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THE INITIAL-BOUNDARY-VALUE PROBLEM OF A GENERALIZED BOUSSINESQ EQUATION ON THE HALF LINE

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Abstract. The local existence of solutions for the initial-boundary value problem of a generalized Boussinesq equation on the half line is considered. The approach consists of replacing the Fourier transform in the initial value problem by the Laplace transform and making use of modern methods for the study of nonlinear dispersive wave equation

1. Introduction

We consider the initial-boundary value problem for the generalized Boussinesq equation on the half line

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + (|u|^{\kappa} u)_{xx} = 0, t > 0, x > 0, \\ u(0,t) = h_1(t), u_x(0,t) = h_2(t), \\ u(x,0) = f(x), u_t(x,0) = \partial_x h(x), \end{cases} \tag{1.1}$$

where $0 < \kappa < 4$, the velocity is an x-derivative function. Equations of type (1.1) are a class of essential model equations appearing in physics and fluid mechanics. It is derived by Boussinesq to describe two-dimensional irrotational flows of an inviscid liquid in a uniform rectangular channel. And it also arises in a large range of physical phenomena including the propagation of ion-sound waves in a plasma and nonlinear lattice wave.

The study of the initial-value problem for the Boussinesq-type equation has recently attracted considerable attention of many mathematicians and physicists (See [1], [2] and references therein). For instance, Bona and Sachs in [2] proved that the initial-value problem of the Boussinesq equation is locally well posed for smooth data by using Kato's abstract theory of quasilinear evolution equation. They proved that for any $f(x) \in H^{s+2}(\mathbb{R})$ and $h(x) \in H^{s+1}(\mathbb{R})$ with some $s > 1/2$, there exists a time $T > 0$ such that the initial-value problem of the Boussinesq equation has a unique solution $u \in C([0,T]; H^{s+2}(\mathbb{R})) \cap C^1([0,T]; H^s(\mathbb{R})) \cap C^2([0,T]; H^{s-2}(\mathbb{R}))$. In the case

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$1 < \kappa < 4$, Linares in [8] established the local and global existence theory for the initial-value problem of the Boussinesq equation when $(f, \partial_x h) \in L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$ or $(f, \partial_x h) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, by the so-called $L^p - L^q$ smoothing effect of the Strichartz type. In [12], Xue considered the case $\kappa > 4$ and proved some local and global existence results when the initial data belong to some suitable Besov spaces.

The practical, quantitative use of the Boussinesq equation and its relatives does not always involve the pure initial-value problem. Instead, initial-boundary value problems either on a finite domain or on the half line often come to the fore. The main difficulty in the study of initial-boundary-value problems is the evaluation of the contribution of the boundary data. In the literature, very few results are available. By using the finite element Galerkin method, Pani and Saranga in [10] got some local existence and uniqueness of the solutions of the initial-boundary-value problems of the Boussinesq equation on a finite domain with homogenous boundary conditions. In [11] Varlamov considered the damped Boussinesq equation with the initial-boundary-value problem in a unit disk, and the global-in-time solvability was obtained by using the Fourier-Bessel series. As far as we know the existence of the initial-boundary-value problem of a generalized Boussinesq equation on the half line was not considered previously. In this paper we consider the local existence of the initial-boundary-value problem for the Boussinesq-type equations on the half line. The approach consists of replacing the Fourier transform in the initial-value problem by the Laplace transform and making use of modern methods for the study of nonlinear dispersive wave equation. The main result obtained in this paper is

Theorem 1.1. *Assume that $f(x) \in L^2(\mathbb{R}^+)$, $h(x) \in H_0^{-1}(\mathbb{R}^+)$, $h_1(t) \in H_0^{\frac{1}{4}}(\mathbb{R}^+)$ and $h_2(t) \in \mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)$. Then there exists a positive constant $T > 0$, which depends only on*

$$\|f\|_{L^2(\mathbb{R}^+)} + \|h\|_{H^{-1}(\mathbb{R}^+)} + \|h_1\|_{H_0^{\frac{1}{4}}(\mathbb{R}^+)} + \|h_2\|_{\mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)},$$

such that the initial-boundary-value problem (1.1) *possess a unique local solution $u(t,x)$ satisfying*

$$u \in C([0,T]; L_x^2(\mathbb{R}^+)) \cap L_t^4([0,T]; L_x^\infty(\mathbb{R}^+)).$$

In the sequel, we denote by $\mathbb{R}^+$ the open right half-line $\{x : x > 0\}$ and denote by C some large constant which may vary from line to line. The notation $A \sim B$ means that there exist two harmless positive constants C_1 and C_2 such that $C_1 A \le B \le C_2 A$. For a Banach space X, we denote by $\|\cdot\|_X$ the norm in X. We also denote by $\chi(x)$ the function satisfying $\chi(x) = 1$ for $x \in \mathbb{R}^+$ and $\chi(x) = 0$ for $x \notin \mathbb{R}^+$.

The rest of this paper is organized as follows. In section 2 we prove some smoothing effects for the linear Boussinesq equation with inhomogeneous initial-boundary data. The existence and the uniqueness of the solution is given in Section 3.

2. Linear estimates

In this section we give some smoothing effects for the linear equation associated to (1.1). These estimates will be the main ingredient in the proof of local existence of the initial-boundary-value problem (1.1). Consideration is first directed to the linear initial-boundary-value problem

$$
\left\{\begin{array}{l} u_{tt} - u_{xx} + u_{xxxx} = 0, t > 0, x > 0, \\ u(0,t) = h_1(t), u_x(0,t) = h_2(t), \\ u(x,0) = 0, u_t(x,0) = 0. \end{array}\right. \tag{2.1}
$$

Denote by

$$
C_0(\mathbb{R}^+) = \{u \in C(\mathbb{R}) : u = 0 \text{ for } x \leq 0\}.
$$

Lemma 2.1. *Assume that $h_1, h_2 \in C_0(\mathbb{R}^+)$. Then the solution, $W_b(t)(h_1, h_2)$, of the linear problem* (2.1) *has an explicit formula*

$$
u(x,t) = W_b(t)(h_1,h_2) = U_1(x,t) + U_2(x,t) + \overline{U_1(x,t)} + \overline{U_2(x,t)},
$$

where

$$
\begin{aligned}
U_1(x,t) &= \frac{1}{2\pi}\int_1^{+\infty} \frac{2\mu^2-1}{\sqrt{\mu^2-1}(\mu+i\sqrt{\mu^2-1})} e^{i\mu\sqrt{\mu^2-1}t} e^{-\mu x} \\
&\quad \left(\int_0^{+\infty} \left(i\sqrt{\mu^2-1}h_1(\xi) - h_2(\xi)\right) e^{-i\mu\sqrt{\mu^2-1}\xi} d\xi\right) d\mu, \\
U_2(x,t) &= -\frac{1}{2\pi}\int_1^{+\infty} \frac{2\mu^2-1}{\sqrt{\mu^2-1}(\mu+i\sqrt{\mu^2-1})} e^{i\mu\sqrt{\mu^2-1}t} e^{i\sqrt{\mu^2-1}x} \\
&\quad \left(\int_0^{+\infty} (\mu h_1(\xi) + h_2(\xi))\, e^{-i\mu\sqrt{\mu^2-1}\xi} d\xi\right) d\mu.
\end{aligned}
$$

Proof. As a potential global solution of (2.1) is defined on a half-line $\mathbb{R}^+$ in each of the two independent variables x and t, it is not unnatural to think of replacing the use of the Fourier transform with the Laplace transform. By taking the Laplace transform with respect to t of both sides of the equation in (2.1), the initial-boundary-value problem is converted to a one-parameter family of four-order, boundary-value problems

$$
\left\{\begin{array}{l} \lambda^2\hat{u}(x,\lambda) - \hat{u}(x,\lambda)_{xx} + \hat{u}(x,\lambda)_{xxxx} = 0, \mathrm{Re}(\lambda) \geq 0, x > 0, \\ \hat{u}(0,\lambda) = \hat{h}_1(\lambda), \hat{u}_x(0,\lambda) = \hat{h}_2(\lambda), \hat{u}(+\infty,\lambda) = 0, \hat{u}_x(+\infty,\lambda) = 0, \end{array}\right. \tag{2.2}
$$

where λ is the dual variable, $\hat{u}(x,\lambda)$, $\hat{h}_1(\lambda)$ and $\hat{h}_2(\lambda)$ are the Laplace transform of $u(x,t)$, $h_1(t)$ and $h_2(t)$ with respect to t, respectively. Let γ_{1A}, γ_{2A}, γ_{3A} and

γ_{4A} be the four roots of the characteristic equation

$$\gamma^4 - \gamma^2 + \lambda^2 = 0,\ \lambda \in A = \{\lambda : \mathrm{Re}(\lambda) > 1/2\},$$

ordered so that $\mathrm{Re}(\gamma_{1A}) < 0$, $\mathrm{Re}(\gamma_{2A}) < 0$, $\mathrm{Re}(\gamma_{3A}) > 0$ and $\mathrm{Re}(\gamma_{4A}) > 0$. It is obvious that γ_{1A}, γ_{2A}, γ_{3A} and γ_{4A} are analytic for $\mathrm{Re}(\lambda) > 1/2$ and continuous for $\mathrm{Re}(\lambda) \geq 1/2$ except at $\lambda = 1/2$. As both $\hat{u}(x,\lambda)$ and $\hat{u}_x(x,\lambda)$ tend to zero as $x \to +\infty$, it is concluded that for any λ with $\mathrm{Re}(\lambda) > 1/2$

$$\hat{u}(x,\lambda) = \frac{1}{\gamma_{2A} - \gamma_{1A}} \left[(\gamma_{2A}\hat{h}_1(\lambda) - \hat{h}_2(\lambda))e^{\gamma_{1A}x} - (\gamma_{1A}\hat{h}_1(\lambda) - \hat{h}_2(\lambda))e^{\gamma_{2A}x}\right].$$

Thus, for any fixed p with $\mathrm{Re}(p) > 1/2$, one has the representation for $x > 0$ and $t > 0$,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} e^{\lambda t}\hat{u}(x,\lambda)d\lambda \\ &= \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} \frac{e^{\lambda t}}{\gamma_{2A} - \gamma_{1A}} \Big[(\gamma_{2A}\hat{h}_1(\lambda) - \hat{h}_2(\lambda))e^{\gamma_{1A}x} \\ &\qquad -(\gamma_{1A}\hat{h}_1(\lambda) - \hat{h}_2(\lambda))e^{\gamma_{2A}x}\Big] d\lambda. \end{aligned} \tag{2.3}$$

A little analysis shows that

$$\overline{\gamma_{1A}(\lambda)} = \gamma_{2A}(\bar{\lambda}), \overline{\gamma_{3A}(\lambda)} = \gamma_{4A}(\bar{\lambda}) \quad \text{for } \mathrm{Re}(\lambda) \geq 1/2,\ \lambda \neq 1/2,$$

and

$$|\gamma_{1A}(\lambda) - \gamma_{2A}(\lambda)| = O(|\lambda - 1/2|^{1/2}) \quad \text{as } \lambda \to 1/2,\ \mathrm{Re}(\lambda) \geq 1/2.$$

As this singularity is integrable, we may let $p \to 1/2$ in (2.3), thereby obtaining for $x > 0$ and $t > 0$,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{e^{\lambda t}}{\gamma_{2A} - \gamma_{1A}} \Big[(\gamma_{2A}\hat{h}_1(\lambda) - \hat{h}_2(\lambda))e^{\gamma_{1A}x} \\ &\qquad -(\gamma_{1A}\hat{h}_1(\lambda) - \hat{h}_2(\lambda))e^{\gamma_{2A}x}\Big] d\lambda. \end{aligned} \tag{2.4}$$

Let γ_{1B}, γ_{2B}, γ_{3B} and γ_{4B} be the four roots of the characteristic equation

$$\gamma^4 - \gamma^2 + \lambda^2 = 0,\ \lambda \in B = \{\lambda : 0 < \mathrm{Re}(\lambda) < 1/2\},$$

ordered so that $\mathrm{Re}(\gamma_{1B}) < 0$, $\mathrm{Re}(\gamma_{2B}) < 0$, $\mathrm{Re}(\gamma_{3B}) > 0$ and $\mathrm{Re}(\gamma_{4B}) > 0$. It is obvious that γ_{1B}, γ_{2B}, γ_{3B} and γ_{4B} are analytic for $\lambda \in B$ and continuous for $0 \leq \mathrm{Re}(\lambda) \leq 1/2$ except at $\lambda = 1/2$ and $\lambda = 0$. By the uniqueness and the continuity of the root of the characteristic equation $\gamma^4 - \gamma^2 + \lambda^2 = 0$ on the half lines $\Gamma_+ = \{\lambda : \mathrm{Re}(\lambda) = 1/2, \mathrm{Im}(\lambda) > 0\}$ and $\Gamma_- = \{\lambda : \mathrm{Re}(\lambda) = 1/2, \mathrm{Im}(\lambda) < 0\}$, we assume, without loss of generality,

$$\gamma_{1B} = \gamma_{1A}, \gamma_{2B} = \gamma_{2A}, \lambda \in \Gamma_+.$$

Then we have

$$\gamma_{1B} = \gamma_{1A}, \gamma_{2B} = \gamma_{2A}, \lambda \in \Gamma_-,$$

or

$$\gamma_{1B} = \gamma_{2A}, \gamma_{2B} = \gamma_{1A}, \lambda \in \Gamma_-.$$

By symmetry we deduce from (2.4) that

$$(2.5) \qquad u(x,t) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{e^{\lambda t}}{\gamma_{2B}-\gamma_{1B}}\Big[(\gamma_{2B}\hat{h}_1(\lambda)-\hat{h}_2(\lambda))e^{\gamma_{1B}x} - (\gamma_{1B}\hat{h}_1(\lambda)-\hat{h}_2(\lambda))e^{\gamma_{2B}x}\Big]\,d\lambda.$$

A little analysis shows that

$$\overline{\gamma_{1B}(\lambda)} = \gamma_{1B}(\bar{\lambda}), \overline{\gamma_{2B}(\lambda)} = \gamma_{2B}(\bar{\lambda}) \text{ for } 0 < \mathrm{Re}(\lambda) < 1/2$$

$$|\gamma_{1B}(\lambda) - \gamma_{2B}(\lambda)| = O(|\lambda - 1/2|^{1/2}), \text{ as } \lambda \to 1/2,\ 0 < \mathrm{Re}(\lambda) \le 1/2,$$

$$|\gamma_{1B}(\lambda) - \gamma_{2B}(\lambda)| = O(1), \text{ as } \lambda \to 0,\ 0 \le \mathrm{Re}(\lambda) < 1/2,$$

and for $\lambda \in B, |\lambda| \to +\infty$,

$$|\gamma_{1B}(\lambda)| \sim |\lambda|^{1/2}, \quad |\gamma_{2B}(\lambda)| \sim |\lambda|^{1/2},$$

$$\mathrm{Re}(\gamma_{1B}(\lambda)) \sim -|\lambda|^{1/2}, \quad \mathrm{Re}(\gamma_{2B}(\lambda)) \sim -|\lambda|^{1/2}.$$

Hence, we are allowed to change the contour on the basis of Cauchy's Theorem and thereby determine from (2.5) that

$$(2.6) \qquad u(x,t) = \frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty} \frac{e^{\lambda t}}{\gamma_{2B}-\gamma_{1B}}\Big[(\gamma_{2B}\hat{h}_1(\lambda)-\hat{h}_2(\lambda))e^{\gamma_{1B}x} - (\gamma_{1B}\hat{h}_1(\lambda)-\hat{h}_2(\lambda))e^{\gamma_{2B}x}\Big]\,d\lambda.$$

Denote by

$$U_1(x,t) = \frac{1}{2\pi i}\int_{0+i0}^{0+i\infty} \frac{e^{\lambda t}}{\gamma_{2B}-\gamma_{1B}}(\gamma_{2B}\hat{h}_1(\lambda)-\hat{h}_2(\lambda))e^{\gamma_{1B}x}d\lambda,$$

$$U_2(x,t) = -\frac{1}{2\pi i}\int_{0+i0}^{0+i\infty} \frac{e^{\lambda t}}{\gamma_{2B}-\gamma_{1B}}(\gamma_{1B}\hat{h}_1(\lambda)-\hat{h}_2(\lambda))e^{\gamma_{2B}x}d\lambda.$$

Note that

$$\overline{\gamma_{1B}(\lambda)} = \gamma_{1B}(\bar{\lambda}), \overline{\gamma_{2B}(\lambda)} = \gamma_{2B}(\bar{\lambda}) \text{ for } 0 \le \mathrm{Re}(\lambda) \le 1/2 \text{ with } \lambda \ne 0, \lambda \ne \frac{1}{2}$$

and

$$\overline{\hat{h}_1(\lambda)} = \hat{h}_1(\bar{\lambda}), \overline{\hat{h}_2(\lambda)} = \hat{h}_2(\bar{\lambda}) \text{ for } 0 \le \mathrm{Re}(\lambda) \le 1/2 \text{ with } \lambda \ne 0, \lambda \ne \frac{1}{2}.$$

By direct computation, it follows that for $x > 0$ and $t > 0$

$$\frac{1}{2\pi i}\int_{0-i\infty}^{0+i0} \frac{e^{\lambda t}}{\gamma_{2B}-\gamma_{1B}}(\gamma_{2B}\hat{h}_1(\lambda)-\hat{h}_2(\lambda))e^{\gamma_{1B}x}d\lambda = \overline{U_1(x,t)}$$

and

$$-\frac{1}{2\pi i}\int_{0-i\infty}^{0+i0} \frac{e^{\lambda t}}{\gamma_{2B}-\gamma_{1B}}(\gamma_{1B}\hat{h}_1(\lambda)-\hat{h}_2(\lambda))e^{\gamma_{2B}x}d\lambda = \overline{U_2(x,t)}.$$

Then we get for $x > 0$ and $t > 0$

$$u(x,t) = W_b(t)(h_1,h_2) = U_1(x,t) + U_2(x,t) + \overline{U_1(x,t)} + \overline{U_2(x,t)}. \tag{2.7}$$

For $\lambda \in \{\lambda : Re(\lambda) = 0,\, Im(\lambda) > 0\}$, it is convenient to make the change of variables

$$\lambda = i\mu\sqrt{\mu^2-1} \text{ for } \mu \geq 1$$

in the characteristic equation $\gamma^4 - \gamma^2 + \lambda^2 = 0$. In terms of μ, the four roots are

$$\gamma_{1B} = -\mu,\ \gamma_{2B} = i\sqrt{\mu^2-1},\ \gamma_{3B} = \mu \text{ and } \gamma_{4B} = -i\sqrt{\mu^2-1},$$

and the integral $U_1(x,t)$ and $U_2(x,t)$ may be rewritten as

$$\begin{aligned} U_1(x,t) &= \frac{1}{2\pi}\int_1^{+\infty} \frac{2\mu^2-1}{\sqrt{\mu^2-1}(\mu+i\sqrt{\mu^2-1})} e^{i\mu\sqrt{\mu^2-1}t}e^{-\mu x} \\ &\quad \left(\int_0^{+\infty}\left(i\sqrt{\mu^2-1}h_1(\xi) - h_2(\xi)\right)e^{-i\mu\sqrt{\mu^2-1}\xi}d\xi\right)d\mu, \end{aligned}$$

$$\begin{aligned} U_2(x,t) &= -\frac{1}{2\pi}\int_1^{+\infty} \frac{2\mu^2-1}{\sqrt{\mu^2-1}(\mu+i\sqrt{\mu^2-1})} e^{i\mu\sqrt{\mu^2-1}t}e^{i\sqrt{\mu^2-1}x} \\ &\quad \left(\int_0^{+\infty}(\mu h_1(\xi) + h_2(\xi))\,e^{-i\mu\sqrt{\mu^2-1}\xi}d\xi\right)d\mu. \end{aligned}$$

We complete the proof. □

For $\alpha, \beta \in \mathbb{R}$ we define

$$\mathcal{H}^{\alpha,\beta}(\mathbb{R}) = \{f : |\xi|^\alpha(1+|\xi|)^{\beta-\alpha}\hat{f}(\xi) \in L^2(\mathbb{R})\}$$

with

$$||f||_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})} = \left\| |\xi|^\alpha(1+|\xi|)^{\beta-\alpha}\hat{f}(\xi)\right\|_{L^2(\mathbb{R})},$$

where $\hat{f}$ is the Fourier transform in x of the function $f(x)$. Let

$$\mathcal{H}_0^{\alpha,\beta}(\mathbb{R}^+) = \{f \in \mathcal{H}^{\alpha,\beta}(\mathbb{R}) : \mathrm{supp} f \subseteq [0,\infty)\}$$

with

$$||f||_{\mathcal{H}_0^{\alpha,\beta}(\mathbb{R}+)} = ||f||_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})},$$

and let

$$\mathcal{H}^{\alpha,\beta}(\mathbb{R}^+) = \{f = F|_{\mathbb{R}^+},\ F \in \mathcal{H}^{\alpha,\beta}(\mathbb{R})\}$$

with

$$||f||_{\mathcal{H}^{\alpha,\beta}(\mathbb{R}+)} = \inf\{||F||_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})},\ f = F|_{\mathbb{R}+}\}.$$

Denote by

$$H_0^\beta(\mathbb{R}^+) = \mathcal{H}_0^{0,\beta}(\mathbb{R}^+),\ H^\beta(\mathbb{R}^+) = \mathcal{H}^{0,\beta}(\mathbb{R}^+)$$

and

$$\dot{H}_0^\alpha(\mathbb{R}^+) = \mathcal{H}_0^{\alpha,0}(\mathbb{R}^+),\ \dot{H}^\alpha(\mathbb{R}^+) = \mathcal{H}^{\alpha,0}(\mathbb{R}^+).$$

For any α and β with $\alpha \le 0$ and $\alpha \le \beta$, we have

$$\|f\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})} \sim \|f\|_{\dot{H}^{\alpha}(\mathbb{R})} + \|f\|_{H^{\beta}(\mathbb{R})} \tag{2.8}$$

because of

$$|\xi|^{\alpha}(1+|\xi|)^{\beta-\alpha} \sim |\xi|^{\alpha} + (1+|\xi|)^{\beta}.$$

Lemma 2.2. *Assume α, β and f satisfy one of the following conditions:*

- $\alpha \in (-\frac{1}{2}, 0]$ *and* $\beta \in (-\frac{1}{2}, \frac{1}{2})$ *with* $\alpha \le \beta$, *and* $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{R})$;
- $\alpha \in (-\frac{1}{2}, 0]$ *and* $\beta \in (\frac{1}{2}, \frac{3}{2})$, $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{R})$ *and* $f(0) = 0$.

Then $\chi(x)f(x) \in \mathcal{H}_0^{\alpha,\beta}(\mathbb{R}^+)$, *and*

$$\|\chi(x)f(x)\|_{\mathcal{H}_0^{\alpha,\beta}(\mathbb{R}^+)} \le C\|f\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})}.$$

Proof. For $0 \le \beta < \frac{1}{2}$ and $f_1(x) \in H^{\beta}(\mathbb{R})$, or for $\frac{1}{2} < \beta < \frac{3}{2}$ and $f_1(x) \in H^{\beta}(\mathbb{R})$ with $f(0) = 0$, that $\chi(x)f_1(x) \in H_0^{\beta}(\mathbb{R}^+)$ and

$$\|\chi(x)f_1(x)\|_{H_0^{\beta}(\mathbb{R}^+)} \le C\|f_1(x)\|_{H^{\beta}(\mathbb{R})} \tag{2.9}$$

comes from Lemma 2.3 and Proposition 2.4 in [4].

Note that $H_0^{\beta}(\mathbb{R}^+)$ is the dual space of $H^{-\beta}(\mathbb{R}^+)$ for $\beta < 0$ (see Proposition 2.1 in [4]). For any $g(x) \in H^{-\beta}(\mathbb{R}^+)$, we can choose $F \in H^{-\beta}(\mathbb{R})$ with $g = F|_{R^+}$ and $\|g\|_{H^{-\beta}(\mathbb{R}^+)} \sim \|F\|_{H^{-\beta}(\mathbb{R})}$. Thus, for $-\frac{1}{2} < \beta < 0$ and $f_1(x) \in H^{\beta}(\mathbb{R})$ we have the following chain of inequalities

$$\begin{aligned}
&|\langle \chi(x)f_1(x), g(x)\rangle| \le |\langle f_1(x), \chi(x)F(x)\rangle| \\
\le\ & \|f_1(x)\|_{H^{\beta}(\mathbb{R})}\|\chi(x)F(x)\|_{H_0^{-\beta}(\mathbb{R})} \le C\|f_1(x)\|_{H^{\beta}(\mathbb{R})}\|\chi(x)F(x)\|_{H_0^{-\beta}(\mathbb{R}^+)} \\
\le\ & C\|f_1(x)\|_{H^{\beta}(\mathbb{R})}\|F(x)\|_{H^{-\beta}(\mathbb{R})} \le C\|f_1(x)\|_{H^{\beta}(\mathbb{R})}\|g(x)\|_{H^{-\beta}(\mathbb{R}^+)},
\end{aligned}$$

the last two inequalities hold because of Lemma 2.3 in [4]. Then, $\chi(x)f_1(x) \in H_0^{\beta}(\mathbb{R}^+)$ and

$$\|\chi(x)f_1(x)\|_{H_0^{\beta}(\mathbb{R}^+)} \le C\|f_1(x)\|_{H^{\beta}(\mathbb{R})}. \tag{2.10}$$

For $-\frac{1}{2} < \alpha \le 0$ and $f_2 \in \dot{H}^{\alpha}(\mathbb{R})$, we have

$$\|f_2\|_{H^{\alpha}(\mathbb{R})} = \|(1+|\xi|)^{\alpha}\hat{f}_2(\xi)\|_{L_\xi^2(\mathbb{R})} \le \|\,|\xi|^{\alpha}\hat{f}_2(\xi)\|_{L_\xi^2(\mathbb{R})} = \|f_2\|_{\dot{H}^{\alpha}(\mathbb{R})},$$

where $\hat{f}_2(\xi)$ is the Fourier transform with respective to x of the function $f_2(x)$. The similar argument to that used in (2.10) yields $\chi(x)f_2(x) \in H_0^{\alpha}(\mathbb{R}^+)$ and

$$\|\chi(x)f_2(x)\|_{H_0^{\alpha}(\mathbb{R}^+)} \le C\|f_2\|_{H^{\alpha}(\mathbb{R})} \le C\|f_2\|_{\dot{H}^{\alpha}(\mathbb{R})},$$

which, together with Proposition 2.8 in [4], implies $\chi(x)f_2(x) \in \dot{H}_0^{\alpha}(\mathbb{R}^+)$ and

$$\|\chi(x)f_2(x)\|_{\dot{H}_0^{\alpha}(\mathbb{R}^+)} \le C\|\chi(x)f_2(x)\|_{H_0{}^{\alpha}(\mathbb{R}^+)} \le C\|f_2\|_{\dot{H}^{\alpha}(\mathbb{R})}. \tag{2.11}$$

Then the lemma follows from a combination of (2.9), (2.10) and (2.11) with (2.8). The proof is completed. $\square$

Lemma 2.3. *There exists a positive constant C such that, for $h_1 \in H_0^{\frac{1}{4}}(\mathbb{R}^+)$ and $h_2 \in \mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)$,*

$$\sup_{t\geq 0} \|W_b(t)(h_1,h_2)\|_{L_x^2(\mathbb{R}^+)} \leq C\left(\|h_1\|_{H_0^{\frac{1}{4}}(\mathbb{R}^+)} + \|h_2\|_{\mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)}\right).$$

Proof. It suffices to prove the lemma for $h_1, h_2 \in C_0^\infty(\mathbb{R}^+)$. It follows from Lemma 3.1 in [3] that

$$\begin{aligned}
&\|U_1(x,t)\|_{L_x^2(\mathbb{R}^+)} \\
&\leq C\left\| \frac{\mu - i\sqrt{\mu^2-1}}{\sqrt{\mu^2-1}} e^{i\mu\sqrt{\mu^2-1}t} \right. \\
&\qquad \left. \times \left(\int_0^{+\infty} (i\sqrt{\mu^2-1}h_1(\xi) - h_2(\xi)) e^{-i\mu\sqrt{\mu^2-1}\xi} d\xi\right)\right\|_{L_\mu^2(1,+\infty)} \\
&\leq C\left\|\sqrt{2\mu^2-1}\int_0^{+\infty} h_1(\xi)e^{-i\mu\sqrt{\mu^2-1}\xi}d\xi\right\|_{L_\mu^2(1,+\infty)} \\
&\quad + C\left\|\frac{\sqrt{2\mu^2-1}}{\sqrt{\mu^2-1}}\int_0^{+\infty} h_2(\xi)e^{-i\mu\sqrt{\mu^2-1}\xi}d\xi\right\|_{L_\mu^2(1,+\infty)} \\
&\leq C\left\|(1+\eta)^{\frac{1}{4}}\int_0^{+\infty} h_1(\xi)e^{-i\eta\xi}d\xi\right\|_{L_\eta^2(0,+\infty)} \\
&\quad + C\left\|\eta^{-\frac{1}{2}}(1+\eta)^{-\frac{3}{4}}\int_0^{+\infty} h_2(\xi)e^{-i\eta\xi}d\xi\right\|_{L_\eta^2(0,+\infty)} \\
&\leq C\left(\|h_1\|_{H_0^{\frac{1}{4}}(\mathbb{R}^+)} + \|h_2\|_{\mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)}\right).
\end{aligned} \tag{2.12}$$

Note that for

$$U(x) = \int_1^\infty g(\mu)e^{i\sqrt{\mu^2-1}x}d\mu = \int_0^\infty g(\mu(s))\frac{s}{\mu(s)}e^{isx}ds$$

where $\mu(s) \geq 0$ is the solution satisfying $s = \sqrt{\mu^2-1}$ for $s \geq 0$, we have

$$\|U(x)\|_{L_x^2(0,\infty)} = \|g(\mu(s))\frac{s}{\mu(s)}\|_{L_s^2(0,\infty)} = \|g(\mu)(\mu^2-1)^{\frac{1}{4}}\mu^{-\frac{1}{2}}\|_{L_\mu^2(1,\infty)}.$$

Then we have

$$\begin{aligned}
&\|U_2(x,t)\|_{L_x^2(\mathbb{R}^+)} \\
&\leq C\left\|\mu^{\frac{1}{2}}\sqrt{2\mu^2-1}(\mu^2-1)^{-\frac{1}{4}}\int_0^{+\infty} h_1(\xi)e^{-i\mu\sqrt{\mu^2-1}\xi}d\xi\right\|_{L_\mu^2(1,+\infty)}
\end{aligned} \tag{2.13}$$

$$
\begin{aligned}
&+C\left\|\mu^{-\frac{1}{2}}\sqrt{2\mu^2-1}(\mu^2-1)^{-\frac{1}{4}}\int_0^{+\infty}h_2(\xi)e^{-i\mu\sqrt{\mu^2-1}\xi}d\xi\right\|_{L^2_\mu(1,+\infty)}\\
&\leq C\left\|(1+\eta)^{\frac{1}{4}}\int_0^{+\infty}h_1(\xi)e^{-i\eta\xi}d\xi\right\|_{L^2_\eta(0,+\infty)}\\
&\quad+C\left\|(1+\eta)^{-\frac{1}{4}}\int_0^{+\infty}h_2(\xi)e^{-i\eta\xi}d\xi\right\|_{L^2_\eta(0,+\infty)}\\
&\leq C\|h_1\|_{H_0^{\frac{1}{4}}(\mathbb{R}^+)}+C\|h_2\|_{H_0^{-\frac{1}{4}}(\mathbb{R}^+)}.
\end{aligned}
$$

The lemma now follows from (2.12) and (2.13) together with Lemma 2.1. We complete the proof. □

Lemma 2.4. *There exists a positive constant C such that, for $h_1 \in H_0^{\frac{1}{4}}(\mathbb{R}^+)$ and $h_2 \in H_0^{-\frac{1}{4}}(\mathbb{R}^+)$,*

$$
\left(\int_0^{\infty}\|W_b(t)(h_1,h_2)\|^4_{L^\infty_x(\mathbb{R}^+)}\,dt\right)^{\frac{1}{4}} \leq C\left(\|h_1\|_{H_0^{\frac{1}{4}}(\mathbb{R}^+)}+\|h_2\|_{H_0^{-\frac{1}{4}}(\mathbb{R}^+)}\right).
$$

Proof. For given real functions $K \in C(\mathbb{R})$ and $h \in C_0(\mathbb{R}^+)$ satisfying $K(-\mu) = \overline{K(\mu)}$, we define

$$
G(K,h)=\int_1^{+\infty}K(\mu)e^{-i\mu\sqrt{\mu^2-1}t}e^{-|\mu|x}\left(\int_0^{+\infty}h(\xi)e^{i\mu\sqrt{\mu^2-1}\xi}d\xi\right)d\mu
$$

and

$$
H(K,h)=\int_1^{+\infty}K(\mu)e^{-i\mu\sqrt{\mu^2-1}t}e^{i\mu\sqrt{1-(1/\mu)^2}x}\left(\int_0^{+\infty}h(\xi)e^{i\mu\sqrt{\mu^2-1}\xi}d\xi\right)d\mu.
$$

Then

$$
\begin{aligned}
&G(K,h)+\overline{G(K,h)}\\
&=\int_{|\mu|>1}K(\mu)e^{-i\mu\sqrt{\mu^2-1}t}e^{-|\mu|x}\left(\int_0^{+\infty}h(\xi)e^{i\mu\sqrt{\mu^2-1}\xi}d\xi\right)d\mu\\
&=\int_{-\infty}^{+\infty}K(\varphi^{-1}(\eta))\frac{d\varphi^{-1}(\eta)}{d\eta}e^{i\eta t}e^{-|\varphi^{-1}(\eta)|x}\left(\int_0^{+\infty}h(\xi)e^{-i\eta\xi}d\xi\right)d\eta,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&H(K,h)+\overline{H(K,h)}\\
&=\int_{-\infty}^{+\infty}K(\varphi^{-1}(\eta))\frac{d\varphi^{-1}(\eta)}{d\eta}e^{i\eta t}e^{-i\varphi^{-1}(\eta)\sqrt{1-(1/\varphi^{-1}(\eta))^2}x}\left(\int_0^{+\infty}h(\xi)e^{-i\eta\xi}d\xi\right)d\eta,
\end{aligned}
$$

where, $\mu = \varphi^{-1}(\eta)$ is the inverse function of $\eta = \varphi(\mu) = \mu\sqrt{\mu^2-1}$. For any $g(x,t) \in L_t^{\frac{4}{3}}(\mathbb{R}, L^1_x(\mathbb{R}^+))$, denoting by

$$
\hat{g}(x,-\eta)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}g(x,t)e^{it\eta}dt,\ \hat{h}(\eta)=\int_0^{+\infty}h(\xi)e^{-i\eta\xi}d\xi,
$$

we deduce by Hölder's inequality and the Sobolev embedding theorem

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^{+\infty}[G(K,h)+\overline{G(K,h)}]g(x,t)dxdt \\
= \;& 2\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K(\varphi^{-1}(\eta))\frac{d\varphi^{-1}(\eta)}{d\eta}e^{-|\varphi^{-1}(\eta)|x}\hat{g}(x,-\eta)\hat{h}(\eta)d\eta dx \\
\leq \;& C\int_{-\infty}^{+\infty}\left|(1+|\eta|)^{\frac{1}{4}}K(\varphi^{-1}(\eta))\frac{d\varphi^{-1}(\eta)}{d\eta}\hat{h}(\eta)\right|\left\|\frac{\hat{g}(x,-\eta)}{(1+|\eta|)^{\frac{1}{4}}}\right\|_{L^1_x(\mathbb{R})}d\eta \\
\leq \;& C\left\|(1+|\eta|)^{\frac{1}{4}}K(\varphi^{-1}(\eta))\frac{d\varphi^{-1}(\eta)}{d\eta}\hat{h}(\eta)\right\|_{L^2_\eta(\mathbb{R})}\|g(x,t)\|_{H^{-\frac{1}{4}}(\mathbb{R},L^1_x(\mathbb{R}^+))} \\
\leq \;& C\left\|(1+|\eta|)^{\frac{1}{4}}K(\varphi^{-1}(\eta))\frac{d\varphi^{-1}(\eta)}{d\eta}\hat{h}(\eta)\right\|_{L^2_\eta(\mathbb{R})}\|g(x,t)\|_{L_t^{\frac{4}{3}}(\mathbb{R},L^1_x(\mathbb{R}^+))},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|G(K,h)+\overline{G(K,h)}\|_{L^4_t(\mathbb{R}^+,L^\infty_x(\mathbb{R}^+))} \\
\leq \;& \|G(K,h)+\overline{G(K,h)}\|_{L^4_t(\mathbb{R},L^\infty_x(\mathbb{R}^+))} \\
\leq \;& C\left\|(1+|\eta|)^{\frac{1}{4}}K(\varphi^{-1}(\eta))\frac{d\varphi^{-1}(\eta)}{d\eta}\hat{h}(\eta)\right\|_{L^2_\eta(\mathbb{R})}. \qquad (2.14)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\|H(K,h)+\overline{H(K,h)}\|_{L^4_t(\mathbb{R}^+,L^\infty_x(\mathbb{R}^+))} \\
&\leq C\left\|(1+|\eta|)^{\frac{1}{4}}K(\varphi^{-1}(\eta))\frac{d\varphi^{-1}(\eta)}{d\eta}\hat{h}(\eta)\right\|_{L^2_\eta(\mathbb{R})}. \qquad (2.15)
\end{aligned}
$$

Rewrite $U_1(x,t)$ and $U_2(x,t)$ as

$$U_1(x,t)=\frac{i}{2\pi}G(\mu,h_1)+\frac{1}{2\pi}G(\sqrt{\mu^2-1},h_1)-\frac{1}{2\pi}G(\frac{\mu}{\sqrt{\mu^2-1}},h_2)+\frac{i}{2\pi}G(1,h_2),$$

and

$$U_2(x,t)=-\frac{1}{2\pi}H(\frac{\mu^2}{\sqrt{\mu^2-1}},h_1)+\frac{i}{2\pi}H(\mu,h_1)-\frac{1}{2\pi}H(\frac{\mu}{\sqrt{\mu^2-1}},h_2)+\frac{i}{2\pi}H(1,h_2).$$

Then we get from (2.14) that

$$
\begin{aligned}
&\|U_1(x,t)+\overline{U_1(x,t)}\|_{L^4_t(\mathbb{R}^+,L^\infty_x(\mathbb{R}^+))} \\
\leq \;& C\left(\|\eta(1+|\eta|)^{-\frac{3}{4}}\hat{h}_1(\eta)\|_{L^2_\eta(\mathbb{R})}+\|\eta^2(1+|\eta|)^{-\frac{7}{4}}\hat{h}_1(\eta)\|_{L^2_\eta(\mathbb{R})}\right) \\
&+C\left(\|(1+|\eta|)^{-\frac{1}{4}}\hat{h}_2(\eta)\|_{L^2_\eta(\mathbb{R})}+\|\eta(1+|\eta|)^{-\frac{5}{4}}\hat{h}_2(\eta)\|_{L^2_\eta(\mathbb{R})}\right) \\
\leq \;& C\left(\|h_1\|_{H_0^{\frac{1}{4}}(\mathbb{R}^+)}+\|h_2\|_{H_0^{-\frac{1}{4}}(\mathbb{R}^+)}\right), \qquad (2.16)
\end{aligned}
$$

and from (2.15),

$$\|U_2(x,t)+\overline{U_2(x,t)}\|_{L_t^4(\mathbb{R}^+,L_x^\infty(\mathbb{R}^+))} \le C\left(\|h_1\|_{H_0^{\frac{1}{4}}(\mathbb{R}^+)} + \|h_2\|_{H_0^{-\frac{1}{4}}(\mathbb{R}^+)}\right). \tag{2.17}$$

Then the lemma follows from (2.16) and (2.17) together with Lemma 2.1. We complete the proof. □

Let $\phi(\xi) = \xi(1+|\xi|^2)^{1/2}$. For $f, h \in \mathcal{S}'(\mathbb{R})$ define

$$V(t)(f) = \int_{-\infty}^{+\infty} e^{it\phi(\xi)} e^{ix\xi} \hat{f}(\xi)d\xi. \tag{2.18}$$

$$V_1(t)(f)) = \frac{1}{2}\int_{-\infty}^{+\infty} [e^{it\phi(\xi)+ix\xi} + e^{-it\phi(\xi)+ix\xi}]\hat{f}(\xi)d\xi \tag{2.19}$$

and

$$V_2(t)(\partial_x h) = \int_{-\infty}^{+\infty} \left[A(\xi)e^{i(-t\phi(\xi)+x\xi)} + B(\xi)e^{i(t\phi(\xi)+x\xi)}\right] d\xi, \tag{2.20}$$

where $A(\xi) = \frac{\hat{h}(\xi)}{2(1+\xi^2)^{1/2}}$ and $B(\xi) = -\frac{\hat{h}(\xi)}{2(1+\xi^2)^{1/2}}$, $\hat{f}$ and $\hat{h}$ are the Fourier transform of f and h respectively. Denote by

$$W_R(t)(f, \partial_x h) = V_1(t)(f) + V_2(t)(\partial_x h).$$

Then $u(x,t) = W_R(t)(f, \partial_x h)$ is the formal solution to the initial-value problem

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0, t > 0, x \in \mathbb{R}, \\ u(x,0) = f(x), u_t(x,0) = \partial_x h. \end{cases} \tag{2.21}$$

Lemma 2.5. *Let $T > 0$. There exists a positive constant C such that, for $f \in L^2(\mathbb{R})$ and $h \in H_0^{-1}(\mathbb{R})$ with $h(x) = -h(-x)$,*

$$\|\chi(t)W_R(t)(f, \partial_x h)|_{x=0}\|_{H_0^{\frac{1}{4}}(\mathbb{R}^+)} \le C\left(\|f\|_{L^2(\mathbb{R})} + \|h\|_{H_0^{-1}(\mathbb{R})}\right),$$

$$\|\chi(t)\partial_x W_R(t)(f, \partial_x h)|_{x=0}\|_{\mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)} \le C\left(\|f\|_{L^2(\mathbb{R})} + \|h\|_{H_0^{-1}(\mathbb{R})}\right)$$

and

$$\left(\int_0^T \|W_R(\tau)(f, \partial_x h)\|_{L_x^\infty(\mathbb{R}^+)}^4 d\tau\right)^{\frac{1}{4}} \le C(1+T^{\frac{1}{4}})\left(\|f\|_{L^2(\mathbb{R})} + \|h\|_{H_0^{-1}(\mathbb{R})}\right).$$

Proof. It suffices to prove the lemma for $f, h \in C_0^\infty(\mathbb{R})$ with $f(0) = 0$ and $h(x) = -h(-x)$. The results in [2] show that $u(x,t) = W_R(t)(f, \partial_x h)$ is the classical solution of the initial-value problem

$$\begin{cases} \partial_{tt}u - \partial_{xx}u + \partial_{xxxx}u = 0, \ -\infty < x < +\infty, \ t > 0 \\ u(x,0) = f(x), \ \partial_t u(x,0) = \partial_x h(x), \ -\infty < x < +\infty, \end{cases}$$

and $W_R(t)(f, \partial_x h) \in C^2(\mathbb{R} \times \mathbb{R})$ with $W_R(t)(f, \partial_x h)|_{(x,t)=(0,0)} = f(0) = 0$.

For given $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \|V(t)(f)\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})} \\
&= \left\| |\tau|^{\alpha}(1+|\tau|)^{\beta-\alpha}\mathcal{F}_{t\to\tau}\left(V(t)(f)\right)(\tau)\right\|_{L^2_\tau(\mathbb{R})} \\
&\le C\left\| |\tau|^{\alpha}(1+|\tau|)^{\beta-\alpha}\hat{f}(\xi)\left|\phi'(\xi)\right|^{-1}|_{\xi=\phi^{-1}(\tau)}\right\|_{L^2_\tau(\mathbb{R})} \\
&\le C\left\| |\phi(\xi)|^{\alpha}(1+|\phi(\xi)|)^{\beta-\alpha}\hat{f}(\xi)\left|\phi'(\xi)\right|^{-\frac{1}{2}}\right\|_{L^2_\xi(\mathbb{R})} \\
&\le C\left\| |\xi|^{\alpha}(1+|\xi|)^{2\beta-\alpha-\frac{1}{2}}\hat{f}(\xi)\right\|_{L^2_\xi(\mathbb{R})} = C\left\|f\right\|_{\mathcal{H}_0^{\alpha,2\beta-\frac{1}{2}}(\mathbb{R})},
\end{aligned}$$

which, together with (2.19) and (2.20), implies

$$\sup_{x\in\mathbb{R}} \|W_R(t)(f,\partial_x h)\|_{H^{\frac{1}{4}}(\mathbb{R})} \le C\left(\|f\|_{L^2(\mathbb{R}^+)} + \|h\|_{H_0^{-1}(\mathbb{R}^+)}\right).$$

Then, $\chi(t)W_R(t)(f,\partial_x h)|_{x=0} \in H_0^{\frac{1}{4}}(\mathbb{R}^+)$ because of Lemma 2.2, and the first inequality in the lemma holds. To prove the second inequality in the lemma, we define

$$v(x,t) = \int_0^t \chi(\tau)\partial_x W_R(\tau)(f,\partial_x h)d\tau,\ (x,t)\in\mathbb{R}\times\mathbb{R}.$$

Obviously $v(0,t)\in C_0(\mathbb{R}^+)$, $\frac{dv(x,t)}{dt} = \chi(t)\partial_x W_R(t)(f,\partial_x h)$, and $v(x,t)$ solves the initial-value problem

$$\begin{cases} \partial_{tt}v - \partial_{xx}v + \partial_{xxxx}v = \partial_{xx}h(x),\ x\in\mathbb{R},\ t\in\mathbb{R}^+, \\ v(x,0) = 0,\ \partial_t v(x,0) = \partial_x f(x). \end{cases}$$

By Duhamel's principle we can rewrite $v(x,t)$ as

$$v(x,t) = V_2(t)(\partial_x f(x)) + \int_0^t V_2(t-\tau)(\partial_{xx}h(x))d\tau,\ t>0.$$

Thus, for $t\ge 0$,

$$\begin{aligned}
&v(0,t) \\
&= V_2(t)(\partial_x f)|_{x=0} + \int_0^t V_2(t-\tau)(\partial_{xx}h)|_{x=0}d\tau \\
&= V_2(t)(\partial_x f)|_{x=0} + \int_0^t\int_{-\infty}^{+\infty}\frac{i\xi sgn(\xi)\hat{h}(\xi)}{2(1+\xi^2)^{\frac{1}{2}}} \\
&\qquad\qquad \times\left(e^{-it\phi(\xi)+i\tau\phi(\xi)} - e^{it\phi(\xi)-i\tau\phi(\xi)}\right)d\xi d\tau
\end{aligned}$$

$$= V_2(t)(\partial_x f)|_{x=0} + \int_{-\infty}^{+\infty} \frac{\hat{h}(\xi)}{2(1+\xi^2)} \left(2 - e^{it\phi(\xi)} - e^{-it\phi(\xi)}\right) d\xi$$

$$= V_2(t)(\partial_x f)|_{x=0} + \int_{-\infty}^{+\infty} \frac{\hat{h}(\xi)}{1+\xi^2} d\xi - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\hat{h}(\xi)}{1+\xi^2} \left(e^{it\phi(\xi)} + e^{-it\phi(\xi)}\right) d\xi$$

$$= V_2(t)(\partial_x f)|_{x=0} - \frac{1}{2}V(t)(g)|_{x=0} - \frac{1}{2}V(-t)(g)|_{x=0},$$

with $g(x) = \int_{-\infty}^{+\infty} \frac{\hat{h}(\xi)}{1+\xi^2} e^{ix\xi} d\xi$, the last inequality holds because of $\hat{h}(-\xi) = \hat{h}(-\xi)$. Denote by

$$J(t) = V_2(t)(\partial_x f)|_{x=0} - \frac{1}{2}V(t)(g)|_{x=0} - \frac{1}{2}V(-t)(g)|_{x=0}.$$

It follows from the similar argument to that used in the proof of the first inequality in the lemma that $J(t) \in H^{\frac{3}{4}}(\mathbb{R})$ and

$$\|J(t)\|_{H^{\frac{3}{4}}(\mathbb{R})} \leq C \left(\|f\|_{L^2(\mathbb{R})} + \|h\|_{H_0^{-1}(\mathbb{R})}\right).$$

Note that $\chi(t)J(t) = v(0,t)$ and $J(0) = v(0,0) = 0$. Lemma 2.2 implies $v(0,t) = \chi(t)J(t) \in H_0^{\frac{3}{4}}(\mathbb{R}^+)$ and

$$\|v(0,t)\|_{H_0^{\frac{3}{4}}(\mathbb{R}^+)} \leq C \left(\|f\|_{L^2(\mathbb{R})} + \|h\|_{H_0^{-1}(\mathbb{R})}\right). \tag{2.22}$$

Then

$$\|\chi(t)\partial_x W_R(t)(f, \partial_x h)|_{x=0}\|_{\mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)} = \left\|\frac{dv(0,t)}{dt}\right\|_{\mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)}$$

$$\leq \|v(0,t)\|_{H_0^{\frac{3}{4}}(\mathbb{R}^+)} \leq C \left(\|f\|_{L^2(\mathbb{R})} + \|h\|_{H_0^{-1}(\mathbb{R})}\right).$$

The second inequality in the lemma is proved. The last inequality in the lemma comes from the results in [8], the proof is omitted. □

For any $f \in L^2(\mathbb{R}^+)$ and $h \in H_0^{-1}(\mathbb{R}^+)$ we define $(\tilde{f}(x), \tilde{h}(x)) = (f(x), h(x))$ for $x \geq 0$ and $(\tilde{f}(x), \tilde{h}(x)) = (0, -h(-x))$ for $x < 0$, and define

$$W_C(t)(f, \partial_x h) = W_R(t)(\tilde{f}, \partial_x \tilde{h}) - W_b(\chi(t)g_1(t), \chi(t)g_2(t)),$$

where

$$g_1(t) = W_R(t)(\tilde{f}, \partial_x \tilde{h})|_{x=0},\ g_2(t) = \partial_x W_R(t)(\tilde{f}, \partial_x \tilde{h})|_{x=0}$$

are the trace of $W_R(t)(\tilde{f}, \partial_x \tilde{h})$ and $\partial_x W_R(t)(\tilde{f}, \partial_x \tilde{h})$ at $x = 0$, respectively. Lemma 2.3 and Lemma 2.5 mean $W_C(t)(f, \partial_x h)$ is well-defined and solves the initial-boundary-value problem

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0, t > 0,\ x > 0, \\ u(0,t) = 0, u_x(0,t) = 0, \\ u(x,0) = f(x), u_t(x,0) = \partial_x h(x). \end{cases} \tag{2.23}$$

The following lemma comes from a combination of Lemma 2.3 and Lemma 2.4 with Lemma 2.5.

Lemma 2.6. *There exists a positive constant C such that, for $f \in L^2(\mathbb{R}^+)$ and $h \in H_0^{-1}(\mathbb{R}^+)$,*

$$\sup_{t\geq 0} \|W_C(t)(f,\partial_x h)\|_{L_x^2(\mathbb{R}^+)} \leq C\left(\|f\|_{L^2(\mathbb{R}^+)} + \|h\|_{H_0^{-1}(\mathbb{R}^+)}\right),$$

$$\left(\int_0^T \|W_C(t)(f,\partial_x h)dt\|^4_{L_x^\infty(\mathbb{R}^+)}\right)^{\frac{1}{4}} \leq C(1+T^{\frac{1}{4}})\left(\|f\|_{L^2(\mathbb{R}^+)} + \|h\|_{H_0^{-1}(\mathbb{R}^+)}\right).$$

Lemma 2.7. *Let $T>0$ and let $g(x,t)\in L^1([0,T],L_x^2(\mathbb{R}^+))$. Then the linear initial-boundary-value problem*

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = \partial_{xx} g, t>0,\ x>0, \\ u(0,t)=0, u_x(0,t)=0, \\ u(x,0)=0, u_t(x,0)=0, \end{cases} \tag{2.24}$$

has a solution $W_I(t)(g)\in C([0,T],L_x^2(\mathbb{R}^+))\cap L_t^4([0,T],L_x^\infty(\mathbb{R}^+))$ such that

$$\sup_{0\leq t\leq T} \|W_I(t)(g)\|_{L_x^2(\mathbb{R}^+)} \leq C\|g\|_{L_t^1([0,T],L_x^2(\mathbb{R}^+))}$$

and

$$\left(\int_0^T \|W_I(t)(g)\|^4_{L_x^\infty(\mathbb{R}^+)}\,dt\right)^{\frac{1}{4}} \leq C(1+T^{\frac{1}{4}})\|g\|_{L_t^1([0,T],L_x^2(\mathbb{R}^+))}.$$

Proof. Choose $g_n(x,t)\in C([0,T],C_0^\infty(\mathbb{R}^+))$ such that $g_n\to g$ in $L^1([0,T], L_x^2(\mathbb{R}^+))$ as $n\to+\infty$. It follows from Duhamel's principle that

$$u_n(x,t) = \int_0^t W_C(t-\tau)(0,\partial_{xx} g_n(x,\tau))d\tau$$

is the solution to the initial-boundary-value problem (2.24) replacing g by g_n. By Lemma 2.6,

$$\sup_{0\leq t\leq T}\|u_n\|_{L_x^2(\mathbb{R}^+)} \leq \int_0^T \|g_n(x,\tau)\|_{L_x^2(\mathbb{R}^+)}d\tau, \tag{2.25}$$

and

$$\left(\int_0^T \|u_n\|^4_{L_x^\infty(\mathbb{R}^+)}\,dt\right)^{\frac{1}{4}} \leq C(1+T^{\frac{1}{4}})\|g\|_{L_t^1([0,T],L_x^2(\mathbb{R}^+))} \tag{2.26}$$

and

$$\sup_{0\leq t\leq T}\|u_n-u_m\|_{L_x^2(\mathbb{R}^+)} + \left(\int_0^T \|u_n-u_m\|^4_{L_x^\infty(\mathbb{R}^+)}\,dt\right)^{\frac{1}{4}} \tag{2.27}$$

$$\leq C(1+T^{\frac{1}{4}})\|g_n-g_m\|_{L_t^1([0,T],L_x^2(\mathbb{R}^+))}.$$

Then $u_n(x,t)$ is a convergent sequence in

$$C([0,T],L_x^2(\mathbb{R}^+))\cap L_t^4([0,T],L_x^\infty(\mathbb{R}^+))$$

because of (2.27). Let $W_I(t)(g) = \lim_{n\to+\infty} u_n(x,t)$. Obviously $W_I(t)(g)$ is a solution of (2.24). Taking $n \to +\infty$ in (2.25) and (2.26) we obtain $W_I(t)(g)$ satisfies the estimates in the lemma. □

3. The local existence of solutions

In this section we give the proof of Theorem 1.1. To prove the local existence we are going to use a contraction mapping argument. For some positive constant $\delta > 0$ and $T > 0$ determined below, we define the set $\mathcal{Z}_T^\delta$ by

$$\mathcal{Z}_T^\delta = \left\{ \begin{array}{c} u \in C([0,T]; L_x^2(\mathbb{R}^+)) \cap L_t^4([0,T]; L_x^\infty(\mathbb{R}^+)) \text{ with} \\ \sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^2(\mathbb{R}^+)} + \|u\|_{L_t^4([0,T];L_x^\infty(\mathbb{R}^+))} \le \delta \end{array} \right\}.$$

Let $\mathcal{X}$ be the product space

$$L^2(\mathbb{R}^+) \times H_0^{-1}(\mathbb{R}^+) \times H_0^{\frac{1}{4}}(\mathbb{R}^+) \times \mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)$$

with the norm

$$\begin{aligned} &\|(f,h,h_1,h_2)\|_{\mathcal{X}} \\ &= \|f(x)\|_{L^2(\mathbb{R}^+)} + \|h(x)\|_{H_0^{-1}(\mathbb{R}^+)} + \|h_1(t)\|_{H_0^{\frac{1}{4}}(\mathbb{R}^+)} + \|h_2(t)\|_{\mathcal{H}_0^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}^+)}. \end{aligned}$$

For $(f,h,h_1,h_2) \in \mathcal{X}$ and $u \in \mathcal{Z}_T^\delta$ we define a mapping $\Phi(u)$ by

$$\Phi(u) = W_C(t)(f,\partial_x h) + W_b(t)(h_1,h_2) - W_I(t)((|u|^\kappa u(\tau))_{xx})d\tau.$$

We only need to prove that $\Phi(u)$ is a contraction mapping from $\mathcal{Z}_T^\delta$ into $\mathcal{Z}_T^\delta$ for suitable $\delta > 0$ and $T > 0$. Denote by $\delta_0 = \|(f,h,h_1,h_2)\|_{\mathcal{X}}$ and denote by

$$|||u|||_T = \sup_{0\le t\le T} \|u\|_{L_x^2(\mathbb{R}^+)} + \left(\int_0^T \|u(x,t)\|_{L_x^\infty(\mathbb{R}^+)}^4 dt\right)^{\frac{1}{4}}.$$

For $u, v \in \mathcal{Z}_T^\delta$, it follows from Lemma 2.3, Lemma 2.4, Lemma 2.6 and Lemma 2.7 that

$$\begin{aligned} &|||\Phi(u)|||_T \\ &\le C(1+T^{\frac{1}{4}})\left(\delta_0 + \int_0^T \left\| |u(\tau)|^{\kappa+1} \right\|_{L_x^2(\mathbb{R}^+)} d\tau\right) \\ &\le C(1+T^{\frac{1}{4}})\left(\delta_0 + T^{\frac{4-\kappa}{4}} \sup_{t\in[0,T]} \|u\|_{L_x^2(\mathbb{R}^+)} |||u|||_T^\kappa\right) \qquad (3.1) \\ &\le C(1+T^{\frac{1}{4}})\left(\delta_0 + T^{\frac{4-\kappa}{4}}\delta^{\kappa+1}\right), \end{aligned}$$

and

(3.2)

$$
\begin{aligned}
&|||\Phi(u) - \Phi(v)|||_T \\
&\leq C(1+T^{\frac{1}{4}}) \int_0^T |||u(\tau)|^\kappa u(\tau) - |v(\tau)|^\kappa v(\tau)||_{L_x^2(\mathbb{R}^+)}\, d\tau \\
&\leq C(1+T^{\frac{1}{4}}) \int_0^T \left(||u(\tau)||^\kappa_{L_x^\infty(\mathbb{R}^+)} + ||v(\tau)||^\kappa_{L_x^\infty(\mathbb{R}^+)} \right) ||u(\tau) - v(\tau)||_{L_x^2(\mathbb{R}^+)}\, d\tau \\
&\leq C(1+T^{\frac{1}{4}}) T^{\frac{4-\kappa}{4}} \sup_{t\in[0,T]} ||u-v||_{L_x^2(\mathbb{R}^+)} \left(|||u|||_T^\kappa + |||v|||_T^\kappa \right) \\
&\leq C\delta^\kappa (1+T^{\frac{1}{4}}) T^{\frac{4-\kappa}{4}} \sup_{t\in[0,T]} ||u-v||_{L_x^2(\mathbb{R}^+)}.
\end{aligned}
$$

For given $\delta_0 > 0$ let $\delta = (2C+1)\delta_0$ and choose $T > 0$ so small that

$$
(3.3) \qquad C(1+T^{\frac{1}{4}})(\delta_0 + T^{\frac{4-\kappa}{4}} \delta^{\kappa+1}) \leq \delta,\ C\delta^\kappa (1+T^{\frac{1}{4}}) T^{\frac{4-\kappa}{4}} \leq \frac{1}{8}.
$$

A combination of (3.1), (3.2) with (3.3) implies that $\Phi(u)$ is a contraction map from $\mathcal{Z}_T^\delta$ into $\mathcal{Z}_T^\delta$, thus we establish the existence and uniqueness of local solution to the initial-boundary-value problem (1.1) in the set $\mathcal{Z}_T^\delta$. In fact the uniqueness holds in a large class $Z = C([0,T]; L^2(\mathbb{R}^+)) \cap L^4([0,T]; L_x^\infty(\mathbb{R}^+))$. Suppose $\tilde{u} \in Z$ satisfying the initial-boundary value data, then it is easy to see that for $T' < T$ sufficiently small $\tilde{u} \in \mathcal{Z}_{T'}^\delta$. Therefore $u = \tilde{u}$ in $[0,T'] \times \mathbb{R}^+$. Reapplying this argument we obtain the desired result.

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