlinear retarded Volterra-Fredholm type integral inequalities in two independent variables are established, which can be used as effective tools in the study of certain integral equations. Some examples of application are also indicated.

1. Introduction

In the study of ordinary differential equations and integral equations one often deals with certain integral inequalities. The Gronwall-Bellman inequality and its various linear and nonlinear generalizations are crucial in the discussion of the existence, uniqueness, continuation, boundedness, oscillation and stability, and other qualitative properties of solutions of differential and integral equations. The literature on such inequalities and their applications is vast; see [1, 2, 12, 20, 24] and the references given therein.

To handle ordinary differential and integral equations with retardation, some delay Volterra-type integral inequalities are needed. During the past few years, some investigators have established some useful and interesting delay Volterra-type integral inequalities in order to achieve various goals; see [3, 5, 10, 11, 13-18, 26] and the references cited therein. Recently, in [25], Pachpatte has established the following useful linear Volterra-Fredholm type integral inequality in two independent variables with retardation:

Theorem 1.1 ([25]). *Let* $u(x,y) \in C(\Delta, R_+)$, $a(x,y,s,t)$, $b(x,y,s,t) \in C(E, R_+)$ *and* $a(x,y,s,t)$, $b(x,y,s,t)$ *be nondecreasing in* x *and* y *for each* $s \in I_1, t \in I_2, \alpha \in C^1(I_1, I_1), \beta \in C^1(I_2, I_2)$ *be nondecreasing with* $\alpha(x) \leq x$ *on*

Received May 22, 2006.

2000 *Mathematics Subject Classification.* 26D15, 35A05.

Key words and phrases. nonlinear retarded inequality, two variables, explicit bound, Volterra-Fredholm type integral equations.

The first author's research was supported by the Research Group Grants Council of the Guangdong University of Foreign Studies (Project No. GW2006-TB-002) of China.

$I_1, \beta(y) \leq y$ *on* I_2 *and suppose that*

$$u(x,y) \leq k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)u(s,t)dtds \tag{1.1}$$
$$+ \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)u(s,t)dtds$$

for $(x,y) \in \Delta$, *where* $k \geq 0$ *is a constant. If*

$$p(x,y) = \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) \exp\left(\int_{\alpha(x_0)}^{\alpha(s)} \int_{\beta(y_0)}^{\beta(t)} a(x,y,s,t)d\tau xd\sigma\right) dtds$$
$$< 1$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq \frac{c}{1-p(x,y)} \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)dtds\right)$$

for $(x,y) \in \Delta$, *where* $I_1 = [x_0, M], I_2 = [y_0, N], \Delta = I_1 \times I_2$ *and* $E = \{(x,y,s,t) \in \Delta^2 : x_0 \leq s \leq x \leq M, y_0 \leq t \leq y \leq N\}$.

In this paper, we consider the explicit bounds on some general versions of (1.1) which the constant k on the right side of (1.1) is replaced by the function $l(x,y)$ and contain some power nonlinear terms with respect to the unknown function $u(x,y)$ on the both side of (1.1). Our results can be used as handy and effective tools in the study of the qualitative behavior of the solutions of certain retarded Volterra-Fredholm type integral equations. To illustrate this, some examples of application are given. Our results also generalize some results in [11].

2. Retarded integral inequalities with power nonlinear

In what follows, R denotes the set of real numbers, $R_+ = [0, +\infty), R_0 = (0, +\infty), I_1 = [x_0, M]$ and $I_2 = [y_0, N]$ are the given subsets of R. Let $\Delta = I_1 \times I_2$ and

$$E = \{(x,y,s,t) \in \Delta^2 : x_0 \leq s \leq x \leq M, y_0 \leq t \leq y \leq N\}.$$

$C^i(M,S)$ denotes the class of all i-times continuously differentiable functions defined on set M with range in the set S $(i = 1, 2, \ldots)$ and $C^0(M,S) = C(M,S)$.

Before giving our main results, we need the following important lemma in our proof.

Lemma 2.1 ([10]). *Let* $a \geq 0, p \geq q \geq 0$ *and* $p \neq 0$. *Then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}$$

for any $K > 0$.

Theorem 2.1. *Let $u(x,y)$ and $l(x,y) \in C(\Delta, R_+)$, $a(x,y,s,t)$ and $b(x,y,s,t) \in C(E, R_+)$, $a(x,y,s,t)$ and $b(x,y,s,t)$ be nondecreasing in x and y for each $s \in I_1$, and $t \in I_2$, $\alpha \in C^1(I_1, I_1)$, $\beta \in C^1(I_2, I_2)$ be nondecreasing with $\alpha(x) \le x$ on $I_1, \beta(y) \le y$ on I_2. If $u(x,y)$ satisfies*

$$\begin{aligned} u^p(x,y) \le l(x,y) &+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t) u^q(s,t) dt ds \\ &+ \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) u^r(s,t) dt ds \end{aligned} \tag{2.1}$$

for $(x,y) \in \Delta$, where $p \ge q \ge 0, p \ge r \ge 0, p, q$ and r are constants and

$$\lambda_1(x,y) = \frac{r}{p} \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) K_2^{\frac{r-p}{p}}(s,t) \exp(A_1(s,t)) dt ds < 1, \tag{2.2}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \le \left[l(x,y) + \frac{\bar{A}_1(x,y) + B_1(x,y)}{1 - \lambda_1(x,y)} \exp\left(A_1(x,y)\right) \right]^{\frac{1}{p}} \tag{2.3}$$

for $(x,y) \in \Delta$ and any $K_i(x,y) \in C(\Delta, R_0) (i = 1,2)$, where

$$A_1(x,y) = \frac{q}{p} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t) K_1^{\frac{q-p}{p}}(s,t) dt ds, \tag{2.4}$$

$$\bar{A}_1(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t) \left[\frac{q}{p} K_1^{\frac{q-p}{p}}(s,t) l(s,t) + \frac{p-q}{p} K_1^{\frac{q}{p}}(s,t) \right] dt ds \tag{2.5}$$

and

$$B_1(x,y) = \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) \left[\frac{r}{p} K_2^{\frac{r-p}{p}}(s,t) l(s,t) + \frac{p-r}{p} K_2^{\frac{r}{p}}(s,t) \right] dt ds \tag{2.6}$$

for $(x,y) \in \Delta$.

Proof. Define a function $v(x,y)$ by

$$\begin{aligned} v(x,y) = &\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t) u^q(s,t) dt ds \\ &+ \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) u^r(s,t) dt ds \end{aligned} \tag{2.7}$$

for $(x,y) \in \Delta$, then

$$u^p(x,y) \le l(x,y) + v(x,y),$$

or

$$u(x,y) \le (l(x,y) + v(x,y))^{\frac{1}{p}}. \tag{2.8}$$

By Lemma 2.1 and (2.8), for any $K_i(x,y) \in C(\Delta, R_0)$ $(i = 1,2)$, we have

$$u^q(x,y) \le (l(x,y)+v(x,y))^{\frac{q}{p}} \le \frac{q}{p}K_1^{\frac{q-p}{p}}(x,y)(l(x,y)+v(x,y)) + \frac{p-q}{p}K_1^{\frac{q}{p}}(x,y),$$

and

$$u^r(x,y) \le (l(x,y)+v(x,y))^{\frac{r}{p}} \le \frac{r}{p}K_2^{\frac{r-p}{p}}(x,y)(l(x,y)+v(x,y)) + \frac{p-r}{p}K_2^{\frac{r}{p}}(x,y).$$

Substituting the last relations into (2.7) we get

(2.9)

$$\begin{aligned} &v(x,y) \\ &\le \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)\Big(\frac{q}{p}K_1^{\frac{q-p}{p}}(s,t)(l(s,t)+v(s,t)) + \frac{p-q}{p}K_1^{\frac{q}{p}}(s,t)\Big)dtds \\ &\quad + \int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)\Big(\frac{r}{p}K_2^{\frac{r-p}{p}}(s,t)(l(s,t)+v(s,t)) + \frac{p-r}{p}K_2^{\frac{r}{p}}(s,t)\Big)dtds \\ &\le \bar{A}_1(x,y) + B_1(x,y) + \frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)K_1^{\frac{q-p}{p}}(s,t)v(s,t)dtds \\ &\quad + \frac{r}{p}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)K_2^{\frac{r-p}{p}}(s,t)v(s,t)dtds, \end{aligned}$$

where $\bar{A}_1(x,y)$ and $B_1(x,y)$ are defined as in (2.5) and (2.6) respectively. It is easy to see that $\bar{A}_1(x,y)$ and $C_1(x,y)$ are nonnegative, continuous and nondecreasing for $(x,y) \in \Delta$.

From the assumptions, we observe that $\alpha'(x) \ge 0$ for $x \in I$. Fixing any arbitrary $(X,Y) \in \Delta$, then for $(x,y) \in \Delta_1 = [x_0, X] \times [y_0, Y]$, from (2.9) we have

$$\begin{aligned} v(x,y) &\le \bar{A}_1(X,Y) + C_1(X,Y) \\ &\quad + \frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(X,Y,s,t)K_1^{\frac{q-p}{p}}(s,t)v(s,t)dtds \\ &\quad + \frac{r}{p}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(X,Y,s,t)K_2^{\frac{r-p}{p}}(s,t)v(s,t)dtds. \end{aligned} \tag{2.10}$$

Define a function $w(x,y), (x,y) \in \Delta_1$ by the right hand side of (2.10). Then for $(x,y) \in \Delta_1, w(x,y)$ is positive and nondecreasing,

$$v(x,y) \le w(x,y), \tag{2.11}$$

$$\begin{aligned} w(x_0,y) &= \bar{A}_1(X,Y) + C_1(X,Y) \\ &\quad + \frac{r}{p}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(X,Y,s,t)K_2^{\frac{r-p}{p}}(s,t)v(s,t)dtds \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}w(x,y) &= \frac{q}{p}\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} a(X,Y,\alpha(x),t)K_1^{\frac{q-p}{p}}(\alpha(x),t)v(\alpha(x),t)dt\\ &\le \frac{q}{p}\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} a(X,Y,\alpha(x),t)K_1^{\frac{q-p}{p}}(\alpha(x),t)w(\alpha(x),t)dt\\ &\le \frac{q}{p}w(\alpha(x),\beta(y))\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} a(X,Y,\alpha(x),t)K_1^{\frac{q-p}{p}}(\alpha(x),t)dt\\ &\le \frac{q}{p}w(x,y)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} a(X,Y,\alpha(x),t)K_1^{\frac{q-p}{p}}(\alpha(x),t)dt,\end{aligned}$$

i.e.,

$$\frac{\frac{\partial w}{\partial x}}{w(x,y)} \le \frac{q}{p}\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} a(X,Y,\alpha(x),t)K_1^{\frac{q-p}{p}}(\alpha(x),t)dt. \tag{2.13}$$

Keeping y fixed in (2.13), setting $x=\tau$ and then integrating from x_0 to $x, x\in[x_0,X]$ and making the change variable $s=\alpha(\tau)$ we get

$$w(x,y) \le w(x_0,y)\exp\left(\frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(X,Y,s,t)K_1^{\frac{q-p}{p}}(s,t)dtds\right), \tag{2.14}$$

where

(2.15)

$$\begin{aligned} w(x_0,y)&\\ &= \bar{A}_1(X,Y)+B_1(X,Y)+\frac{r}{p}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(X,Y,s,t)K_2^{\frac{r-p}{p}}(s,t)v(s,t)dtds\\ &\le \bar{A}_1(X,Y)+B_1(X,Y)+\frac{r}{p}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(X,Y,s,t)K_2^{\frac{r-p}{p}}(s,t)w(s,t)dtds\end{aligned}$$

for $(x,y)\in\Delta_1$. Using (2.14) to the right side of (2.15) we have

$$\begin{aligned} w(x_0,y) \le\ & \bar{A}_1(X,Y)+B_1(X,Y)\\ &+w(x_0,y)\frac{r}{p}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(X,Y,s,t)K_2^{\frac{r-p}{p}}(s,t)\times\\ &\times\exp\left(\frac{q}{p}\int_{\alpha(x_0)}^{\alpha(s)}\int_{\beta(y_0)}^{\beta(t)} a(X,Y,\sigma,\tau)K_1^{\frac{q-p}{p}}(s,t)d\tau d\sigma\right)dtds.\end{aligned}$$

Since $(X,Y)\in\Delta$ is arbitrary, from (2.11), (2.14) and the last inequality with X and Y replaced by x and y we have

$$v(x,y)\le w(x,y), \tag{2.16}$$

$$w(x,y) \le w(x_0,y)\exp\left(\frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)K_1^{\frac{q-p}{p}}(s,t)dtds\right) \tag{2.17}$$

and

$$w(x_0,y) \le \bar{A}_1(x,y) + B_1(x,y) + w(x_0,y)\frac{r}{p}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)K_2^{\frac{r-p}{p}}(s,t) \times \exp\left(\frac{q}{p}\int_{\alpha(x_0)}^{\alpha(s)}\int_{\beta(y_0)}^{\beta(t)} a(s,t,\sigma,\tau)K_1^{\frac{q-p}{p}}(\sigma,\tau)d\tau d\sigma\right) dtds$$

for$(x,y) \in \Delta$. Now in view of (2.2), (2.4) and the last inequality we have

$$w(x_0,y) \le \frac{\bar{A}_1(x,y) + B_1(x,y)}{1-\lambda_1(x,y)}. \tag{2.18}$$

Using (2.18) in (2.17) and combining with (2.16) and (2.8) we get the desired inequality in (2.3). □

When $p = 2, q = r = 1$ in Theorem 2.1 we get a Volterra-Fredholm-Ou-Iang type inequality in two variables as following. About Ou-Iang type inequality and its generalizations and applications, one can see [22-24].

Corollary 2.2. *Let $u(x,y), a(x,y,s,t), b(x,y,s,t), \alpha(x)$ and $\beta(y)$ be defined as in Theorem 2.1. If $u(x,y)$ satisfies*

$$u^2(x,y) \le l(x,y) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)u(s,t)dtds + \int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(x,y;s,t)u(s,t)dtds \tag{2.19}$$

for $(x,y) \in \Delta$, where

(2.20)
$$\lambda_{11}(x,y) = \frac{1}{2}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)K_2^{-\frac{1}{2}}(s,t)\exp(A_{11}(s,t))dtds < 1$$

for $(x,y) \in \Delta$, then

$$u(x,y) \le \left[l(x,y) + \frac{\bar{A}_{11}(x,y) + B_{11}(x,y)}{1-\lambda_{11}(x,y)}\exp\left(A_{11}(x,y)\right)\right]^{\frac{1}{2}} \tag{2.21}$$

for $(x,y) \in \Delta$ and any $K_i(x,y) > 0 (i = 1,2)$, where

$$A_{11}(x,y) = \frac{1}{2}\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)K_1^{-\frac{1}{2}}(s,t)dtds, \tag{2.22}$$

(2.23)
$$\bar{A}_{11}(x,y) = \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)\left[\frac{1}{2}K_1^{-\frac{1}{2}}(s,t)l(s,t) + \frac{1}{2}K_1^{\frac{1}{2}}(s,t)\right] dtds$$

and
(2.24)
$$B_{11}(x,y) = \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) \left[\frac{1}{2}K_2^{-\frac{1}{2}}(s,t)l(s,t) + \frac{1}{2}K_2^{\frac{1}{2}}(s,t)\right] dtds$$
for $(x,y) \in \Delta$.

When $p = q = n = 1$ we get an interesting result as follows

Corollary 2.3. *Let* $u(x,y), a(x,y,s,t), b(x,y,s,t), \alpha(x)$ *and* $\beta(y)$ *be defined as in Theorem 2.1. If* $u(x,y)$ *satisfies*

$$\begin{aligned} u(x,y) \le l(x,y) &+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)u(s,t)dtds \\ &+ \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)u(s,t)dtds \end{aligned} \tag{2.25}$$

for $(x,y) \in \Delta$, *where*

$$\lambda_{12}(x,y) = \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) \exp(A_{12}(s,t))dtds < 1, \tag{2.26}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \le l(x,y) + \frac{\bar{A}_{12}(x,y) + B_{12}(x,y)}{1 - \lambda_{12}(x,y)} \exp\left(A_{12}(x,y)\right) \tag{2.27}$$

for $(x,y) \in \Delta$, *where*

$$A_{12}(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)dtds, \tag{2.28}$$

$$\bar{A}_{12}(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)l(s,t)dtds \tag{2.29}$$

and

$$B_{12}(x,y) = \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)l(s,t)dtds \tag{2.30}$$

for $(x,y) \in \Delta$.

Remark 2.1. (i) When $l(x,y) \equiv k \ge 0$ (k is a constant), the inequality (2.25) has been studied in Theorem 1.1, but in this special case, under same conditions as in Theorem 1.1, a new estimate to the solution of (2.25) is established in (2.27), which is incomparable with the result given in Theorem 1.1.

(ii) Using the similar procedures of proof of Theorem 2.1, we can get a more general result as following.

Theorem 2.4. *Let $u(x,y)$ and $l(x,y) \in C(\Delta, R_+), a_i(x,y,s,t)$ and $b_j(x,y,s,t) \in C(E, R_+), a_i(x,y,s,t)$ and $b_j(x,y,s,t)$ be nondecreasing in x and y for each $s \in I_1$, and $t \in I_2, \alpha_{1i}, \alpha_{2j} \in C^1(I_1, I_1), \beta_{1i}, \beta_{2j} \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_{1i}(x), \alpha_{2j}(x) \leq x$ on $I_1, \beta_{1i}(y), \beta_{2j}(y) \leq y$ on $I_2, i = 1, 2, \ldots, m_1$, $j = 1, 2, \ldots, m_2$ (m_1 and m_2 are some positive integers). If $u(x,y)$ satisfies*

$$\begin{aligned} u^p(x,y) \leq l(x,y) &+ \sum_{i=1}^{m_1} \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(x,y,s,t) u^{q_i}(s,t) dt ds \\ &+ \sum_{j=1}^{m_2} \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(M)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(N)} b_j(x,y,s,t) u^{r_j}(s,t) dt ds \end{aligned} \tag{2.31}$$

for $(x,y) \in \Delta$, where $p \geq q_i \geq 0, p \geq r_j \geq 0, p, q_i, r_i$ and n_j are constants, and

$$\lambda_2(x,y) = \sum_{j=1}^{m_2} \frac{r_j}{p} \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(M)} \int_{\beta 2j(y_0)}^{\beta_{2j}(N)} b_j(x,y,s,t) K_{2j}^{\frac{r_j - p}{p}}(s,t) \exp(A_2(s,t)) dt ds < 1, \tag{2.32}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \leq \left[l(x,y) + \frac{\bar{A}_2(x,y) + B_2(x,y)}{1 - \lambda_2(x,y)} \exp\left(A_2(x,y)\right) \right]^{\frac{1}{p}} \tag{2.33}$$

for $(x,y) \in \Delta$ and $K_{1i}(s,t), K_{2j}(s,t) \in C(\Delta, R_0)$, $i = 1, 2, \ldots, m_1$; $j = 1, 2, \ldots, m_2$, where

$$A_2(x,y) = \sum_{i=1}^{m_1} \frac{q_i}{p} \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(x,y,s,t) K_{1i}^{\frac{q-p}{p}}(s,t) dt ds, \tag{2.34}$$

$$\begin{aligned} &\bar{A}_2(x,y) \\ &= \sum_{i=1}^{m_1} \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(x,y,s,t) \left[\frac{q_i}{p} K_{1i}^{\frac{q_i - p}{p}}(s,t) l(s,t) + \frac{p - q_i}{p} K_{1i}^{\frac{q_i}{p}}(s,t) \right] dt ds \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} &B_2(x,y) \\ &= \sum_{j=1}^{m_2} \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(M)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(N)} b_j(x,y,s,t) \left[\frac{r_j}{p} K_{2j}^{\frac{r_j - p}{p}}(s,t) l(s,t) + \frac{p - r_j}{p} K_{2j}^{\frac{r_j}{p}}(s,t) \right] dt ds \end{aligned} \tag{2.36}$$

for $(x,y) \in \Delta$.

Remark 2.2. (i) When $m_1 = 2, p = q_1 = q_2 = 1, a_1(x,y,s,t) = a(s,t)$, $a_2(x,y,s,t) = b(s,t), \alpha_{11}(x) = x, \beta_{11}(y) = y, b_j(x,y,s,t) = 0, j = 1, 2, \ldots, m_2$, from Theorem 2.4 we can get Theorem 3.1 given in [11]; (ii)When $m_1 = 2, p > 1, q_1 = q_2 = 1, a_1(x,y,s,t) = a(s,t), a_2(x,y,s,t) = b(s,t), \alpha_{11}(x) = x, \beta_{11}(y) = y, b_j(x,y,s,t) = 0, j = 1, 2, \ldots, m_2$, from Theorem 2.4 (let $K_{11} = K_{12} = l(x,y)$) we can get Theorem 3.2 given in [11].

Theorem 2.5. *Let $u(x,y), l(x,y), a(x,y,s,t), b(x,y,s,t), \alpha(x)$ and $\beta(y)$ be defined as in Theorem 2.1. If $u(x,y)$ satisfies*

$$u^p(x,y) \le l(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)u^q(s,t)dtds + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)V(s,t,u(s,t))dtds \tag{2.37}$$

for $(x,y) \in \Delta$, where $p \ge q \ge 0, p \ge r \ge 0, p \ge n \ge 0, p, q$ and r are constants, $U, V \in C(R_+^3, R_+)$ satisfying

$$0 \le V(s,t,x) - V(s,t,y) \le U(s,t,y)(x-y), \tag{2.38}$$

and

$$\lambda_2(x,y) = \frac{1}{p} \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)U\left(s,t,\frac{p-1}{p} + \frac{1}{p}l(s,t)\right) \exp(A_1(s,t))dtds < 1 \tag{2.39}$$

then

$$u(x,y) \le \left[l(x,y) + \frac{\bar{A}_1(x,y) + \overline{V}}{1 - \lambda_2(x,y)} \exp(A_1(x,y)) \right]^{\frac{1}{p}} \tag{2.40}$$

for $(x,y) \in \Delta$ and any $K_1(x,y) \in C(\Delta, R_0)$, where $A_1(x,y)$ and $\bar{A}_1(x,y)$ are defined as in (2.4) and (2.5), respectively, and

$$\overline{V} = \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} V\left(s,t,\frac{p-1}{p} + \frac{1}{p}l(s,t)\right) dtds. \tag{2.41}$$

Proof. Define a function $\bar{v}(x,y)$ by

$$\bar{v}(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)u^q(s,t)dtds + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)V(s,t,u(s,t))dtds. \tag{2.42}$$

Then we have

$$u^p(x,y) \le l(x,y) + \bar{v}(x,y)$$

or

$$u(x,y) \le (l(x,y) + \bar{v}(x,y))^{\frac{1}{p}}.$$

By Lemma 2.1, for any $K_1(x,y) \in C(\Delta, R_0)$, we have

$$u(x,y) \le (l(x,y) + \bar{v}(x,y))^{\frac{1}{p}} \le \frac{1}{p}(l(x,y) + \bar{v}(x,y)) + \frac{p-1}{p},$$

and

$$u^q(x,y) \le (l(t) + \bar{v}(x,y))^{\frac{q}{p}} \le \frac{q}{p} K_1^{\frac{q-p}{p}}(x,y)(l(x,y) + \bar{v}(x,y)) + \frac{p-q}{p} K_1^{\frac{q}{p}}(x,y).$$

Substituting the last relations into(2.42)and using (2.38), we find that

$$
\begin{aligned}
(2.43)\quad &\bar{v}(x,y)\\
&\le \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)\left(\frac{q}{p}K_1^{\frac{q-p}{p}}(s,t)(l(s,t)+\bar{v}(s,t))+\frac{p-q}{p}K_1^{\frac{q}{p}}(s,t)\right)dtds\\
&\quad+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} V\left(s,t,\frac{p-1}{p}+\frac{1}{p}(l(s,t)+\bar{v}(s,t))\right)dtds\\
&\quad-\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} V\left(s,t,\frac{p-1}{p}+\frac{1}{p}l(s,t)\right)dtds\\
&\quad+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} V\left(s,t,\frac{p-1}{p}+\frac{1}{p}l(s,t)\right)dtds\\
&\le \bar{A}_1(x,y)+\overline{V}+\frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)K_1^{\frac{q-p}{p}}(s,t)\bar{v}(s,t)dtds\\
&\quad+\frac{1}{p}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)U\left(s,t,\frac{p-1}{p}+\frac{1}{p}l(s,t)\right)\bar{v}(s,t)dtds,
\end{aligned}
$$

where $\overline{V}$ is defined in (2.41). Obviously, $\bar{A}_1(x,y)+\overline{V}$ are nonnegative, continuous and nondecreasing for $(x,y)\in\Delta$. Taking similar procedure from (2.9) to (2.18) in the proof of Theorem 2.1 to (2.43), we can get the desired inequality (2.40). □

Remark 2.3. As in Theorem 2.4, using similar arguments in the proof of Theorem 2.5, we can get a more general version of (2.37), but, for space-saving, the details are omitted here.

3. Applications

Consider retarded Volterra-Fredholm integral equations in two independent variables of the form

$$
\begin{aligned}
(3.1)\quad z^p(x,y)&=f(x,y)+\int_{x_0}^{x}\int_{y_0}^{y} F(x,y,s,t,z(s-h_1(s),t-h_2(t))dtds\\
&\quad+\int_{x_0}^{M}\int_{y_0}^{N} G(x,y,s,t,z(s-h_1(s),t-h_2(t))dtds,
\end{aligned}
$$

where $z,f\in C(\Delta,R), F,G\in C(E\times R,R)$ and $h_1\in C(I_1,R_+), h_2\in C(I_2,R_+)$, are nonincreasing, $x-h_1(x)\ge 0, y-h_2(y)\ge 0, x-h_1(x)\in C^1(I_1,I_1), y-h_2(y)\in C^1(I_2,I_2), h_1'(x)<1, h_2'(y)<1, h(x_0)=h_2(y_0)=0, p\ge 1$ is a constant.

Theorem 3.1. *Assume that the functions F and G in* (3.1) *satisfy the conditions*

$$(3.2)\qquad |F(x,y,s,t,z)|\le a(x,y,s,t)|z|^q,$$

and

$$|G(x,y,s,t,z)| \le b(x,y,s,t)|z|^r, \tag{3.3}$$

where $a(x,y,s,t), b(x,y,s,t)$ *are as in Theorem 2.1,* q *and* r *satisfying* $0 < q \le p, 0 < r \le p$ *are constants. Let*

$$M_1 = \max_{x\in I_1} \frac{1}{1-h_1'(x)},\ M_2 = \max_{y\in I_2} \frac{1}{1-h_2'(y)} \tag{3.4}$$

and

$$\begin{aligned} \bar{\lambda}_1(x,y) = \frac{r}{p}\int_{\phi(x_0)}^{\phi(M)}\int_{\psi(y_0)}^{\psi(N)} & \bar{b}(x,y,s,t)K_2^{\frac{r-p}{p}}(s,t) \\ & \times \exp\left(\int_{\phi(x_0)}^{\phi(s)}\int_{\psi(y_0)}^{\psi(t)} \bar{a}(s,t,\sigma,\tau)d\tau d\sigma\right) dtds < 1, \end{aligned} \tag{3.5}$$

where $\phi(x) = x - h_1(x), x \in I_1, \psi(y) = y - h_2(y), y \in I_2$ *and*

$$\begin{aligned} \bar{a}(x,y,\sigma,\tau) &= M_1M_2a(x,y,\sigma+h_1(s),\tau+h_2(t)), \\ \bar{b}(x,y,\sigma,\tau) &= M_1M_2b(x,y,\sigma+h_1(s),\tau+h_2(t)). \end{aligned}$$

If $z(x,y)$ *is any solution of* (3.1)-(3.3)*, then*

$$|z(x,y)| \le \left[|f(x,y)| + \frac{\bar{A}_1^*(x,y)+B_1^*(x,y)}{1-\bar{\lambda}_1(x,y)}\exp\left(A_1^*(x,y)\right)\right]^{\frac{1}{p}} \tag{3.6}$$

for $(x,y) \in \Delta$ *and any* $K_i(x,y) \in C(\Delta, R_0)$ $(i = 1,2)$*, where*

$$A_1^*(x,y) = \frac{q}{p}\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,s,t)K_1^{\frac{q-p}{p}}(s,t)dtds, \tag{3.7}$$

(3.8)

$$\bar{A}_1^*(x,y) = \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,s,t)\left[\frac{q}{p}K_1^{\frac{q-p}{p}}(s,t)|f(s,t)| + \frac{p-q}{p}K_1^{\frac{q}{p}}(s,t)\right]dtds$$

and
(3.9)

$$B_1^*(x,y) = \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{b}(x,y,s,t)\left[\frac{r}{p}K_2^{\frac{r-p}{p}}(s,t)|f(s,t)| + \frac{p-r}{p}K_2^{\frac{r}{p}}(s,t)\right]dtds$$

for $(x,y) \in \Delta$.

Proof. Let $z(x,y)$ be a solution of (3.1). Using conditions (3.2) and (3.3) in (3.1), we have

$$\begin{aligned} |z(x,y)| \le |f(x,y)| &+ \int_{x_0}^{x}\int_{y_0}^{y} a(x,y,s,t)|z(s-h_1(s),t-h_2(t))|^q dtds \\ &+ \int_{x_0}^{M}\int_{y_0}^{N} b(x,y,s,t)|z(s-h_1(s),t-h_2(t))|^r dtds. \end{aligned} \tag{3.10}$$

Now, by making the change of variables on the right side of (3.10), we have

$$|z(x,y)| \le |f(x,y)| + \int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,\sigma,\tau)|z(\sigma,\tau)|^q d\tau d\sigma \tag{3.11}$$
$$+ \int_{\phi(x_0)}^{\phi(M)} \int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,\sigma,\tau)|z(\sigma,\tau)|^r d\tau d\sigma.$$

Now a suitable application of the inequality given in Theorem 2.1 to (3.11) yields (3.6). The right-hand side of (3.6) give us the bound on the solution $z(x,y)$ of (3.1) in terms of the known functions. □

Theorem 3.2. *Assume that the functions F and G in* (3.1) *satisfy the conditions*

$$|F(x,y,s,t,z) - F(x,y,s,t,\bar{z})| \le a(x,y,s,t)|z^p - \bar{z}^p|, \tag{3.12}$$

$$|G(x,y,s,t,z) - G(x,y,s,t,\bar{z})| \le b(x,y,s,t)|z^p - \bar{z}^p|, \tag{3.13}$$

and

$$\bar{\lambda}_2(x,y) = \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} \bar{b}(x,y,s,t)\exp(A_1^*(s,t))dtds < 1,$$

where

$$A_1^*(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \bar{a}(x,y,s,t)dtds, \tag{3.14}$$

functions $a(x,y,s,t)$ and $b(x,y,s,t)$ are as in Theorem 2.1. Let M_1, M_2, ϕ, ψ, $\bar{a}$, $\bar{b}$ and p be as in Theorem 3.1. Then the equation (3.1) *has at most one solution on Δ.*

Proof. Let $z(x,y)$ and $\bar{z}(x,y)$ be two solutions of (3.1) on Δ. Then using (3.12) and (3.13) in (3.1) we have

$$|z^p(x,y) - \bar{z}^p(x,y)|$$
$$\le \int_{x_0}^{x}\int_{y_0}^{y} a(x,y,s,t)|z^p(s-h_1(s),t-h_2(t)) - \bar{z}^p(s-h_1(s),t-h_2(t))|dtds$$
$$+ \int_{x_0}^{M}\int_{y_0}^{N} b(x,y,s,t)|z^p(s-h_1(s),t-h_2(t)) - \bar{z}^p(s-h_1(s),t-h_2(t))|dtds.$$

Making changes of variables to the right side of the last inequality, we have

$$|z^p(x,y) - \bar{z}^p(x,y)| \le \int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,s,t)|z^p(\sigma,\tau) - \bar{z}^p(\sigma,\tau)|dtds$$
$$+ \int_{\phi(x_0)}^{\phi(M)} \int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,s,t)|z^p(\sigma,\tau) - \bar{z}^p(\sigma,\tau)|dtds.$$

Now a suitable application of the inequality given in Corollary 2.3 to the last inequalities yields

$$|z^p(x,y) - \bar{z}^p(x,y)| \le 0$$

for $(x, y) \in \Delta$. Hence $z = \bar{z}$ on Δ. □

Finally, we investigate the continuous dependence of the solutions of (3.1) on the functions f, F and G. For this we consider the following variation of (3.1):

$$(\overline{3.1}) \quad \begin{aligned} z^p(x,y) = \bar{f}(x,y) &+ \int_{x_0}^{x}\int_{y_0}^{y} \overline{F}(x,y,s,t,z(s-h_1(s),t-h_2(t))dtds \\ &+ \int_{x_0}^{M}\int_{y_0}^{N} \overline{G}(x,y,s,t,z(s-h_1(s),t-h_2(t))dtds \end{aligned}$$

for $(x,y) \in \Delta, z, \bar{f} \in C(\Delta, R), \overline{F}, \overline{G} \in C(E \times R, R), p, h_1$ and h_2 are as in Theorem 3.1.

Theorem 3.3. *Consider* (3.1) *and* $(\overline{3.1})$. *If*

(i)
$$|F(x,y,s,t,z_1) - F(x,y,s,t,z_2)| \le a(x,y,s,t)|z_1^p - z_2^p|$$

and

$$|G(x,y,s,t,z_1) - G(x,y,s,t,z_2)| \le b(x,y,s,t)|z_1^p - z_2^p|;$$

(ii)
$$|f(x,y) - \bar{f}(x,y)| \le \frac{\varepsilon}{3};$$

(iii)
$$\bar{\lambda}_2(x,y) = \int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} \bar{b}(x,y,s,t)\exp(A_1^*(s,t))dtds < 1;$$

(iv) *for all solutions* $\overline{z}$ *of* $(\overline{3.1})$,

$$\int_{x_0}^{x}\int_{y_0}^{y}\Big|F\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big) - \overline{F}\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big)\Big|dtds \le \frac{\varepsilon}{3}$$

and

$$\int_{x_0}^{M}\int_{y_0}^{N}\Big|G\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big) - \overline{G}\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big)\Big|dtds \le \frac{\varepsilon}{3}$$

for all $(x,y) \in \Delta$ *and* $z_1, z_2 \in R$, *where* $\varepsilon > 0$ *is an arbitrary constant, then*

$$(3.15) \quad |z^p(x,y) - \bar{z}^p(x,y)| \le \varepsilon\left[1 + \frac{\bar{A}_2^*(x,y) + B_2^*(x,y)}{1 - \bar{\lambda}_2(x,y)}\exp\left(A_1^*(x,y)\right)\right]$$

for $(x,y) \in \Delta$, *where*

$$(3.16) \quad A_1^*(x,y) = \bar{A}_2^*(x,y) = \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,s,t)dtds,$$

and

$$B_2^*(x,y) = \int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{b}(x,y,s,t)dtds \tag{3.17}$$

for $(x,y) \in \Delta$.

Proof. Let $z(x,y)$ and $\overline{z}$ be the solutions of (3.1) and $\overline{(3.1)}$. Hence

$$\begin{aligned}
&|z^p(x,y) - \bar{z}^p(x,y)| \\
&\le |f(x,y) - \bar{f}(x,y)| + \int_{x_0}^{x}\int_{y_0}^{y} \Big| F\Big(x,y,s,t,z(s-h_1(s),t-h_2(t))\Big) \\
&\qquad - \overline{F}\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big)\Big| dtds \\
&\quad + \int_{x_0}^{M}\int_{y_0}^{N} \Big| G\Big(x,y,s,t,z(s-h_1(s),t-h_2(t))\Big) \\
&\qquad - \overline{G}\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big)\Big| dtds \\
&\le \frac{\varepsilon}{3} + \int_{x_0}^{x}\int_{y_0}^{y} \Big| F\Big(x,y,s,t,z(s-h_1(s),t-h_2(t))\Big) \\
&\qquad - F\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big)\Big| dtds \\
&\quad + \int_{x_0}^{x}\int_{y_0}^{y} \Big| F\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big) \\
&\qquad - \overline{F}\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big)\Big| dtds \\
&\quad + \int_{x_0}^{M}\int_{y_0}^{N} \Big| G\Big(x,y,s,t,z(s-h_1(s),t-h_2(t))\Big) \\
&\qquad - G\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big)\Big| dtds \\
&\quad + \int_{x_0}^{M}\int_{y_0}^{N} \Big| G\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big) \\
&\qquad - \overline{G}\Big(x,y,s,t,\overline{z}(s-h_1(s),t-h_2(t))\Big)\Big| dtds \\
&\le \varepsilon + \int_{x_0}^{x}\int_{y_0}^{y} a(x,y,s,t)\Big| z^p(s-h_1(s),t-h_2(t)) - \bar{z}^p(s-h_1(s),t-h_2(t))\Big| dtds \\
&\quad + \int_{x_0}^{M}\int_{y_0}^{N} b(x,y,s,t)\Big| z^p(s-h_1(s),t-h_2(t)) - \bar{z}^p(s-h_1(s),t-h_2(t))\Big| dtds
\end{aligned}$$

by the assumptions (i)-(iv). Now by making the change of variables on the right side of the last inequality we have

(3.17)

$$
\begin{aligned}
|z^p(x,y) - \bar{z}^p(x,y)| \le \varepsilon &+ \int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,\sigma,\tau)|z^p(\sigma,\tau) - \bar{z}^p(\sigma,\tau)|d\tau d\sigma \\
&+ \int_{\phi(x_0)}^{\phi(M)} \int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,\sigma,\tau)|z^p(\sigma,\tau) - \bar{z}^p(\sigma,\tau)|d\tau d\sigma
\end{aligned}
$$

for $(x,y) \in \Delta$. A suitable application of Corollary 2.3 to (3.17) yields the desired estimate in (3.15). Evidently, if function

$$
\frac{\bar{A}_2^*(x,y) + B_2^*(x,y)}{1 - \bar{\lambda}_2(x,y)} \exp\left(A_1^*(x,y)\right)
$$

is bounded on Δ, so

$$
|z^p(x,y) - \bar{z}^p(x,y)| \le \varepsilon K
$$

for some $K > 0$ and $(x,y) \in \Delta$. Hence z^p depends continuously on f, F and G. □

Acknowledgment. The authors are grateful to the referee for his very helpful and detailed comments in improving this paper.

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