

# INTEGRAL REPRESENTATIONS IN ELECTRICAL IMPEDANCE TOMOGRAPHY USING BOUNDARY INTEGRAL OPERATORS

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ABSTRACT. Electrical impedance tomography (EIT) problem with anisotropic anomalous region is formulated in a few different ways using boundary integral operators. The Fréchet derivative of Neumann-to-Dirichlet map is computed also by using boundary integral operators and the boundary of the anomalous region is approximated by trigonometric expansion with Lagrangian basis. The numerical reconstruction is done in case that the conductivity of the anomalous region is isotropic.

## 1. Introduction

EIT is to find interior conductivity profile from measured voltage or electric potential and given current density. Like various imaging modality, EIT is applied to clinical and nonclinical cases including the detection of pulmonary emboli in the lungs [3]. Generally, the conductivity of tissues has anisotropic characteristics, i.e., the value of conductivity is different in each directions.

Let  $H^s(U)$  be the Sobolev space of order  $s$  in the subset  $U$  contained in  $\mathbb{R}^n$ ,  $n = 2, \dots$  and  $H_0^s(U) = \{h \in H^s(U) \mid \int_U h = 0\}$ . Remind that  $H^0(U) = L^2(U)$ . Consider the following elliptic equation:

$$(1.1a) \quad \nabla \cdot ((A + (B - A)\chi_D)\nabla u) = 0 \text{ in } \Omega,$$

$$(1.1b) \quad \nu_\Omega \cdot A\nabla u = g \text{ on } \partial\Omega,$$

$$(1.1c) \quad \int_{\partial\Omega} u = 0,$$

where  $\Omega$  and  $D$  are domains contained in  $\mathbb{R}^n$ ,  $n = 2, \dots$  and  $D$  is contained in  $\Omega$  such that  $\Omega \setminus \overline{D}$  is connected,  $g \in L_0^2(\partial\Omega)$ ,  $A$  and  $B$  are positive-definite matrix functions, and  $\nu_\Omega$  is the outward normal vector on  $\partial\Omega$ . Assume that  $\partial\Omega$  and  $\partial D$  are sufficiently smooth. It is well known that there exists a unique function in  $H^1(\Omega)$  satisfying (1.1) [10].

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Let  $u|_{\Omega \setminus \overline{D}} = u^+$  and  $u|_D = u^-$ , then (1.1) can be equivalently represented as

$$(1.2a) \quad \nabla \cdot (A \nabla u^+) = 0 \text{ in } \Omega \setminus \overline{D},$$

$$(1.2b) \quad \nabla \cdot (B \nabla u^-) = 0 \text{ in } D,$$

$$(1.2c) \quad \nu_\Omega \cdot A \nabla u^+ = g \text{ on } \partial\Omega,$$

$$(1.2d) \quad \int_{\partial\Omega} u^+ = 0,$$

$$(1.2e) \quad u^+ = u^- \text{ on } \partial D,$$

$$(1.2f) \quad \nu_D \cdot A \nabla u^+ = \nu_D \cdot B \nabla u^- \text{ on } \partial D,$$

where  $\nu_D$  is the unit outer normal vector on  $\partial D$ . By using trace formula  $f := u|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$  and (1.1c), we obtain  $f \in H_0^{1/2}(\partial\Omega)$  [9]. Therefore, we can define the Neumann-to-Dirichlet operator as  $\Lambda_{D,B} : H_0^{-1/2}(\partial\Omega) \rightarrow H_0^{1/2}(\partial\Omega)$  such that  $\Lambda_{D,B}(g) = f := u|_{\partial\Omega}$ .

Then EIT is mathematically formulated to find  $D$  and  $B$  when we know  $A$  and Neumann-to-Dirichlet map  $(g, \Lambda_{D,B}g)$  for infinite number of  $g$  (infinite measurements) or finite number of  $g$  (finite measurements). The unique determination of  $D$  and the distribution of  $B$  on  $\partial D$  when infinite measurements is given, is proved in [16, 17, 25, 26]. However, there are many kinds of conductivity distributions inside  $B$  giving the same Neumann-to-Dirichlet map [25]. This shows the nonuniqueness for the infinite measurement EIT with anomalous anomaly, which is easily induced from the researches in [21, 27, 28, 29, 30, 34, 35]. When  $A$  and  $B$  are constant positive matrices, the uniqueness of  $B$  for infinite measurement is shown [16, 26]. In this paper, we will take the following cases:

$$(1.3) \quad \begin{aligned} &A \text{ and } B \text{ are constant positive definite matrices} \\ &\text{and } B - A \text{ is positive definite or negative definite.} \end{aligned}$$

EIT is nonlinear and ill-posed. We will give a simple example showing how severe the nonlinearity of EIT is. Let  $B$  be a positive constant and  $\Omega$  and  $D$  be concentric disks centered at the origin with radius 1 and  $r_0 < 1$ , respectively. Consider polar coordinate  $(r, \theta)$  with center at the origin. If  $g(\theta) = \cos(k\theta)$  for positive integer  $k$ , then the solution  $u^+, u^-$  of (1.2), the Dirichlet data  $f = u^+|_{\partial\Omega}$ , and the Fréchet derivative  $\frac{\partial f}{\partial r_0}$  are computed as follows:

$$(1.4a) \quad u^+(r, \theta) = \left( \frac{1}{k(1+\mu)} r^k - \frac{\mu}{k(1+\mu)} \frac{1}{r^k} \right) \cos(k\theta) \quad \text{if } r \geq r_0,$$

$$(1.4b) \quad u^-(r, \theta) = \frac{1-\lambda}{k(1+\mu)} r^k \cos(k\theta) \quad \text{if } 0 \leq r \leq r_0,$$

$$(1.4c) \quad f(\theta) = u^+(1, \theta) = \frac{1-\mu}{k(1+\mu)} \cos(k\theta),$$

$$(1.4d) \quad \frac{\partial f}{\partial r_0} = \frac{-4\mu}{r_0(1+\mu)^2} \cos(k\theta),$$

where  $\lambda = \frac{B-1}{B+1} < 1$  and  $\mu = \lambda r_0^{2k} < 1$ . Using Taylor series expansion for  $\mu$ , (1.4c) and (1.4d) are represented by  $f(\theta) = \frac{\cos(k\theta)}{k} + O(r_0^{2k})$  and  $\frac{\partial f}{\partial r_0} = O(r_0^{2k})$ . Let  $\Lambda_0$  be the Neumann-to-Dirichlet map when  $D = \phi$ . Then  $\Lambda_0(\cos(k\theta)) = \frac{\cos(k\theta)}{k}$  and  $\Lambda_D(\cos(k\theta)) = f(\theta) = \frac{\cos(k\theta)}{k} + O(r_0^{2k})$  for  $k = 1, 2, \dots$ . Therefore,  $\Lambda_D g - \Lambda_0 g$  and  $\frac{\partial \Lambda_D}{\partial r_0} g$  are of order  $O(r_0^{2k})$  and it is difficult to discriminate small object from the background, especially if we use high frequency current density. The analysis in (1.4) is done in [11] to seek the best currents to distinguish between the conductivity inside the body and a conjectured conductivity.

There are many numerical approaches for the numerical implementation of EIT such as backprojection method [1, 32], layer stripping method [33, 36], one-step Newton method [2], point source method [7], and least square method using finite element method as a direct solver [6, 22, 38, 39]. Least square method using boundary integral equation method as a direct solver is used in [13, 19, 20] for the isotropic case. In this paper, we extend the approach in [13, 19, 20] to the anisotropic case (1.3) and Fréchet derivative of the Neumann-to-Dirichlet map with respect to  $D$  is computed for the numerical development. And trigonometric expansion with Lagrangian basis used in [23] will be used for the discretization and approximation of  $\partial D$ .

In the next section, three different representation formulas for the solution of (1.1) are given. Direct problem to find the Dirichlet data  $u|_{\partial\Omega}$  is formulated by the system of boundary integral equations. The equivalence of this system of boundary integral equation and (1.1) is addressed. In Section 3, EIT is formulated to find the conductivity to minimize the difference from the measured voltage on the boundary. Fréchet derivatives of Neumann -to-Dirichlet map is computed solving the system of boundary integral operators. In Section 4, boundary integral operators, their Fréchet derivatives, and the boundary of  $D$  are discretized and approximated. In Section 5, the  $L^2$  error convergence of the direct problem with respect to the number of division for  $\partial\Omega$  and  $\partial D$  is given for concentric disks. And the numerical reconstruction of EIT for various anomalous regions  $D$ , proposed in this paper, is given.

## 2. Direct problem

In the direct problem of EIT, we seek to  $f := \Lambda_{D,B}(g)$  from current density  $g \in L_0^2(\partial\Omega)$ . We will use integral equation method for the Direct problem. Before introducing the representation formulas, we consider the following change of variables.

Let  $C$  be an invertible constant  $n \times n$  matrix. Then by the transformation  $\tilde{y} = Cy$

$$(2.5) \quad \nabla_y \cdot (A \nabla_y u) = 0, \text{ in } \Omega$$

is changed into

$$(2.6) \quad \nabla_{\tilde{y}} \cdot (\tilde{A} \nabla_{\tilde{y}} \tilde{u}) = 0, \text{ in } \tilde{\Omega}$$

where

$$(2.7) \quad \tilde{\Omega} = C(\Omega), \tilde{u}(\tilde{x}) = u(x), \tilde{A} = C^t A C.$$

And the followings hold:

$$(2.8) \quad \nu_{\tilde{\Omega}}(\tilde{y}) = C\nu_{\Omega}(y), \quad \nu_{\tilde{\Omega}} \cdot \tilde{A} \nabla_{\tilde{y}} \tilde{u} = \nu_{\Omega} \cdot A \nabla_x u.$$

Let  $z$  be a  $2\pi$  periodic function whose image  $Z$  is contained in  $\Omega$ , then

$$(2.9) \quad ds_{C(Z)}(Cz(t)) = |\det(C)| ds_Z(z(t)).$$

In Section 2.1, we find the solution  $u$  of (1.1) by using integral equation method using single and double layer potentials. In Section 2.2, we formulate the boundary data  $f := u|_{\partial\Omega}$  using boundary integral operators.

## 2.1. Representation using single and double layer potentials

In what follows, we notate  $x, \xi \in \partial\Omega$ ,  $z, \zeta \in \partial D$  and  $y, \eta \in \mathbb{R}^n \setminus (\partial\Omega \cup \partial D)$ . And let  $\phi \in L^2(\partial\Omega)$  and  $\psi \in L^2(\partial D)$  are functions on  $\partial\Omega$  and  $\partial D$ , respectively. Let  $\Gamma$  be the fundamental solution of  $\Delta$  in  $\mathbb{R}^n$  ( $n = 2, \dots$ ) such that

$$\Gamma(y, \eta) := \begin{cases} \frac{1}{2\pi} \log \left( \frac{1}{|y-\eta|} \right) & \text{for } n = 2, \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|y-\eta|^{n-2}} & \text{for } n \geq 3, \end{cases}$$

where  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$  ( $n = 2, 3, \dots$ ). For symmetric positive-definite constant matrix  $A$ , let

$$\Gamma^A(y, \eta) := \Gamma(A^{-1/2}y, A^{-1/2}\eta)$$

then  $\Gamma^A$  is the fundamental solution of the operator  $\nabla \cdot (A\nabla)$  by (2.6).

Let us single and double layer potential operators as follows:

$$(2.10) \quad \begin{aligned} (\mathcal{S}_{\Sigma}\phi)(y) &= \int_{\partial\Sigma} \Gamma^F(y, \xi)\phi(\xi) |\det(F^{-1/2})| ds(\xi), \\ (\mathcal{D}_{\Sigma}\phi)(y) &= \int_{\partial\Sigma} \nu_{\Sigma}(\xi) \cdot F \nabla \Gamma^F(y, \xi)\phi(\xi) |\det(F^{-1/2})| ds(\xi), \end{aligned}$$

for connected domain  $\Sigma$ , symmetric positive definite matrix  $F$ , and  $y \in \mathbb{R}^n \setminus (\partial\Sigma)$ . In this paper, we will replace  $(\Sigma, \partial\Sigma, F)$  with

$$(2.11) \quad (\Omega, \partial\Omega, A), (D^+ := \mathbb{R}^n \setminus \bar{D}, \partial D, A), \text{ or } (D^- := D, \partial D, B).$$

Let  $u^+$  and  $u^-$  be the solution of (1.2). Then by the Green's formula

$$(2.12a) \quad \begin{aligned} u^+ &= -\mathcal{D}_\Omega u^+|_{\partial\Omega} + \mathcal{S}_\Omega g + \mathcal{D}_{D^+} u^+|_{\partial D} - \mathcal{S}_{D^+}(\nu_D \cdot A \nabla u^+) \\ &\quad \text{in } \Omega \setminus \overline{D}, \end{aligned}$$

$$(2.12b) \quad \begin{aligned} 0 &= -\mathcal{D}_\Omega u^+|_{\partial\Omega} + \mathcal{S}_\Omega g + \mathcal{D}_{D^+} u^+|_{\partial D} - \mathcal{S}_{D^+}(\nu_D \cdot A \nabla u^+) \\ &\quad \text{in } \mathbb{R}^n \setminus \overline{[\Omega \setminus \overline{D}]}, \end{aligned}$$

$$(2.12c) \quad u^- = -\mathcal{D}_D u^-|_{\partial D} + \mathcal{S}_D(\nu_D \cdot B \nabla u^-) \quad \text{in } D,$$

$$(2.12d) \quad 0 = -\mathcal{D}_D u^-|_{\partial D} + \mathcal{S}_D(\nu_D \cdot B \nabla u^-) \quad \text{in } \mathbb{R}^n \setminus \overline{D}.$$

## 2.2. Representations using boundary integral operators

We will use the following boundary integral operators:

$$(2.13) \quad \begin{aligned} (S_\Sigma \phi)(x) &= \int_{\partial\Sigma} \Gamma^F(x, \xi) \phi(\xi) |\det(F^{-1/2})| ds(\xi), \\ (K_\Sigma \phi)(x) &= \int_{\partial\Sigma} \nu_\Sigma(\xi) \cdot F \nabla \Gamma^F(x, \xi) \phi(\xi) |\det(F^{-1/2})| ds(\xi), \\ (K_\Sigma^* \phi)(x) &= \int_{\partial\Sigma} \nu_\Sigma(x) \cdot F \nabla \Gamma^F(x, \xi) \phi(\xi) |\det(F^{-1/2})| ds(\xi), \\ (T_\Sigma \phi)(x) &= \nu_\Sigma(x) \cdot F \nabla \int_{\partial\Sigma} \nu_\Sigma(x) \cdot F \nabla \Gamma^F(x, \xi) \phi(\xi) |\det(F^{-1/2})| ds(\xi) \end{aligned}$$

for  $x \in \partial\Sigma$  and  $(\Sigma, \partial\Sigma, F)$  can be replaced by (2.11). By [4, 37], the following operators are bounded:

$$(2.14a) \quad S_\Sigma : L^2(\partial\Sigma) \rightarrow H^1(\partial\Sigma)$$

$$(2.14b) \quad K_\Sigma : L^2(\partial\Sigma) \rightarrow L^2(\partial\Sigma)$$

$$(2.14c) \quad K_\Sigma^* : L^2(\partial\Sigma) \rightarrow L^2(\partial\Sigma)$$

when  $\Sigma$  is bounded connected domain. For the operator  $T$ , the following Maue formula is known [5, 23]

$$(2.15) \quad T_\Sigma(x) = \det(F) \frac{\partial}{\partial s(x)} S_\Sigma \frac{\partial \phi}{\partial s}.$$

Thus using (2.14a) and (2.15), we get the boundedness of the operator

$$(2.16) \quad T_\Sigma : H^1(\partial\Sigma) \rightarrow L^2(\partial\Sigma).$$

To use limiting value to  $\partial\Sigma$  from (2.12), let us state the following jump relation which relates the single and double layer potentials (2.10) and the boundary

integral operators (2.13):

$$\begin{aligned}
(2.17) \quad & \lim_{t \rightarrow 0^+} \mathcal{S}_\Sigma \phi(x \pm t\nu_\Sigma(x)) = \mathcal{S}_\Sigma \phi(x), \\
& \lim_{t \rightarrow 0^+} \mathcal{D}_\Sigma \phi(x \pm t\nu_\Sigma(x)) = \pm \frac{1}{2} \phi(x) + K_\Sigma \phi(x), \\
& \lim_{t \rightarrow 0^+} \nu_\Sigma \cdot F \nabla \mathcal{S}_\Sigma \phi(x \pm t\nu_\Sigma(x)) = \mp \frac{1}{2} \phi(x) + K_\Sigma^* \phi(x), \\
& \lim_{t \rightarrow 0^+} \nu_\Sigma \cdot F \nabla \mathcal{D}_\Sigma \phi(x \pm t\nu_\Sigma(x)) = T_\Sigma(x).
\end{aligned}$$

Let us consider the following system of boundary integral equations:

$$(2.18a) \quad \begin{aligned} & -T_\Omega q_1 + \nu_\Omega \cdot A \nabla \mathcal{D}_{D^+} q_2 - \nu_\Omega \cdot A \nabla \mathcal{S}_{D^+} q_3 \\ & = \left( \frac{1}{2} I - K_\Omega^* \right) g \text{ on } \partial\Omega, \end{aligned}$$

$$(2.18b) \quad \begin{aligned} & -\mathcal{D}_\Omega q_1 + (K_{D^+} + K_{D^-}) q_2 - (S_{D^+} + S_{D^-}) q_3 \\ & = -S_\Omega g \text{ on } \partial D, \end{aligned}$$

$$(2.18c) \quad \begin{aligned} & -\nu_D \cdot A \nabla \mathcal{D}_\Omega q_1 + (T_{D^+} + T_{D^-}) q_2 - (K_{D^+}^* + K_{D^-}^*) q_3 \\ & = -\nu_D \cdot A \nabla S_\Omega g \text{ on } \partial\Omega, \end{aligned}$$

$$(2.18d) \quad \int_{\partial\Omega} q_1 = 0$$

for  $q_1 \in H_0^1(\partial\Omega)$ ,  $q_2 \in H^1(\partial D)$ , and  $q_3 \in L^2(\partial D)$ . Let  $u^+$  and  $u^-$  are the unique solution of (1.2) for  $g \in L_0^2(\partial\Omega)$ . Then by trace formula  $u^+|_{\partial\Omega} \in H_0^{1/2}(\partial\Omega)$ ,  $u^+|_{\partial D} \in H^{1/2}(\partial D)$ . Using the result in [8] we obtain  $\nu_D \cdot (A \nabla u^+)|_{\partial D} \in L_0^2(\partial D)$ . Therefore,  $u^+|_{\partial\Omega}$ ,  $u^+|_{\partial D}$ , and  $\nu_D \cdot (A \nabla u^+)|_{\partial D}$  satisfy (2.18) replacing  $q_1$ ,  $q_2$ , and  $q_3$ . Furthermore, the following theorem tells us that

$$(2.19a) \quad q_1 = u^+|_{\partial\Omega},$$

$$(2.19b) \quad q_2 = u^+|_{\partial D} = u^-|_{\partial D},$$

$$(2.19c) \quad q_3 = \nu_D \cdot A \nabla u^+ = \nu_D \cdot B \nabla u^-,$$

are the unique solution of (2.18).

**Theorem 2.1.** *Let  $g \in L_0^2(\partial\Omega)$  be given. Then (2.18) has a unique solution  $q_1 \in H_0^{1/2}(\partial\Omega)$ ,  $q_2 \in H^{1/2}(\partial D)$ , and  $q_3 \in L_0^2(\partial D)$  satisfying (2.19).*

*Proof.* Let  $q_1, q_2$ , and  $q_3$  be given by (2.19). Then inserting the limit value of (2.12a) on  $\partial\Omega$  and (2.12c) on  $\partial D$  into (1.2c)-(1.2f) and using (2.17) we get (2.18).

Next let  $q_1, q_2$ , and  $q_3$  be the solution of (2.18) and define

$$(2.20a) \quad v^+ = -\mathcal{D}_\Omega q_1 + \mathcal{S}_\Omega g + \mathcal{D}_{D^+}^+ q_2 - \mathcal{S}_{D^+}^+ q_3,$$

$$(2.20b) \quad v^- = -\mathcal{D}_{D^-}^- q_2 + \mathcal{S}_{D^-}^- q_3.$$

Then  $v^+$  and  $v^-$  satisfies (1.2) and by the uniqueness of (1.2)  $u^+ = v^+$  and  $u^- = v^-$ .  $u^+$  and  $u^-$  is represented by (2.12a) and (2.12c), respectively. Comparing

the jump of  $u^+$  and  $v^+$  along  $\partial\Omega$ , we get  $q_1 = u^+|_{\partial\Omega}$ . Considering the jump of  $u^-$ ,  $v^-$ , and their vertical derivatives along  $\partial D$ , we get  $q_2 = u^+|_{\partial D} = u^-|_{\partial D}$  and  $q_3 = \nu_D \cdot A\nabla u^+ = \nu_D \cdot B\nabla u^-$ .  $\square$

Instead of (2.18a), let us consider the following boundary integral equation:

$$(2.21) \quad -\left(\frac{1}{2}I + K_\Omega\right) q_1 + \mathcal{D}_{D^+} q_2 - \mathcal{S}_{D^+} q_3 = -S_\Omega g \text{ on } \partial\Omega.$$

**Theorem 2.2.** *Let  $g \in L_0^2(\partial\Omega)$  be given. Then (2.21), (2.18b)-(2.18d) has a unique solution  $q_1 \in H_0^{1/2}(\partial\Omega)$ ,  $q_2 \in H^{1/2}(\partial D)$ , and  $q_3 \in L_0^2(\partial D)$  satisfying (2.19).*

*Proof.* Let  $q_1, q_2$ , and  $q_3$  be given by (2.19). Then we get (2.18b)-(2.18d) as in the above Theorem 2.1. (2.21) comes from (2.19a) and (2.12a).

Let us define  $v^+$  and  $v^-$  as in (2.20) for  $q_1, q_2$ , and  $q_3$  which is the solution of (2.18). Then  $v^+$  and  $v^-$  satisfies (1.2a), (1.2b), (1.2d), (1.2e), and (1.2f). By (2.17) and (2.21), the limiting value of  $v^+$  to  $\partial\Omega$  from the outside of  $\Omega$  is 0. Let  $B_R$  be the ball of radius  $R$  centered at the origin. Then for sufficiently large  $R$  such that  $B_R \supset \Omega$ , the following hold

$$(2.22) \quad \int_{B_R \setminus \bar{\Omega}} A\nabla v^+ \nabla v^+ = \int_{\partial B_R} v^+ \nu_{B_R} \cdot A\nabla v^+,$$

which is induced from the fact that  $v^+ = 0$  on  $\partial\Omega$  and  $\nabla \cdot A\nabla v^+ = 0$  in  $\mathbb{R}^n \setminus \Omega$ . We will show that

$$(2.23) \quad \int_{\partial B_R} v^+ \nu_{B_R} \cdot A\nabla v^+ = O(|y|^{2-n}) \rightarrow 0$$

for  $n \geq 2$ .

In the case that  $n \geq 3$ ,  $v^+(y) = O(|y|^{2-n})$  and  $\nabla v^+(y) = O(|y|^{1-n})$  for large  $|y|$ . Therefore, (2.23) is derived. Whereas  $n = 2$ ,  $\int_{\partial\Omega} g = 0$  and  $\int_{\partial(\Omega \setminus \bar{D})} \nu_{\Omega \setminus \bar{D}} \cdot A\nabla u = 0$ , where  $\nu_{\Omega \setminus \bar{D}} = \nu_\Omega$  on  $\partial\Omega$  and  $\nu_{\Omega \setminus \bar{D}} = -\nu_D$  on  $\partial D$ . Hence, we obtain  $\int_{\partial D} q_3 = 0$ . And  $\log|y-x| - \log|y| \rightarrow 0$  as  $|y| \rightarrow \infty$ . Thus in two-dimension, we get  $v^+(y) = o(1)$  and  $\nabla v^+(y) = O(|y|^{-1})$ . (2.23) is derived using these facts.

By (2.22) and (2.23),  $\nabla v^+ = 0$ . Therefore,  $v^+ = 0$  in  $\mathbb{R}^n \setminus \bar{\Omega}$  due to the vanishing limit value of  $v^+$  onto  $\partial\Omega$  from the outside. Considering the limiting value along  $\partial\Omega$  for  $u^+$  and  $v^+$ , and reminding  $v^+ = 0$  in  $\mathbb{R}^n \setminus \bar{\Omega}$ ,  $v^+$  satisfies (1.2c). Thus as in Theorem 2.1, we get (2.19).  $\square$

Instead of (2.18b) and (2.18c), let us consider the following two equations:

$$(2.24a) \quad -\mathcal{D}_\Omega q_1 + \left(-\frac{1}{2}I + K_{D^+}\right) q_2 - \mathcal{S}_{D^+} q_3 = -S_\Omega g \text{ on } \partial D,$$

$$(2.24b) \quad \left(\frac{1}{2}I - K_{D^-}\right) q_2 + \mathcal{S}_{D^-} q_3 = 0 \quad \text{on } \partial D.$$

**Theorem 2.3.** *Let  $g \in L_0^2(\partial\Omega)$  be given. Then (2.21), (2.24), (2.18d) has a unique solution  $q_1 \in H_0^{1/2}(\partial\Omega)$ ,  $q_2 \in H^{1/2}(\partial D)$ , and  $q_3 \in L_0^2(\partial D)$  satisfying (2.19).*

*Proof.* Let  $q_1, q_2$ , and  $q_3$  satisfies (2.19). Then we get (2.21), (2.24a), and (2.24b) from (2.19a), (2.19b), and (2.19c), respectively.

Let  $v^+$  and  $v^-$  be defined by (2.20) for  $q_1, q_2$ , and  $q_3$ , the solution of (2.18). Then  $v^+$  and  $v^-$  satisfy (1.2a) and (1.2b), respectively. Thus we get

$$(2.25a) \quad \begin{aligned} v^+ &= -\mathcal{D}_\Omega v^+|_{\partial\Omega} + \mathcal{S}_\Omega(\nu_\Omega \cdot A\nabla v^+) + \mathcal{D}_D^+ v^+|_{\partial D} \\ &\quad - \mathcal{S}_D^+(\nu_D \cdot A\nabla v^+) \quad \text{in } \Omega \setminus \overline{D}, \end{aligned}$$

$$(2.25b) \quad v^- = -\mathcal{D}_D^- v^-|_{\partial D} + \mathcal{S}_D^-(\nu_D \cdot B\nabla v^-) \quad \text{in } D.$$

Since (2.24) implies

$$(2.26) \quad v^+|_{\partial\Omega} = q_1, v^+|_{\partial D} = v^-|_{\partial D} = q_2,$$

subtracting (2.24) from (2.25) we get

$$(2.27a) \quad \begin{aligned} \mathcal{S}_\Omega(g - \nu_\Omega \cdot A\nabla v^+|_{\partial\Omega}) + \mathcal{S}_D^+(\nu_D \cdot A\nabla v^+|_{\partial D} - q_3) \\ = 0 \text{ in } \Omega \setminus \overline{D}, \end{aligned}$$

$$(2.27b) \quad \mathcal{S}_D^-(\nu_D \cdot B\nabla v^- - q_3) = 0 \text{ in } D.$$

(2.27b) implies  $q_3 = \nu_D \cdot B\nabla v^-|_{\partial D}$ . From the invertibility of  $\mathcal{S}_{\Omega \setminus \overline{D}}$  if we define  $\mathcal{S}_{\Omega \setminus \overline{D}}(p_1, p_2) = \mathcal{S}_\Omega p_1 - \mathcal{S}_D p_2$  for  $p_1 \in L_0^2(\partial\Omega)$  and  $p_2 \in L_0^2(\partial D)$ , we get  $g = \nu_\Omega \cdot A\nabla v^+|_{\partial\Omega}$  and  $q_3 = \nu_D \cdot B\nabla v^-$ . Thus  $v^+$  and  $v^-$  satisfies (1.2) and by the uniqueness of the solution of (1.2),  $v^+ = u^+$  and  $v^- = u^-$ . Hence the theorem is proved.  $\square$

### 2.3. Isotropic case

In this section, a special case of conductivity  $B$  will be considered. Suppose that  $B = kA$  for positive constant. By the condition (1.3),  $k \neq 1$ . And by (2.6), the conductivity  $\gamma = A + (B - A)\chi_D$  in (1.1) is converted into  $\tilde{\gamma} = 1 + (k - 1)\chi_{\overline{D}}$  by the transformation  $\tilde{y} = A^{-1/2}y$ . Usually, the case  $B = kA$  is called the isotropic case. In isotropic case, we get

$$(2.28a) \quad \Gamma^B = \begin{cases} -\frac{1}{4\pi} \log k + \Gamma^A & \text{for } n = 2 \\ k^{\frac{n-2}{2}} \Gamma^A & \text{for } n \geq 3, \end{cases}$$

$$(2.28b) \quad B\nabla\Gamma^B = k^{\frac{n}{2}} A\nabla\Gamma^A \text{ for } n \geq 2.$$

And the followings hold for single and double layer potentials by using  $\det(B^{-\frac{1}{2}}) = k^{-\frac{n}{2}}$

$$(2.29a) \quad \mathcal{S}_D^- \psi = \begin{cases} -\frac{1}{4\pi} \frac{\log k}{k} \int_{\partial D} \psi + \frac{1}{k} \mathcal{S}_D^+ \psi & \text{for } n = 2 \\ \frac{1}{k} \mathcal{S}_D^+ \psi & \text{for } n \geq 3, \end{cases}$$

$$(2.29b) \quad \mathcal{D}_D^- \psi = \mathcal{D}_D^+ \psi \text{ for } n \geq 2$$



in  $\mathbb{R}^n \setminus \partial D$ . Since  $\int_{\partial D} \nu_D \cdot A \nabla u^+ = \int_{\partial D} \nu_D \cdot B \nabla u^- = 0$ , (2.29a) is changed into

$$(2.30) \quad \mathcal{S}_{D^-} [\nu_D \cdot B \nabla u^-] = \frac{1}{k} \mathcal{S}_{D^+} [\nu_D \cdot A \nabla u^+] \text{ for } n \geq 2.$$

By using (1.2e), (1.2f), (2.12b), (2.12d), (2.29b), and (2.30), we obtain

$$(2.31a) \quad \begin{aligned} \mathcal{D}_{D^+} u^+ &= \mathcal{D}_{D^-} u^- = \mathcal{S}_{D^-} [\nu_D B \nabla u^-] \\ &= \frac{1}{k} \mathcal{S}_{D^+} [\nu_D A \nabla u^+] \text{ in } \mathbb{R}^n \setminus \overline{D}, \end{aligned}$$

$$(2.31b) \quad \begin{aligned} \mathcal{D}_{D^-} u^- &= \mathcal{D}_{D^+} u^+ = \mathcal{S}_{D^+} [\nu_D A \nabla u^+] - H \\ &= k \mathcal{S}_{D^-} [\nu_D B \nabla u^-] - H \text{ in } D, \end{aligned}$$

where  $H = -\mathcal{D}_{\Omega} u^+|_{\partial \Omega} + \mathcal{S}_{\Omega} g$ .

Inserting (2.31) into (2.12a) and (2.12c), removing double layer potentials on  $\partial D$ , and using (2.30), we can represent

$$(2.32) \quad u^+ = H + \left( \frac{1}{k} - 1 \right) \mathcal{S}_{D^+} [\nu_D \cdot A \nabla u^+] \text{ in } \Omega \setminus \overline{D},$$

$$(2.33) \quad u^- = u^+ \text{ in } D.$$

For the more study about the representation (2.32), see [18, 19]. If we remove single layer potentials on  $\partial D$  using (2.31) and (2.29b), we get

$$(2.34) \quad u^+ = H + (1 - k) \mathcal{D}_{D^+} u^+ \text{ in } \Omega \setminus \overline{D},$$

$$(2.35) \quad u^- = \frac{1}{k} u^+ \text{ in } D.$$

Similar approaches are found in [20]. Using (2.32) and (2.34), we can simplify equations (2.18) and (2.21) by removing one of the boundary integral operators containing  $q_2$  or  $q_3$ . Since the representation (2.32) and (2.34) satisfies (1.2e) and (1.2f), respectively, the equations (2.18b) and (2.18c) are redundant equation for the representation (2.32) and (2.34), respectively. Among many possible representations, we choose the following representation,

$$(2.36a) \quad - \left( \frac{1}{2} I + K_{\Omega} \right) q_1 + \left( \frac{1}{k} - 1 \right) \mathcal{S}_{D^+} q_3 = -S_{\Omega} g \quad \text{on } \partial \Omega,$$

$$(2.36b) \quad - \frac{\partial \mathcal{D}_{\Omega}}{\partial \nu_{\Omega}} q_1 + \left( \frac{1}{k} - 1 \right) \left( \frac{k+1}{2(k-1)} I + K_{D^+}^* \right) q_3 = \frac{\partial S_{\Omega}}{\partial g} \quad \text{on } \partial D,$$

$$(2.36c) \quad \int_{\partial \Omega} q_1 = 0.$$

**Corollary 2.4.** *Let  $g \in L_0^2(\partial \Omega)$  be given. Then (2.36) has a unique solution  $q_1$  and  $q_3$  satisfying (2.19).*

*Proof.* The corollary derived using similar analyses in Theorem 2.3 for the representation (2.32).  $\square$

### 3. Inverse problem

In the previous section, we have considered the direct problem to find Dirichlet data from the given Neumann data  $g$ , an obstacle  $D$ , and its conductivity  $B$ . In this section, we will consider the problem of finding an obstacle  $D$  and its conductivity  $B$  creating the measured voltage  $f$  for given current density  $g$ . Thus the inverse problem is formulated as the following minimization problem

$$(3.37) \quad [D, B] = \operatorname{argmin}_{D, B} \|\Lambda_{D, B} g - f\|_{L^2(\partial\Omega)}$$

where  $\Lambda_{D, B} := q_1$  and  $q_1$  is the solution of (2.18) for given  $D$  and  $B$ . One of most popular methods for the minimization problem (3.37) are the Newtonian methods. Among various kinds of Newtonian methods, we will use the trust region method which is known efficient constrained large scale optimization problem as our case. The most important part when we use Newtonian method is finding Fréchet derivative of  $\Lambda_{D, B}$  with respect to  $D$  and  $B$ . We will treat this in the next section.

#### 3.1. Fréchet derivative

In this paper, the Fréchet derivative is computed as the solution of an integral equations system using the Fréchet derivatives of integral operators as in [15, 31]. The Fréchet derivatives computed here is applied for the general representation formulas developed in Section 2.2. Similar approaches are found in inverse scattering problem [12, 14].

For example, we will compute the Fréchet derivatives for the integral equation system (2.36). By differentiating (2.36), we get formally

$$(3.38a) \quad \begin{aligned} & - \left( \frac{1}{2}I + K_\Omega \right) q'_1 + \left( \frac{1}{k} - 1 \right) \mathcal{S}_{D+} q'_3 \\ & = - \left( \frac{1}{k} - 1 \right) \mathcal{S}'_{D+} q_3 \quad \text{on } \partial\Omega, \end{aligned}$$

$$(3.38b) \quad \begin{aligned} & - \frac{\partial \mathcal{D}_\Omega}{\partial \nu_D} q'_1 + \left( \frac{1}{k} - 1 \right) \left( \frac{k+1}{2(k-1)} + K_{D+}^* \right) q'_3 \\ & = - \frac{\partial \mathcal{S}_\Omega'}{\partial \nu_\Omega} g + \frac{\partial \mathcal{D}_\Omega'}{\partial \nu_D} q_1 - \left( \frac{1}{k} - 1 \right) (K_{D+}^*)' q_3 \quad \text{on } \partial D, \end{aligned}$$

$$(3.38c) \quad \int_{\partial\Omega} q'_1 = 0.$$

**Theorem 3.1.** *Let  $g \in L_0^2(\partial\Omega)$ . Then there is a unique solution of (2.36),  $q'_1$  and  $q'_3$ , satisfying*

$$(3.39) \quad q'_1 = [\Lambda_{D, B} g]', q'_3 = [\nu_D \cdot A \nabla u]'$$

*Proof.* The existence of the Fréchet derivative of integral operators is shown in [31]. If  $q'_1$  and  $q'_3$  is given by (3.39), by considering the derivatives of (2.36), we

get (3.38) for  $q'_1$  and  $q'_3$ . Since the left hand side of (3.38) is the same as (2.36), the uniqueness of the system (3.38) follows from the uniqueness of the system (2.36). Thus there is a unique solution  $q'_1$  and  $q'_3$  of (2.36) satisfying (3.39).  $\square$

Assume that  $B$  is fixed. Then the following injectivity is known for the Fréchet derivative of Neumann-to-Dirichlet map [14].

If  $\frac{\partial \Lambda_D}{\partial D}(h) = 0$ , then  $h \cdot \nu_D = 0$ .

#### 4. Discretization of the boundary integral operators and the boundary of the obstacle

In this section, we discretize the boundary integral operators, their Fréchet derivatives, and the boundary of the obstacle of interest. For simplicity, we consider the isotropic case. Then by the argument in Section 2.3, it is enough to consider the conductivity  $\gamma = 1 + (k - 1)\chi_D, k > 0, k \neq 1$ . Assume that  $\Omega$  is the unit disk in two-dimensional plane and the obstacle  $D$  is parameterized by  $2\pi$  periodic vector function  $z(t), (0 \leq t \leq 2\pi)$  such that  $z \in V^\infty = \{a \in C^2([0, 2\pi]) \times C^2([0, 2\pi]) | a(0) = a(2\pi)\}$ . The parametric function  $z$  imply the obstacle  $D$ , from the rest of this paper. The perturbation of the obstacle  $z$  is parameterized by the  $2\pi$  periodic function  $h$  contained in the function space  $V^l = \{a \in V_\infty | \|a\|_{C^2([0, 2\pi]) \times C^2([0, 2\pi])} \leq l\}$  for some small positive number  $l$ .

##### 4.1. Approximation of the integral operators

Let us define  $(v_1, v_2)^\perp = (v_2, -v_1)$ , then  $\nu_\Omega(x(s)) = x(s) = (\cos s, \sin s)$  and  $\nu_D(z(t)) = (z'(t))^\perp$ . The Fréchet-derivatives of the integral operators appeared in (2.36) can be computed as follows:

$$\begin{aligned}
 (4.40) \quad & \left( \frac{\partial}{\partial \nu_D} \mathcal{S}_\Omega \phi(z(t)) \right)' = \int_{\partial \Omega} \Gamma_{i,j}(z(t), x(\sigma)) (z'(t))_i^\perp h_j(t) \phi(x(\sigma)) d\sigma, \\
 & \left( \frac{\partial}{\partial \nu_D} \mathcal{D}_\Omega \phi(z(t)) \right)' = \int_{\partial \Omega} \Gamma_{i,j,k}(z(t), x(\sigma)) (z'(t))_i^\perp (x'(\sigma))_j^\perp h_k(t) \phi(x(\sigma)) d\sigma, \\
 (\mathcal{S}'_D \psi)(x) &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{h(\tau) \cdot (z(\tau) - x)}{|z(\tau) - x|^2} |z'(\tau)| \psi(\tau) d\tau \\
 & \quad - \frac{1}{2\pi} \int_0^{2\pi} \log(|z(\tau) - x|) \frac{h'(\tau) \cdot z'(\tau)}{|z'(\tau)|} \psi(\tau) d\tau, \\
 ((K'_D)^* \psi)(t) &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{(h(t) - h(\tau)) \cdot (z'(t))^\perp + (z(t) - z(\tau)) \cdot (h')^\perp(t)}{|z(t) - z(\tau)|^2} \frac{|z'(\tau)|}{|z'(t)|} \psi(\tau) d\tau \\
 & \quad + \frac{1}{2\pi} \int_0^{2\pi} \frac{2(z(t) - z(\tau)) \cdot (h(t) - h(\tau)) (z(t) - z(\tau)) \cdot (z'(t))^\perp}{|z(t) - z(\tau)|^4} \frac{|z'(\tau)|}{|z'(t)|} \psi(\tau) d\tau
 \end{aligned}$$

$$-\frac{1}{2\pi} \int_0^{2\pi} \frac{(z(t) - z(\tau)) \cdot (z')^\perp(t)}{|z(t) - z(\tau)|^2} \left( \frac{h'(\tau) \cdot z'(\tau)}{|z'(\tau)||z'(\tau)|} - \frac{h'(t) \cdot z'(t)|z'(\tau)|}{|z'(t)|^3} \right) \psi(\tau) d\tau,$$

where  $\Gamma_{i,j} = \frac{\partial^2}{\partial y_i \partial y_j} \Gamma(y, \eta)$  and  $\Gamma_{i,j,k} = \frac{\partial^3}{\partial y_i \partial y_j \partial y_k} \Gamma(y, \eta)$ . The Eisenstein notation is used for the indices  $i, j$ , and  $k$ .

All the integral operators to be discretized in the integral equation system (2.36) are as follows:

$$(4.41a) \quad \frac{\partial}{\partial \nu_D} \mathcal{S}_\Omega \phi(z), \frac{\partial}{\partial \nu_D} \mathcal{D}_\Omega \phi(z), \left( \frac{\partial}{\partial \nu_D} \mathcal{S}_\Omega(z) \right)' \phi, \left( \frac{\partial}{\partial \nu_D} \mathcal{D}_\Omega(z) \right)' \phi,$$

$$(4.41b) \quad \mathcal{S}_D \psi(x), \mathcal{S}'_D \psi(x),$$

$$(4.41b) \quad K_\Omega \phi(x), S_\Omega \phi(x), K_D^* \psi(z), (K_D^*)' \psi(z),$$

where  $\phi \in L^2(\partial\Omega)$  and  $\psi \in L^2(\partial D)$ .

Let us divide  $\partial\Omega$  by  $2N_\Omega$  times and  $\partial D$  by  $2N_D$  times. Then, the integral operators in (4.41a) can be approximated by trapezoidal rule, since there is no singular point in the region of integration.

Whereas, the integral operators in (4.41b) should be treated differently. Let us approximate the boundary integral operators one by one. Since  $\Omega$  is the unit disk,

$$\frac{\partial \Gamma(x, \xi)}{\partial \nu(\xi)} = \frac{-1}{4\pi} \quad \text{for } x, \xi \in \partial\Omega.$$

Therefore, we get

$$(4.42) \quad K_\Omega q_1 = K_\Omega q'_1 = 0$$

by (2.36c) and (3.38c).

The integration of  $S_\Omega \phi(x)$  is approximated by  $S_\Omega^{N_\Omega}$  using trigonometric interpolation [23].

$$(4.43) \quad S_\Omega^{N_D} \phi(x(s)) = -\frac{1}{2N_D} \sum_{j=0}^{2N_D-1} \phi(x(s_j)) R_j^{N_D}(s),$$

$$R_j^{N_D}(s) = -\sum_{k=1}^{N_D-1} \frac{1}{k} \cos k(s - s_j) - \frac{1}{2N_D} \cos N_D(s - s_j).$$

Now let us approximate  $K_D^* \psi(z)$  and  $(K_D^*)' \psi(z)$ . Using kernel  $k^*(t, \tau)$  and  $(k^*)'(t, \tau)$ , they are represented as follows:

$$K_D^* \psi(z(t)) = -\frac{1}{2\pi} \int_0^{2\pi} k^*(t, \tau) \psi(z(\tau)) d\tau,$$

$$(K_D^*)' \psi(z(t)) = -\frac{1}{2\pi} \int_0^{2\pi} (k^*)'(t, \tau) \psi(z(\tau)) d\tau,$$

where

(4.44)

$$\begin{aligned}
 k^*(t, \tau) &= \frac{(z')^\perp(t) \cdot (z(t) - z(\tau)) |z'(\tau)|}{|z(t) - z(\tau)|^2 |z'(t)|} \\
 (k^*)'(t, \tau) &= \frac{(h(\tau) - h(t)) \cdot (z')^\perp(\tau) + (z(\tau) - z(t)) \cdot (h')^\perp(\tau) |z'(\tau)|}{|z(\tau) - z(t)|^2 |z'(t)|} \\
 &\quad - \frac{2(z(\tau) - z(t)) \cdot (h(\tau) - h(t)) (z(\tau) - z(t)) \cdot (z')^\perp(\tau) |z'(\tau)|}{|z(\tau) - z(t)|^4 |z'(t)|} \\
 &\quad + \frac{(z(t) - z(\tau)) \cdot (z')^\perp(t) |h'(\tau) \cdot z'(\tau)|^2 - h'(t) \cdot z'(t) |z'(\tau)|^2}{|z(t) - z(\tau)|^2 |z'(t)|^3 |z'(\tau)|}
 \end{aligned}$$

when  $t \neq \tau$ . To get the limiting values  $k^*(t, t) := \lim_{\tau \rightarrow t} k^*(t, \tau)$  and

$$(k^*)'(t, t) := \lim_{\tau \rightarrow t} (k^*)'(t, \tau),$$

let us use the mean value theorem to (4.44). For all  $t, \tau$ , there are numbers  $\tau', \tau'', \tau'_h, \tau''_h$  between  $t$  and  $\tau$  such that

(4.45)

$$\begin{aligned}
 k^*(t, \tau) &= - \frac{(z')^\perp(t) \cdot (z'(t)(\tau - t) + \frac{z''(\tau'')}{2}(\tau - t)^2) |z(\tau)|}{|z'(\tau')|^2 |\tau - t|^2 |z(t)|} \\
 (k^*)'(t, \tau) &= - \frac{(h'(t)(\tau - t) + h''(\tau''_h)(\tau - t)^2/2) \cdot (z')^\perp(t) |z'(\tau)|}{|z'(\tau')|^2 |t - \tau|^2 |z'(t)|} \\
 &\quad + \frac{(z'(t)(\tau - t) + z''(\tau'')(\tau - t)^2/2) \cdot (h')^\perp(t) |z'(\tau)|}{|z'(\tau')|^2 |t - \tau|^2 |z'(t)|} \\
 &\quad - \frac{2z'(\tau') \cdot h'(\tau'_h)(z'(t)(\tau - t) + z''(\tau'')(\tau - t)^2/2) \cdot (z')^\perp(t) |z'(\tau)|}{|z'(\tau')|^4 |t - \tau|^2 |z'(t)|} \\
 &\quad + \frac{(z(t) - z(\tau)) \cdot (z')^\perp(t) |h'(\tau) \cdot z'(\tau)|^2 - h'(t) \cdot z'(t) |z'(\tau)|^2}{|z(t) - z(\tau)|^2 |z'(t)|^3 |z'(\tau)|}.
 \end{aligned}$$

Taking limit values of (4.45) we get

(4.46)

$$\begin{aligned}
 k^*(t, t) &= - \frac{(z')^\perp(t) \cdot z''(t)}{2|z'(t)|^2} \\
 (k^*)'(t, t) &= - \frac{h''(t) \cdot (z')^\perp(t) + z''(t) \cdot (h')^\perp(t)}{2|z'(t)|^2} + \frac{z'(t) \cdot h'(t) z''(t) \cdot (z')^\perp(t)}{|z'(t)|^4}.
 \end{aligned}$$

In the last equality of (4.46), the equation  $h'(t) \cdot (z')^\perp(t) + z'(t) \cdot (h')^\perp(t) = 0$  is used, which is induced from the fact that  $h'(t) \cdot (z')^\perp(t) + z'(t) \cdot (h')^\perp(t) = |h'(t)| |z'(t)| (\cos(90 - \theta) + \cos(90 + \theta)) = 0$  where  $\theta = \arg(h'(t), z'(t))$ .

Thus,  $K_D^* \psi(z(t_i))$  and  $(K_D^*)' \psi(z(t_i))$ ,  $i = 1, \dots, 2N_D$  are discretized as follows

$$\begin{aligned} (K_D^*)^{N_D} \psi(z(t_i)) &= -\frac{1}{2N} \sum_{j=1}^{2N_D} \psi(z(t_j)) k^*(t_i, t_j) \\ ((K_D^*)')^{N_D} \psi(z(t_i)) &= -\frac{1}{2N} \sum_{j=1}^{2N_D} \psi(z(t_j)) (k^*)'(t_i, t_j), \end{aligned}$$

where the values of  $k^*$  and  $(k^*)'$  are chosen from (4.44) when  $i \neq j$  and from (4.46) when  $i = j$ .

The overall discretization of each integral operators is at least  $O(N_D^{-2} + N_\Omega^{-2})$ , since (4.42) don't need any discretization, the discretization order of (4.43) is  $O(\exp(-N_\Omega))$ , and Trapezoidal rules are used for the boundary integral operators except  $K_\Omega, K'_\Omega, S_\Omega$ .

Although the boundary integral operators are discretized, the boundary of the obstacle  $z$  and its derivatives  $z'$  and  $z''$  are not discretized, which will be treated in the next section.

## 4.2. Approximation of obstacles using the trigonometric expansion

In this section, the boundary of the obstacle  $z(t)$  and its derivatives  $z'(t)$  and  $z''(t)$  are discretized and approximated using the trigonometric expansion. Let us introduce  $W^\infty$  and  $W^l$  spaces, instead of the spaces  $V^\infty$  and  $V^l$  in Section 4.1 as follows:

$$\begin{aligned} W^\infty &= \{a \in V^\infty | a(t) = r(t)(\cos t, \sin t), r \in C^2([0, 2\pi])\} \\ (4.47) \quad W^\lambda &= V^l \cap W^\infty. \end{aligned}$$

Assume that  $z \in W^\infty$  and  $h \in W^l$  for some small number  $l > 0$ . If  $z \in W^\infty$ , the obstacle  $D$  parameterized by  $z$  is star convex.

To approximate  $r(t)$ , let us consider the following  $2N_D$ -dimensional subspace containing trigonometric polynomials:

$$(4.48) \quad T_{N_D} = \left\{ \sum_{l=0}^{N_D} a_l \cos lt + \sum_{l=1}^{N_D-1} b_l \sin lt \right\}.$$

Define the trigonometric interpolation operator  $P_{N_D}$  from  $V^\infty$  to  $T_{N_D}$  is given by

$$(4.49) \quad P_{N_D}(\varphi)(t) = \sum_{j=1}^{2N_D} \varphi(s_j) L_j(t),$$

where  $\varphi$  is a periodic function and  $L_j(t)$  is the Lagrangian basis function of  $T_{N_D}$  given by

$$(4.50) \quad L_j(t) = \frac{1}{2N_D} \left( 1 + 2 \sum_{k=1}^{N_D-1} \cos k(t - t_j) + \cos N_D(s - s_j) \right),$$

where  $t_j = \frac{j\pi}{N_D}$ , ( $j = 0, \dots, 2N_D - 1$ ). Note that the trigonometric interpolation operator  $P_{N_D}$  satisfies

$$P_{N_D}(\varphi)(t_j) = \varphi(t_j), \text{ for } j = 0, \dots, 2N_D - 1.$$

Assuming that  $z(t) = (z_1(t), z_2(t))$  is known only on  $2N_D$  points  $z(t_j)$ , ( $j = 1, \dots, 2N_D$ ),  $z, z'$ , and  $z''$  are approximated by

$$P_{N_D}(z) = P_{N_D}(r)(\cos(t), \sin(t)), \quad P_{N_D}(z') = (P_{N_D}(z))',$$

and  $P_{N_D}(z'') = (P_{N_D}(z))''$  such that

$$\begin{aligned} P_{N_D}(z_1)(t) &= P_{N_D}(r)(t) \cos(t), \\ P_{N_D}(z'_1)(t) &= P_{N_D}(r')(t) \cos(t) - P_{N_D}(r)(t) \sin(t), \\ P_{N_D}(z''_1)(t) &= P_{N_D}(r'')(t) \cos(t) - 2P_{N_D}(r')(t) \sin(t) \\ &\quad - P_{N_D}(r)(t) \cos(t), \\ P_{N_D}(z_2)(t) &= P_{N_D}(r)(t) \sin(t), \\ P_{N_D}(z'_2)(t) &= P_{N_D}(r')(t) \sin(t) + P_{N_D}(r)(t) \cos(t), \\ P_{N_D}(z''_2)(t) &= P_{N_D}(r'')(t) \sin(t) + 2P_{N_D}(r')(t) \cos(t) \\ &\quad - r(t) \sin(t). \end{aligned} \tag{4.51}$$

$P_{N_D}(r), P_{N_D}(r')$ , and  $P_{N_D}(r'')$  can be found by differentiating each side of (4.49) and inserting  $r$  in place of  $\varphi$  such that

$$\begin{aligned} P_{N_D}(r)(t) &= \sum_{j=1}^{2N_D} r(t_j) L_j(t), \\ (P_{N_D}(r))'(t) &= \sum_{j=1}^{2N_D} r(t_j) L'_j(t), \\ (P_{N_D}(r))''(t) &= \sum_{j=1}^{2N_D} r(t_j) L''_j(t). \end{aligned} \tag{4.52}$$

In the following numerical approximation, it is sufficient to know the value of the functions in (4.52) only on the points  $t_k, (k = 1, \dots, 2N_D)$ . Since

$$L_j(t_k) = L_0(t_{k-j}), L'_j(t_k) = L'_0(t_{k-j}), L''_j(t_k) = L''_0(t_{k-j}),$$

it is enough to compute only the values of  $L_0(t_k), L'_0(t_k), L''_0(t_k), k = 1, \dots, 2N_D$  in order to compute the values of all interpolation functions at  $t_k, (k = 1, \dots, 2N_D)$ .

**Lemma 4.1.** *Let  $L_0$  be the Lagrange function given in (4.50) and  $t_k = \frac{k\pi}{N_D}$ , then the following equations hold.*

$$\begin{aligned}
(4.53a) \quad L_0(t_k) &= \delta_{k,0} \\
L'_0(t_k) &= \frac{1}{2N_D} [\delta_{k,0} \operatorname{Re}(F'(0)) + (1 - \delta_{k,0}) \operatorname{Re}(G'(t_k))] \\
(4.53b) \quad &= (1 - \delta_{k,0}) (-1)^k \frac{\cot(t_k/2)}{2} \\
L''_0(t_k) &= \frac{1}{2N_D} [\delta_{k,0} \operatorname{Re}(F''(0)) + (1 - \delta_{k,0}) \operatorname{Re}(G''(t_k))] \\
(4.53c) \quad &= -\delta_{k,0} \frac{2N_D^2 + 1}{6} + (1 - \delta_{k,0}) \frac{-(-1)^k \operatorname{cosec}^2(t_k/2)}{2},
\end{aligned}$$

where  $\delta_{k,l}$  is a Kronecker delta.

*Proof.* Define

$$\begin{aligned}
F(t) &:= 1 + 2 \sum_{l=1}^{N_D-1} e^{ilt} + e^{iN_D t} \\
G(t) &:= i(1 - e^{iN_D t}) \cot(t/2).
\end{aligned}$$

Then as in [24],  $L_0(t) = \frac{1}{2N_D} \operatorname{Re}(F(t))$  for all  $t$  and  $F(t) = G(t)$  for  $t \neq 0$ . About  $F$ , the following equation can be obtained easily

$$\begin{aligned}
F'(0) &= 2i \sum_{l=1}^{N_D-1} l + iN_D = iN_D^2 \\
F''(0) &= -2 \sum_{l=1}^{N_D-1} l^2 - N_D^2 = -\frac{N_D(2N_D^2 + 1)}{3}.
\end{aligned}$$

Differentiating  $G$ , we obtain

$$\begin{aligned}
G'(t) &= N_D e^{iN_D t} \cot(t/2) - i/2(1 - e^{iN_D t}) \operatorname{cosec}^2(t/2) \\
G''(t) &= iN_D^2 e^{iN_D t} \cot(t/2) - N_D e^{iN_D t} \operatorname{cosec}^2(t/2) \\
&\quad + i/2(1 - e^{iN_D t}) \operatorname{cosec}^2(t/2) \cot(t/2).
\end{aligned}$$

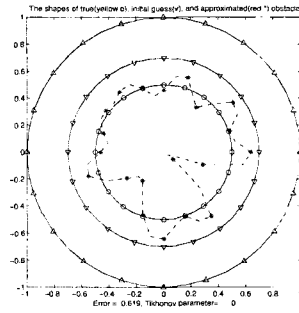
Hence, using  $e^{iN_D t_k} = (-1)^k$ , we get the lemma.  $\square$

## 5. Numerical implementation

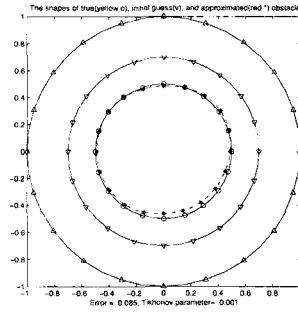
In this section, the numerical implementation of direct and inverse problem using boundary integral representation is done for the various obstacles, the boundary of which is parameterized by

- (1) Circle:  $z(t) = r(t)(\cos t, \sin t)$ ,  $r(t) = r_0$
- (2) Bean:  $z(t) = r(t)(\cos t, \sin t)$ ,  $r(t) = \frac{1+0.9 \cos t + 0.1 \sin 2t}{2(1+0.75 \cos t)}$
- (3) Peanut:  $z(t) = r(t)(\cos t, \sin t)$ ,  $r(t) = \frac{1}{2} \sqrt{\cos^2 t + 0.25 \sin^2 t}$

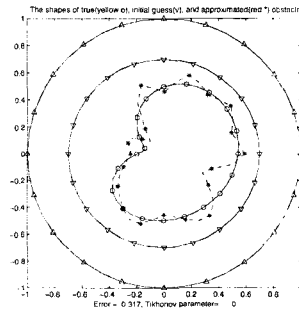




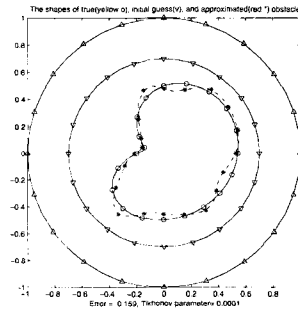
Circle, 20 iterations,  
 $N_{\Omega} = N_D = 10$ , 2% noise,  
 $\alpha = 0$



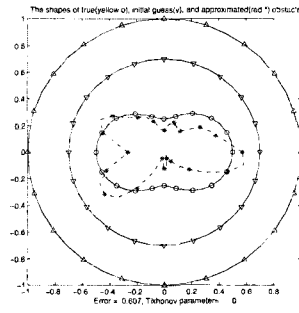
Circle, 20 iterations,  
 $N_{\Omega} = N_D = 10$ , 2% noise,  
 $\alpha = 10^{-3}$



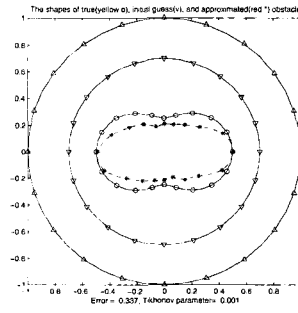
Bean, 20 iterations,  
 $N_{\Omega} = N_D = 10$ , 1% noise,  
 $\alpha = 0$



Bean, 20 iterations,  
 $N_{\Omega} = N_D = 10$ , 1% noise,  
 $\alpha = 10^{-4}$



Peanut, 20 iterations,  
 $N_{\Omega} = N_D = 10$ , 1% noise,  
 $\alpha = 0$



Peanut, 20 iterations,  
 $N_{\Omega} = N_D = 10$ , 1% noise,  
 $\alpha = 10^{-3}$

FIGURE 1. Reconstruction of Circle, Bean, and Peanut.

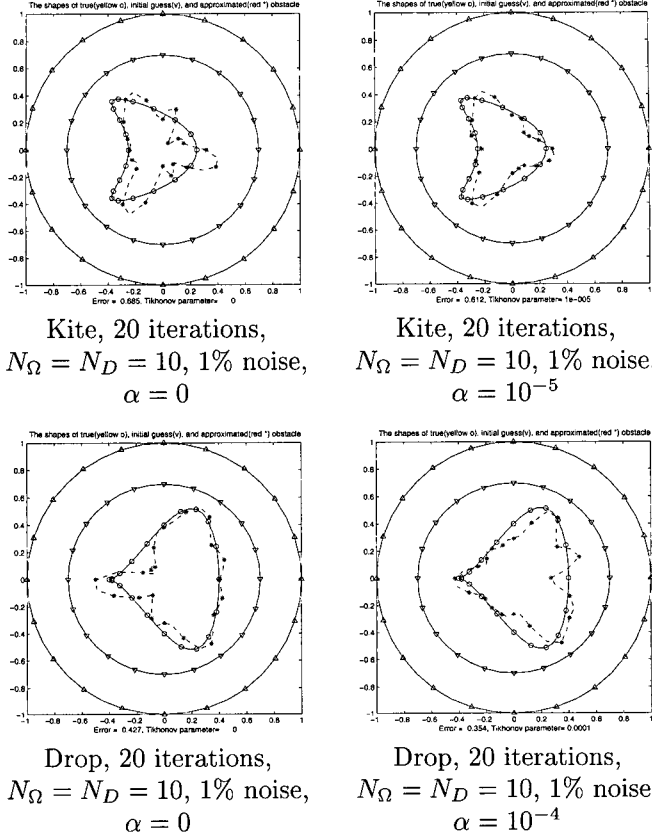


FIGURE 2. Reconstruction of Kite and Drop

- (4) Kite:  $z(t) = (z_1(t), z_2(t))$ ,  $z_1(t) = 0.25 \cos t + 0.1625 \cos 2t - 0.1625$ ,  
 $z_2(t) = 0.375 \sin t$
- (5) Drop:  $z(t) = (z_1(t), z_2(t))$ ,  $z_1(t) = 0.4 \cos t$ ,  $z_2(t) = 0.4 \sin t + 0.2 \sin(2t)$
- (6) Square:  $z(t) = (z_1(t), z_2(t))$ ,
- $z_1(t) = \frac{1}{2\sqrt{2}}$ ,  $z_2(t) = \frac{1}{2} \sin(t)$ , if  $0 \leq t < \frac{\pi}{4}$
  - $z_1(t) = \frac{1}{2} \cos(t)$ ,  $z_2(t) = \frac{1}{2\sqrt{2}}$ , if  $\frac{\pi}{4} \leq t < \frac{3\pi}{4}$
  - $z_1(t) = -\frac{1}{2\sqrt{2}}$ ,  $z_2(t) = \frac{1}{2} \sin(t)$ , if  $\frac{3\pi}{4} \leq t < \frac{5\pi}{4}$
  - $z_1(t) = \frac{1}{2} \cos(t)$ ,  $z_2(t) = -\frac{1}{2\sqrt{2}}$ , if  $\frac{5\pi}{4} \leq t < \frac{7\pi}{4}$
  - $z_1(t) = \frac{1}{2\sqrt{2}}$ ,  $z_2(t) = \frac{1}{2} \sin(t)$ , if  $\frac{7\pi}{4} \leq t \leq 2\pi$
- (7) Wedge:  $z(t) = (z_1(t), z_2(t))$ ,
- $z_1(t) = \frac{1-\sin(t)}{2\sqrt{3}}$ ,  $z_2(t) = 0.5 * \sin(t)$ , if  $0 \leq t < \frac{\pi}{2}$
  - $z_1(t) = \frac{\sin(t)-1}{2\sqrt{3}}$ ,  $z_2(t) = 0.5 * \sin(t)$ , if  $\frac{\pi}{2} \leq t < \frac{7\pi}{6}$
  - $z_1(t) = -\frac{3\sqrt{3}}{\pi}(\frac{3\pi}{2} - t)$ ,  $z_2(t) = -\frac{3}{4\pi}(\frac{3\pi}{2} - t)$ , if  $\frac{7\pi}{6} \leq t < \frac{3\pi}{2}$

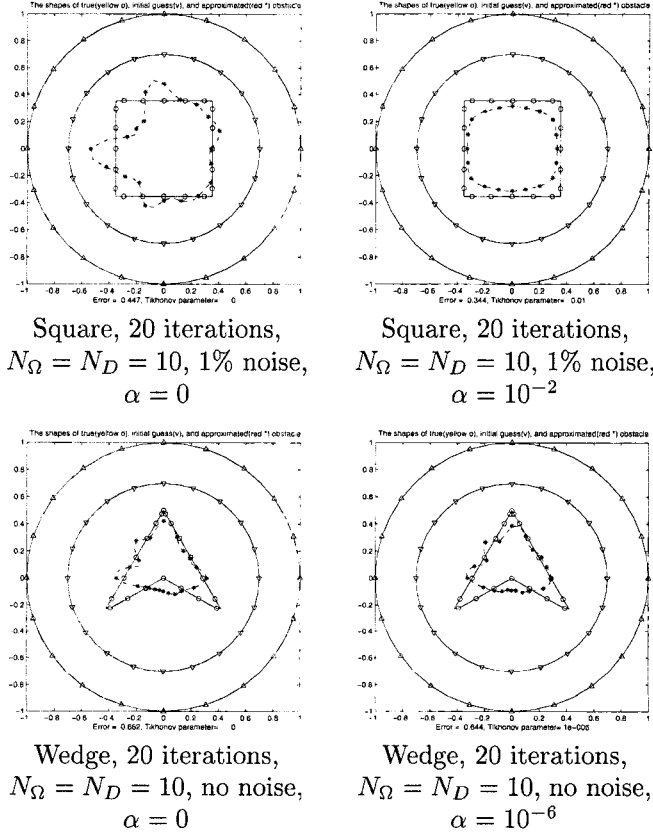


FIGURE 3. Reconstruction of Square and Wedge

- $z_1(t) = \frac{3\sqrt{3}}{\pi}(t - \frac{3\pi}{2})$ ,  $z_2(t) = -\frac{3}{4\pi}(t - \frac{3\pi}{2})$ , if  $\frac{3\pi}{2} \leq t < \frac{11\pi}{6}$
- $z_1(t) = \frac{1}{2\sqrt{3}}(1 - \sin(t))$ ,  $z_2(t) = \frac{1}{2} \sin(t)$ ; if  $\frac{11\pi}{6} \leq t \leq 2\pi$ .

### 5.1. Direct problem

Let us consider the efficiency of direct problem solver in (2.36). If  $g(x, y) = x$  and  $D$  is a circle with center  $(0, 0)$  and radius  $r_0 \in (0, 1)$ , then the solution of (1.1) is represented by (1.4) with  $k = 1$ .

Let us approximate  $q_1^{N_\Omega, N_D}$  be the solution of (2.36) when we divide  $\partial\Omega$  and  $\partial D$  by  $2N_\Omega$  and  $2N_D$  times. Define

$$(5.54) \quad \rho_{N_\Omega/2N_\Omega, N_D} = \log_2 \left( \frac{\|q_1^{N_\Omega, N_D} - f\|}{\|q_1^{2N_\Omega, 2N_D} - f\|} \right).$$

TABLE 1.  $\|q_1 - f\|_{L^2(\partial\Omega)}, r_0 = 0.5$ 

$N_\Omega \backslash N_D$	5	10	20	40	80
5	0.0387	0.0327	0.0327	0.0327	0.0327
10	0.0082	0.0082	0.0082	0.0082	0.0082
20	0.0020	0.0020	0.0020	0.0020	0.0020
40	0.0005	0.0005	0.0005	0.0005	0.0005
80	0.0002	0.0001	0.0001	0.0001	0.0001

TABLE 2.  $\rho_{N_\Omega/2N_\Omega, N_D}, r_0 = 0.5.$ 

$\rho \backslash N_D$	5	10	20	40	80
$\rho_{5/10, N_D}$	2.2396	1.9986	2.0001	2.0001	2.0001
$\rho_{10/20, N_D}$	1.9999	2.0020	2.0000	2.0000	2.0000
$\rho_{20/40, N_D}$	1.9640	2.0000	2.0000	2.0000	2.0000
$\rho_{40/80, N_D}$	1.5799	2.0000	2.0000	2.0000	2.0000

TABLE 3. The error of  $q_1$  when  $D$  is a circle of radius  $0.1, \dots, 0.9$  centered at the origin and  $N_\Omega = 10, N_D = 10$ .

$r$	$\ q_1 - f\ $
0.1	0.0088
0.2	0.0087
0.3	0.0086
0.4	0.0084
0.5	0.0082
0.6	0.0083
0.7	0.0167
0.8	0.1252
0.9	0.6657

The  $L^2$  error between  $q_1^{N_\Omega, N_D}$  and the true solution  $f$  in (1.4c) is shown in Table 1. The error attenuated significantly as  $N_\Omega$  increase but stayed as  $N_D$  increase, which is due to the fact that  $f$  is a function on  $\partial\Omega$ . Since we have used Trapezoidal rule approximating integral operators, we get  $O(N_\Omega^{-2})$  approximations as in Table 2. The dependence of the error on the radius of the obstacle is shown in Table 3. For the disk with radius less than 0.6, the error was not changed significantly, but the error blows when the radius is greater than 0.6.

## 5.2. Inverse problem

We considered the efficiency of the boundary integral representation (2.36) which is used for the forward solver for the trust region method for the inverse minimization problem (3.37) to find the obstacle  $D$  for given  $k$ . 2%, 0%, and 1% multiplicative noises are added for the implementation of Circle, Wedge, and all the other cases, respectively. The circle of radius 0.7 centered at the origin is chosen as an initial guess. The approximated obstacle is plotted with dotted line. The reconstructed obstacles are plotted when we use Tikhonov regularization with Tikhonov parameter  $\alpha$ , and it is compared with the result when we don't use any regularization.

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