J. Korean Math. Soc. **45** (2008), No. 1, pp. 63–78

HOLOMORPHIC FUNCTIONS ON THE MIXED NORM SPACES ON THE POLYDISC

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Abstract. We generalize several integral inequalities for analytic functions on the open unit polydisc $U^n = \{z \in \mathbb{C}^n \mid |z_j| < 1,\ j = 1, \dots, n\}$. It is shown that if a holomorphic function on U^n belongs to the mixed norm space $\mathcal{A}^{p,q}_{\vec{\omega}}(U^n)$, where $\omega_j(\cdot)$, $j = 1, \dots, n$, are admissible weights, then all weighted derivations of order $|k|$ (with positive orders of derivations) belong to a related mixed norm space. The converse of the result is proved when, $p, q \in [1, \infty)$ and when the order is equal to one. The equivalence of these conditions is given for all $p, q \in (0, \infty)$ if $\omega_j(z_j) = (1 - |z_j|^2)^{\alpha_j}$, $\alpha_j > -1$, $j = 1, \dots, n$ (the classical weights.) The main results here improve our results in Z. Anal. Anwendungen **23** (3) (2004), no. 3, 577–587 and Z. Anal. Anwendungen **23** (2004), no. 4, 775–782.

1. Introduction

Let $U^1 = U$ be the unit disk in the complex plane $\mathbb{C}$, $dm(\cdot) = \frac{1}{\pi} r dr d\theta$ the normalized area measure on U, $D(a, r_0)$ the disk in $\mathbb{C}$ centered at a with radius r_0, U^n the unit polydisc in the complex vector space $\mathbb{C}^n$, $r, \rho, \delta \in (0, \infty)^n$ and $\alpha \in (-1, \infty)^n$. If we write $0 \le r < 1$, where $r = (r_1, \dots, r_n)$ it means $0 \le r_j < 1$ for $j = 1, \dots, n$, and $r + 2$ stands for $(r_1 + 2, \dots, r_n + 2)$. For $z, w \in \mathbb{C}^n$ we write $z \cdot w = (z_1 w_1, \dots, z_n w_n)$; $e^{i\theta}$ is an abbreviation for $(e^{i\theta_1}, \dots, e^{i\theta_n})$; $dt = dt_1 \cdots dt_n$; $d\theta = d\theta_1 \cdots d\theta_n$. Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a multi-index, γ_k being nonnegative integers, we write

$$|\gamma| = \gamma_1 + \cdots + \gamma_n, \qquad \gamma! = \gamma_1! \cdots \gamma_n!, \qquad z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}.$$

For a holomorphic function f we denote

$$D^\gamma f = \frac{\partial^{|\gamma|} f}{\partial z_1^{\gamma_1} \cdots \partial z_n^{\gamma_n}}.$$

Let

$$P^n(w, r) = \{\, z \in \mathbb{C}^n \mid |z_j - w_j| < r_j, j = 1, \dots, n \,\}$$

Received May 12, 2006.

2000 *Mathematics Subject Classification.* Primary 32A10; Secondary 32A46.

Key words and phrases. holomorphic function, mixed norm space, polydisc, weighted derivations, admissible weight.

be a polydisc in $\mathbb{C}^n$ and let $H(P^n(w,r))$ be the class of all holomorphic functions f defined on $P^n(w,r)$.

For $f \in H(U^n)$ and $p \in (0,\infty)$ we usually write

$$M_p(f,r) = \left(\frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} |f(r\cdot e^{i\theta})|^p d\theta\right)^{1/p},$$

$0 \le r < 1$, for the integral means of f.

Let $\omega(s)$, $0 \le s < 1$, be a weight function which is positive and integrable on $(0,1)$. We extend ω on U by setting $\omega(z) = \omega(|z|)$. We may assume that our weights are normalized so that $\int_0^1 \omega(s)ds = 1$.

Let $\mathcal{L}^p_{\vec{\omega}} = \mathcal{L}^p_{\vec{\omega}}(U^n)$ denotes the class of all measurable functions defined on U^n such that

$$(1) \qquad ||f||^p_{\mathcal{L}^p_{\vec{\omega}}} = \int_{U^n} |f(z)|^p \prod_{j=1}^n \omega_j(z_j)dm(z_j) < \infty,$$

where $\omega_j(z_j)$, $j = 1,\dots,n$, are admissible weights (see, Definition 1) on the unit disk U. The weighted Bergman space $\mathcal{A}^p_{\vec{\omega}} = \mathcal{A}^p_{\vec{\omega}}(U^n)$ is the intersection of $\mathcal{L}^p_{\vec{\omega}}$ and $H(U^n)$. For $\omega_j(z_j) = (\alpha_j+1)(1-|z_j|^2)^{\alpha_j}$, $\alpha_j > -1$, $j = 1,\dots,n$, we obtain the classical Bergman space $\mathcal{A}^p_{\alpha}$, see [1, p.33] and Lebesgue space $\mathcal{L}^p_{\alpha}$.

Let $\mathcal{L}^{p,q}_{\vec{\omega}} = \mathcal{L}^{p,q}_{\vec{\omega}}(U^n)$ denotes the class of all measurable functions defined on U^n such that

$$(2) \qquad ||f||^q_{\mathcal{L}^{p,q}_{\vec{\omega}}} = \int_{[0,1)^n} M^q_p(f,r) \prod_{j=1}^n \omega_j(r_j)dr_j < \infty,$$

and $\mathcal{A}^{p,q}_{\vec{\omega}} = \mathcal{A}^{p,q}_{\vec{\omega}}(U^n)$ be the intersection of $\mathcal{L}^{p,q}_{\vec{\omega}}$ and $H(U^n)$. When $p = q$ we denote $\mathcal{A}^{p,q}_{\vec{\omega}}$ by $\mathcal{A}^p_{\vec{\omega}}$. This space is called the mixed norm space. If $\omega_j(z_j) = (\alpha_j+1)(1-|z_j|^2)^{\alpha_j}$, $\alpha_j > -1$, $j = 1,\dots,n$, then the space will be denoted by $\mathcal{A}^{p,q}_{\alpha}(U^n)$ (the classical mixed norm space).

Using polar coordinates and by some elementary calculations it is easy to see that in the case $p = q$, norms (1) and (2) are equivalent on the space $H(U^n)$.

Recently there is a huge interest in studying the weighted Bergman spaces of analytic functions of one variable see, for example, [4, 5, 6, 7, 8, 15, 16, 24], and the weighted Bergman spaces of analytic and harmonic functions on the unit ball $B \subset \mathbb{C}^n$ see, for example, in [1, 3, 9, 10, 12, 13, 14, 22, 23] (see, also the references therein).

In [1] and [18] the authors proved the following theorem.

Theorem A. *Let $p \in (0,\infty)$, $\alpha = (\alpha_1,\dots,\alpha_n)$, with $\alpha_j > -1$ for $j = 1,\dots,n$, m be a fixed positive integer and let $\mathbf{k} = (k_1,\dots,k_n) \in (\mathbb{Z}_+)^n$. Let $f \in H(U^n)$, then $f \in \mathcal{A}^p_{\alpha}(U^n)$ if and only if*

$$I_{\mathbf{k}} = \Big[\prod_{j=1}^n (1-|z_j|^2)^{k_j}\Big]\frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z) \in \mathcal{L}^p_{\alpha}, \quad \textit{for every } \mathbf{k},\ |\mathbf{k}| = m.$$

Moreover,

$$||f||_{\mathcal{A}^p_\alpha} \asymp \sum_{|\mathbf{k}|=0}^{m-1} \left|D^{\mathbf{k}} f(0)\right| + \sum_{|\mathbf{k}|=m} \| I_{\mathbf{k}} \|_{\mathcal{L}^p_\alpha} .$$

The expression $A \asymp B$ means that there are finite positive constants C and C' such that $CA \leq B \leq C'A$.

In the proof of Theorem A, when $p \in [1, \infty)$, G. Benke and D. C. Chang used the weighted Bergman projection $\mathbf{B}_\alpha : \mathcal{L}^2_\alpha \to \mathcal{A}^2_\alpha$, which can be extended as a bounded operator from $\mathcal{L}^p_\alpha$ onto $\mathcal{A}^p_\alpha$. Case $p \in (0,1]$ was considered by a quite different method in [18] by the author of this paper. Closely related results on the unit disc and the unit ball in $\mathbb{C}^n$ or $\mathbb{R}^n$ can be found in [1, 2, 4, 5, 12, 14, 15, 16, 17, 21, 22, 24].

Motivated by paper [22], in [19] we proved the following result:

Theorem B. *Let $p \in (0,\infty)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_j > -1$ for $j = 1, \ldots, n$, and $f \in H(U^n)$. Then $f \in \mathcal{A}^p_\alpha(U^n)$ if and only if the functions*

$$T_S f = \prod_{j \in S} (1 - |z_j|^2) \frac{\partial^{|S|} f}{\prod_{j \in S} \partial z_j} (\chi_S(1) z_1, \chi_S(2) z_2, \ldots, \chi_S(n) z_n),$$

belong to the space $\mathcal{L}^p_\alpha(U^n)$, for every $S \subseteq \{1, 2, \ldots, n\}$, where $\chi_S(\cdot)$ is the characteristic function of S, $|S|$ is the cardinal number of S, and $\prod_{j \in S} \partial z_j = \partial z_{j_1} \cdots \partial z_{j_{|S|}}$, where $j_k \in S$, $k = 1, \ldots, |S|$.

Moreover, $||\cdot||_{\mathcal{A}^p_\alpha}$ and the following norm

$$||f||_* = |f(\vec{0})| + \sum_{S \subseteq \{1,\ldots,n\}, S \neq \emptyset} ||T_S f||_{\mathcal{L}^p_\alpha},$$

$||\cdot||_*$ are equivalent on $\mathcal{A}^p_\alpha(U^n)$.

From now on $||f||_*$ will denote the following quantity

$$|f(\vec{0})| + \sum_{S \subseteq \{1,\ldots,n\}, S \neq \emptyset} ||T_S f||_{\mathcal{L}^{p,q}_\alpha}.$$

Note that Theorems A and B are both characterizations for a function f to belong to $\mathcal{A}^p_\alpha(U^n)$. The main purpose of this paper is to generalize Theorems A and B in the case of the mixed norm space.

For a given weight ω the function

$$\psi(r) = \psi_\omega(r) \overset{def}{=} \frac{1}{\omega(r)} \int_r^1 \omega(u) du, \quad 0 \leq r < 1,$$

is called the *distortion function* of ω. We put $\psi(z) = \psi(|z|)$ for $z \in B$.

Definition 1 ([15]). We say that a weight ω is *admissible* if it satisfies the following conditions:

(a) There is a positive constant $A = A(\omega)$ such that

$$\omega(r) \geq \frac{A}{1-r}\int_r^1 \omega(u)du \quad \text{for} \quad 0 \leq r < 1;$$

(b) ω is differentiable and there is a positive constant $B = B(\omega)$ such that

$$\omega'(r) \leq \frac{B}{1-r}\omega(r) \quad \text{for} \quad 0 \leq r < 1;$$

(c) For each sufficiently small positive δ there is a positive constant $C = C(\delta, \omega)$ such that

$$\sup_{0\leq r<1} \frac{\omega(r)}{\omega(r + \delta\psi(r))} \leq C.$$

Observe that (a) implies $A\psi(r) \leq 1 - r$ thus for sufficiently small positive δ we have $r + \delta\psi(r) < 1$ and the quantity in the denominator of the fraction in (c) is well defined. It is easy to see that the classical weight $\omega(r) = (1 - r^2)^\alpha$, $\alpha > -1$ is admissible. Some other examples of admissible weights can be found in [15, pp.660–663].

In this paper we prove the following results.

Theorem 1. *Let* $\mathbf{k} = (k_1, \dots, k_n) \in (\mathbb{Z}_+)^n$, f *be a holomorphic function defined on* U^n *in* $\mathbb{C}^n$ *and* $\omega_j(z_j)$, $j = 1, \dots, n$ *are admissible weights on the unit disk* U, *with distortion functions* $\psi_j(z_j)$, $j = 1, \dots, n$.

(a) *If* $f \in \mathcal{A}^{p,q}_{\vec{\omega}}(U^n)$ *with* $p, q > 0$, *then*

$$I_{\mathbf{k},\vec{\omega}} = \Big[\prod_{j=1}^n \psi_j^{k_j}(z_j)\Big]\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z) \in \mathcal{L}^{p,q}_{\vec{\omega}}(U^n). \tag{3}$$

Moreover, let m *be a fixed positive integer. Then there is a positive constant* $C = C(p, q, \omega_j, m, n)$ *such that*

$$||f||_{\mathcal{A}^{p,q}_{\vec{\omega}}} \geq C\Big(\sum_{|\mathbf{k}|=0}^{m-1} |D^{\mathbf{k}}f(\vec{0})| + \sum_{|\mathbf{k}|=m} ||\, I_{\mathbf{k},\vec{\omega}}\, ||_{\mathcal{L}^{p,q}_{\vec{\omega}}}\Big). \tag{4}$$

(b) *If* $p, q \in [1, \infty)$ *and for all* $j = 1, \dots, n$, $\psi_j(z_j)\frac{\partial f}{\partial z_j}(z) \in \mathcal{L}^{p,q}_{\vec{\omega}}$, *then* $f \in \mathcal{A}^{p,q}_{\vec{\omega}}$ *and there is a positive constant* $C = C(p, q, \omega_j, n)$ *such that*

$$||f||_{\mathcal{A}^{p,q}_{\vec{\omega}}} \leq C\left(|f(\vec{0})| + \sum_{j=1}^n \left|\left|\psi_j\frac{\partial f}{\partial z_j}\right|\right|_{\mathcal{L}^{p,q}_{\vec{\omega}}}\right).$$

Theorem 1 (b) was proved in [20] so that we give here only a sketch of the proof for the benefit of the reader. It is an open problem whether Theorem 1 (b) holds if p or q belong to the interval $(0, 1]$. A partial answer to the question gives the following main result of this paper, which concerns the classical weight case.

Theorem 2. *Let $p,q \in (0,\infty)$, $\alpha = (\alpha_1,\dots,\alpha_n)$, with $\alpha_j > -1$ for $j = 1,\dots,n$, m be a fixed positive integer and let $\mathbf{k} = (k_1,\dots,k_n) \in (\mathbb{Z}_+)^n$. Let $f \in H(U^n)$, then the following conditions are equivalent*

(a) $f \in \mathcal{A}^{p,q}_\alpha(U^n)$;

(b)

$$I_{\mathbf{k}} = \Big[\prod_{j=1}^n (1-|z_j|^2)^{k_j}\Big] \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z) \in \mathcal{L}^{p,q}_\alpha \qquad \text{for all } \mathbf{k},\ |\mathbf{k}| = m;$$

(c) *The functions*

$$T_S f = \prod_{j\in S}(1-|z_j|^2)\frac{\partial^{|S|} f}{\prod_{j\in S}\partial z_j}(\chi_S(1)z_1, \chi_S(2)z_2,\dots,\chi_S(n)z_n),$$

for every $S \subseteq \{1,2,\dots,n\}$, are in $\mathcal{L}^{p,q}_\alpha(U^n)$.

Moreover,

$$||f||_{\mathcal{A}^{p,q}_\alpha} \asymp \sum_{|\mathbf{k}|=0}^{m-1} |D^{\mathbf{k}} f(\vec{0})| + \sum_{|\mathbf{k}|=m} \| I_{\mathbf{k}} \|_{\mathcal{L}^{p,q}_\alpha} \asymp ||f||_*.$$

We would like to point out that Theorem 2 cannot be easily obtained from the results in our papers [18] and [19].

The organization of the paper is as follows: In Section 2 we prove several auxiliary results, which we use in the proofs of the main results. The main results of the paper, i.e., Theorems 1 and 2 are proved in Section 3.

We have to say that throughout the rest of the paper C will denote a constant not necessarily the same at each occurrence.

2. Auxiliary results

In this section we prove several auxiliary results which we use in proving Theorems 1 and 2 in the subsequent section.

Lemma 1 ([18, p.579]). *Let $f \in H(U^n)$, γ be a multi-index and $p > 0$. Then*

$$|D^\gamma f(w)|^p \le \frac{C}{r^{\gamma p}\prod_{j=1}^n r_j^2}\int_{P^n(w,r)} |f|^p \prod_{j=1}^n dm(z_j), \tag{5}$$

whenever $P^n(w,r) \subset U^n$, where C is a constant depending only on p,γ and n.

Lemma 2. *Let β be a multi-index and $a \in U^n$. Then the point evaluations $\Lambda_{a,\beta}(f) = D^\beta f(a)$ are bounded linear functionals on $\mathcal{A}^{p,q}_\omega(U^n)$ for all $p,q \in (0,\infty)$.*

Proof. Choose $\overline{P^n(a,\delta)} \subset U^n$. Let $m = \min_{z\in\overline{P^n(a,\delta)}} \prod_{j=1}^n \omega_j(z_j) > 0$ and $d = \max_{j\in\{1,\dots,n\}} (|a_j| + \delta_j)$. Note that $d < 1$. By Lemma 1, using polar coordinates and the monotonicity of the integral means $M_p^p(f, r)$, we have

$$
\begin{aligned}
|D^\beta f(a)|^p &\le \frac{C}{\delta^{\beta p}\prod_{j=1}^n \delta_j^2}\int_{\overline{P^n(a,\delta)}} |f(z)|^p \prod_{j=1}^n dm(z_j)\\
&\le \frac{C}{m\delta^{\beta p+2}}\int_{\overline{P^n(0,d)}} |f(z)|^p \prod_{j=1}^n \omega_j(z_j)dm(z_j)\\
&\le \frac{C2^n}{m\delta^{\beta p+2}}\int_0^d \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} |f(r\cdot e^{i\theta})|^p d\theta \prod_{j=1}^n \omega_j(r_j) r_j dr_j \qquad (6)\\
&\le \frac{2^n C}{m\delta^{\beta p+2}} M_p^p(f,d)\int_0^d \prod_{j=1}^n \omega_j(r_j) r_j dr_j\\
&\le \frac{C}{m\delta^{\beta p+2}} M_p^p(f,r)
\end{aligned}
$$

for $r \in [d, 1)$. □

Raising (6) to the q/pth power, then multiplying obtained inequality by $\prod_{j=1}^n \omega(r_j)$ and integrating over $r \in [d, 1)^n$, we obtain

$$
|D^\beta f(a)|^q \int_{[d,1)^n} \prod_{j=1}^n \omega(r_j)dr_j \le C\int_{[d,1)^n} M_p^q(f,r)\prod_{j=1}^n \omega(r_j)dr_j,
$$

from which the result follows.

Lemma 3. *Let* $f \in H(U^n)$, *and*

$$
M_p^p(f, r_k) = \frac{1}{2\pi}\int_0^{2\pi} |f(\dots, r_k e^{i\theta_k}, \dots)|^p d\theta_k.
$$

Then there are positive constants C_1 *and* C_2 *independent of* f, z_j, $j \ne k$, ρ_k *and* r_k, *such that*

(a) *If* $p \in (0, 1]$, *then*

$$
M_p^p(f,\rho_k) - M_p^p(f,r_k) \le C_1(\rho_k - r_k)^p M_p^p\Big(\frac{\partial f}{\partial z_k}, \rho_k\Big).
$$

(b) *If* $p \ge 1$, *then*

$$
M_p(f,\rho_k) - M_p(f,r_k) \le C_2(\rho_k - r_k) M_p\Big(\frac{\partial f}{\partial z_k}, \rho_k\Big).
$$

Proof. Let $l = \min\{1, p\}$. Using Minkowski's inequality in the case $p \ge 1$ or the following elementary inequality $(x + y)^p \le x^p + y^p$, $x, y \ge 0$, when $p \in (0, 1]$, we have

$$\begin{aligned} & M_p^l(f,\rho_k) - M_p^l(f,r_k) \\ \le\ & \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\dots,\rho_k e^{i\theta_k},\dots) - f(\dots,r_k e^{i\theta_k},\dots)|^p d\theta_k\right)^{l/p} \\ \le\ & \frac{(\rho_k - r_k)^l}{2\pi}\left(\int_0^{2\pi} \sup_{r_k<t<\rho_k}\left|\frac{\partial f}{\partial z_k}(\dots,te^{i\theta_k},\dots)\right|^p d\theta_k\right)^{l/p} \\ \le\ & C(\rho_k - r_k)^l M_p^l\Big(\frac{\partial f}{\partial z_k},\rho_k\Big). \end{aligned}$$

In the last inequality we have used Hardy-Littlewood maximal theorem, see, for example, [4, Theorem 1.9]. □

Lemma 4. *Let $f \in H(U^n)$, $p,q \in (0,\infty)$ and $\beta_{k,j} \in \mathbb{R}$, $k,j = 1,\dots,n$. Then there is a constant $C = C(p,q,\beta_{k,j},n)$ such that*

$$\max_{z\in\overline{D^n(0,1/2)}} |f(z)|^q \le C\Big(|f(\vec{0})|^q + \sum_{k=1}^n \int_0^1 M_p^q\Big(\frac{\partial f}{\partial z_k},r\Big)\prod_{j=1}^n (1-r_j^2)^{\beta_{k,j}} dr_j\Big),$$

for all $k = 1,\dots,n$.

Proof. Without loss of generality we may assume that $n = 2$. The case $n \ge 3$ is only technically complicated. Since

$$f(z_1,z_2) - f(0,0) = \int_0^1 \frac{d}{dt}(f(tz_1,z_2))dt + \int_0^1 \frac{d}{dt}(f(0,tz_2))dt,$$

by some well-known inequalities we obtain

(7)
$$|f(z_1,z_2)|^p \le c_p\left(|f(0,0)|^p + \max_{|\zeta_1|\le 1/2}\left|\frac{\partial f}{\partial z_1}(\zeta_1,z_2)\right|^p + \max_{|\zeta_2|\le 1/2}\left|\frac{\partial f}{\partial z_2}(0,\zeta_2)\right|^p\right),$$

for all $z_1,z_2 \in \overline{D(0,1/2)}$, where $c_p = 1$ for $0<p<1$ and $c_p = 3^{p-1}$ for $p \ge 1$.

On the other hand, from (5), (7), by polar coordinates and the monotonicity of $M_p^p\Big(\frac{\partial f}{\partial z_k},r_1,r_2\Big)$, we obtain

$$|f(z_1,z_2)|^p \le C\left(|f(0,0)|^p + \sum_{k=1}^2 M_p^p\Big(\frac{\partial f}{\partial z_k},r_1,r_2\Big)\right) \tag{8}$$

for all $z_1,z_2 \in \overline{D(0,1/2)}$, and $r_1,r_2 \in [3/4,1)$.

Let $m_j = \min_{r\in[0,7/8]^2}\prod_{j=1}^2(1-r_j^2)^{\beta_{k,j}}$, $j = 1,2$. Raising (8) to the q/pth power and by some simple calculation, it follows that

$$|f(z_1,z_2)|^q \le C\left(|f(0,0)|^q + \sum_{k=1}^2 \frac{\prod_{j=1}^2(1-r_j^2)^{\beta_{k,j}}}{m_j} M_p^q\Big(\frac{\partial f}{\partial z_k},r_1,r_2\Big)\right).$$

Integrating this inequality over $[3/4,7/8]^2$ with respect to r_1 and r_2 and dividing by the obtained constant standing nearby $|f(z_1,z_2)|^q$, it follows that

$$|f(z_1,z_2)|^q \le C\left(|f(0,0)|^q + \sum_{k=1}^{2}\int_{[3/4,7/8]^2} M_p^q\Big(\frac{\partial f}{\partial z_k}, r_1, r_2\Big)\prod_{j=1}^{2}(1-r_j^2)^{\beta_{k,j}}dr_j\right)$$

for every $z_1,z_2 \in \overline{D(0,1/2)}$, from which the result follows. □

Using the change $r \to (1+r)/2$ and some well known elementary inequalities the following lemma can be proved (see [11]).

Lemma 5. *Let $g(r)$ be a nonnegative continuous function on the interval $[0,1)$, $b>0$ and let $a>-1$. Then there is a constant $C=C(a,b)$ such that*

$$\int_0^1 g^b(r)(1-r)^a dr \le C\left(\max_{r\in[0,1/2]} g^b(r) + \int_0^1 \left|g\Big(\frac{1+r}{2}\Big) - g(r)\right|^b (1-r)^a dr\right).$$

Lemma 6. *Suppose $p,q\in[1,\infty)$ and $f\in H(U^n)$. Then*

$$\frac{d}{dt}M_p^q(f,tr) \le qM_p^{q-1}(f,tr)\sum_{i=1}^{n} r_i M_p\Big(\frac{\partial f}{\partial z_i}, tr\Big), \tag{9}$$

almost everywhere.

Proof. Let first $p=q$. For $f\equiv 0$ the result is obvious. If $f\not\equiv 0$, at points where f is not zero, it is easy to see that

$$\frac{d}{dt}\Big(|f(tr\cdot e^{i\theta})|^p\Big) \le p|f(tr\cdot e^{i\theta})|^{p-1}\sum_{i=1}^{n} r_i\left|\frac{\partial f}{\partial z_i}(tr\cdot e^{i\theta})\right|. \tag{10}$$

From (10) and by the dominated convergence theorem we obtain

$$\frac{d}{dt}M_p^p(f,tr) \le \frac{p}{(2\pi)^n}\sum_{i=1}^{n} r_i \int_{[0,2\pi]^n} |f(tr\cdot e^{i\theta})|^{p-1}\left|\frac{\partial f}{\partial z_i}(tr\cdot e^{i\theta})\right| d\theta.$$

If $p=1$ the assertion is clear. If $p>1$, applying on the last integral Hölder's inequality with exponents $p/(p-1)$ and p we obtain the result.

If $p\ne q$, computing $\frac{d}{dt}M_p^q(f,tr)$ and then using the case $p=q$, the result follows. □

3. Proof of the main results

In this section we prove the main results in this paper.

Proof of Theorem 1. (a) Let γ be a multi-index, such that $|\gamma| = m$. Let $f \in H(U^n)$ and $z = (z_1, \dots, z_n) \in U^n$. Applying Lemma 1 to the functions $f(z \cdot e^{i\theta})$, where $\theta_j \in [0, 2\pi), j = 1, \dots, n$, when $\rho > r$, we get

(11)
$$|D^\gamma f(r \cdot e^{i\theta})|^p \le \frac{C}{(\rho - r)^{\gamma p} \prod_{j=1}^n (\rho_j - r_j)^2} \int_{P^n(r, \rho - r)} |f(\omega \cdot e^{i\theta})|^p \prod_{j=1}^n dm(\omega_j).$$

Integrating (11) over $[0, 2\pi]^n$ and then using Fubini's theorem, we obtain

$$(12) \quad M_p^p(D^\gamma f, r) \le \frac{C}{(\rho - r)^{\gamma p + 2}} \int_{P^n(r, \rho - r)} \int_{[0, 2\pi]^n} |f(\omega \cdot e^{i\theta})|^p d\theta \prod_{j=1}^n dm(\omega_j).$$

Assume first that $q \ge p$. Raising both sides of inequality (12) to the q/pth power and applying Jensen's inequality, it follows that

(13)
$$M_p^q(D^\gamma f, r) \le \frac{C}{(\rho - r)^{\gamma q + 2}} \int_{P^n(r, \rho - r)} \left(\int_{[0, 2\pi]^n} |f(\omega \cdot e^{i\theta})|^p d\theta \right)^{q/p} \prod_{j=1}^n dm(\omega_j).$$

If $p \ge q$, then using Minkowski's inequality to inequality (11), where instead of p stands q, we also obtain inequality (13). By the monotonicity of the integral means, 2π-periodicity of the function $|f(r \cdot e^{i\theta})|$ in each variable θ_j, $j \in \{1, \dots, n\}$ and (13), it follows that

$$(14) \qquad (\rho - r)^{\gamma q} M_p^q(D^\gamma f, r) \le C \left(\int_{[0, 2\pi]^n} |f(\rho_1 e^{i\theta_1}, \dots, \rho_n e^{i\theta_n})|^p d\theta \right)^{q/p}.$$

Put $\rho_j = \rho_j(r_j) = r_j + \delta_j \psi_j(r_j)$, $0 \le r_j < 1$, in (14), where δ_j are chosen in the following way. First note that, if $\delta \in (0, A)$ we have $r_j < \rho_j(r_j) < 1$ for $r_j \in [0, 1)$. On the other hand by conditions (b) and (c) of Definition 1 we obtain

$$\rho_j'(r_j) = 1 - \delta_j - \delta_j \frac{\omega_j'(r_j)}{\omega_j(r_j)} \psi_j(r_j) \ge 1 - \delta_j \left(1 + \frac{B}{A} \right).$$

We choose $\delta_j \in (0, A)$ such that $\rho_j'(r_j) > c_0 > 0$ for $r_j \in [0, 1)$. Putting $\rho_j = \rho_j(r_j)$ in (14), then multiplying obtained inequality by $\prod_{j=1}^n \omega_j(r_j)$, using condition (c) in Definition 1 and the fact that $\rho_j'(r_j) > c_0 > 0$ for $r_j \in [0, 1)$ and every $j \in \{1, \dots, n\}$, we obtain

$$\prod_{j=1}^n (\delta_j \psi_j(r_j))^{\gamma_j q} \omega_j(r_j) M_p^q(D^\gamma f, r)$$
$$\le C \Big(\int_{[0, 2\pi]^n} |f(\rho \cdot e^{i\theta})|^p d\theta \Big)^{q/p} \prod_{j=1}^n \omega_j(\rho_j(r_j)) \rho_j'(r_j).$$

Integrating this inequality over $[0,1)^n$ and making the changes $t_j = \rho_j(r_j)$, $j=1,\dots,n$, it follows that

$$\begin{aligned}
&\prod_{j=1}^n \delta_j^{\gamma_j q} \int_{[0,1)^n} M_p^q(D^\gamma f, r) \prod_{j=1}^n \psi_j^{\gamma_j q}(r_j)\omega_j(r_j)dr_j \\
\le\ & C\int_{[\rho_j(0),1)^n} \left(\int_{[0,2\pi]^n} |f(t_1e^{i\theta_1},\dots,t_ne^{i\theta_n})|^p d\theta\right)^{q/p} \prod_{j=1}^n \omega_j(t_j)dt_j \\
\le\ & C\int_{[0,1)^n} M_p^q(f,t)\prod_{j=1}^n \omega_j(t_j)dt_j,
\end{aligned}$$

from which inequality (3) follows.

Let β be a multi-index. By Lemma 2 we know that the linear functional $L(f) = D^\beta f(\vec{0})$, is bounded. Hence $|D^\beta f(\vec{0})|^q \le C\,||f||^q_{\mathcal{A}^{p,q}_{\vec{\omega}}}$ for all $f\in H(U^n)$ and for some $C = C(p,q,\beta,\vec{\omega}) > 0$. Hence inequality (4) holds.

(b) Without loss of generality, we may assume that $n=2$, and $f(0,0)=0$. Also we assume that f is not constant and all integrals are finite. In order to avoid some complicated notations we use $M_p^q(r_1t,r_2t)$ instead of $M_p^q(f,r_1t,r_2t)$. We have

$$\begin{aligned}
(15)\quad & ||f||^q_{\mathcal{A}^{p,q}_{\vec{\omega}}} \\
=\ & \int_0^1\int_0^1 \left(\int_0^1 \frac{d}{dt}M_p^q(r_1t,r_2t)dt\right)\omega_1(r_1)\omega_2(r_2)dr_1dr_2 \\
\le\ & q\int_0^1\int_0^1\left(\int_0^1 M_p^{q-1}(r_1t,r_2)M_p\Big(\frac{\partial f}{\partial z_1},r_1t,r_2\Big)r_1dt\right)\omega_1(r_1)\omega_2(r_2)dr_1dr_2 \\
& +q\int_0^1\int_0^1\left(\int_0^1 M_p^{q-1}(r_1,r_2t)M_p\Big(\frac{\partial f}{\partial z_2},r_1,r_2t\Big)r_2dt\right)\omega_1(r_1)\omega_2(r_2)dr_1dr_2 \\
\le\ & q\int_0^1\int_0^1\left(\int_0^{r_1} M_p^{q-1}(s,r_2)M_p\Big(\frac{\partial f}{\partial z_1},s,r_2\Big)ds\right)\omega_1(r_1)\omega_2(r_2)dr_1dr_2 \\
& +q\int_0^1\int_0^1\left(\int_0^{r_2} M_p^{q-1}(r_1,\tau)M_p\Big(\frac{\partial f}{\partial z_2},r_1,\tau\Big)d\tau\right)\omega_1(r_1)\omega_2(r_2)dr_1dr_2 \\
\le\ & q\int_0^1\int_0^1 M_p^{q-1}(s,r_2)M_p\Big(\frac{\partial f}{\partial z_1},s,r_2\Big)\psi_1(s)\omega_1(s)\omega_2(r_2)dsdr_2 \\
& +q\int_0^1\int_0^1 M_p^{q-1}(r_1,\tau)M_p\Big(\frac{\partial f}{\partial z_2},r_1,\tau\Big)\psi_2(\tau)\omega_2(\tau)\omega_1(r_1)d\tau dr_1 = I_1+I_2.
\end{aligned}$$

If $q>1$, by Hölder inequality with the exponents $q/(q-1)$ and q, we get

$$(16)\qquad I_1 \le ||f||^{q-1}_{\mathcal{A}^{p,q}_{\vec{\omega}}}\left(\int_0^1\int_0^1 M_p^q\Big(\frac{\partial f}{\partial z_1},s,r_2\Big)\psi_1^q(s)\omega_1(s)\omega_2(r_2)dsdr_2\right)^{1/q}.$$

Similar inequality holds for I_2. From the inequality, (15) and (16) we obtain the result in this case. For $q = 1$ the result follows from (15). If f is constant the result is clear. To remove the restriction of the finiteness of the integrals we consider holomorphic functions $f_\rho(z) = f(\rho z), \rho \in (0,1)$ and use the Monotone Convergence Theorem, when $\rho \to 1$. □

Proof of Theorem 2. (a)⇒(b), (c). Implication (a)⇒(b) is a consequence of Theorem 1, when $\omega_j(z_j) = (1-|z_j|^2)^{\alpha_j}$, $j = 1, \dots, n$. Indeed, in this case $\psi_j(z_j) \asymp (1-|z_j|^2)$, $j = 1, \dots, n$. (a)⇒(c) follows if we take the points $(\chi_S(1)z_1, \dots, \chi_S(n)z_n)$, $S \subseteq \{1, \dots, n\}$, into the functions

$$\Big[\prod_{j=1}^{n}(1-|z_j|^2)^{\chi_S(j)}\Big]\frac{\partial^{|S|}f}{\partial z_1^{\chi_S(1)}\cdots\partial z_n^{\chi_S(n)}}.$$

(b)⇒(a). Without loss of generality, we may assume that $n = 2$. By some simple calculation, it is easy to see that

$$||f||^q_{A^{p,q}_\alpha} \asymp \int_0^1 (1-r_2)^{\alpha_2}\int_0^1 M_p^q(f,r_1,r_2)(1-r_1)^{\alpha_1}dr_1dr_2.$$

Let $l = \min\{1,p\}$. By Lemmas 3 and 5, and since $M_p^l(f,r_1,r_2)$ is nondecreasing in r_1, we obtain

$$\begin{aligned}
&\int_0^1 M_p^q(f,r_1,r_2)(1-r_1)^{\alpha_1}dr_1\\
= &\int_0^1 (M_p^l(f,r_1,r_2))^{q/l}(1-r_1)^{\alpha_1}dr_1\\
\le &\; C\Big((M_p^l(f,1/2,r_2))^{q/l}\\
&\quad+\int_0^1 \Big|M_p^l\Big(f,\frac{1+r_1}{2},r_2\Big)-M_p^l\Big(f,r_1,r_2\Big)\Big|^{q/l}(1-r_1)^{\alpha_1}dr_1\Big)\\
\le &\; C\left(M_p^q(f,1/2,r_2)+\int_0^1 M_p^q\Big(\frac{\partial f}{\partial z_1},\frac{1+r_1}{2},r_2\Big)(1-r_1)^{\alpha_1+q}dr_1\right).
\end{aligned}$$

Hence

$$\begin{aligned}
(17)\qquad &||f||^q_{A^{p,q}_\alpha}\\
\le &\; C\Big(\int_0^1 (1-r_2)^{\alpha_2}M_p^q(f,1/2,r_2)dr_2\\
&+\int_0^1(1-r_2)^{\alpha_2}\int_0^1 M_p^q\Big(\frac{\partial f}{\partial z_1},\frac{1+r_1}{2},r_2\Big)(1-r_1)^{\alpha_1+q}dr_1dr_2\Big).
\end{aligned}$$

Since $M_p^q\left(\frac{\partial f}{\partial z_1}, \frac{1+r_1}{2}, r_2\right)$ is nondecreasing in r_2 and applying the changes $\frac{1+r_j}{2} \to r_j$, $j = 1, 2$, we obtain

$$\begin{aligned} &\int_0^1 (1-r_2)^{\alpha_2} \int_0^1 M_p^q\Big(\frac{\partial f}{\partial z_1}, \frac{1+r_1}{2}, r_2\Big)(1-r_1)^{\alpha_1+q} dr_1 dr_2 \\ (18) \quad \leq\ & C \int_0^1 \int_0^1 M_p^q\Big(\frac{\partial f}{\partial z_1}, r_1, r_2\Big)(1-r_1^2)^{\alpha_1+q} r_1 dr_1 (1-r_2^2)^{\alpha_2} r_2 dr_2. \end{aligned}$$

Using again Lemmas 3 and 5, and since $M_p^q(f, 1/2, r_2)$ is nondecreasing in r_2 we get

(19)

$$\begin{aligned} &\int_0^1 (1-r_2)^{\alpha_2} M_p^q(f, 1/2, r_2) dr_2 \\ &\leq C\bigg(M_p^q(f, 1/2, 1/2) + \int_0^1 (1-r_2)^{\alpha_2} \Big| M_p^l\Big(f, \frac{1}{2}, \frac{1+r_2}{2}\Big) - M_p^l\Big(f, \frac{1}{2}, r_2\Big)\Big|^{q/l} dr_2\bigg) \\ &C\bigg(\max_{|z_1| \leq 1/2, |z_2| \leq 1/2} |f(z_1, z_2)|^q + \int_0^1 (1-r_2)^{\alpha_2+q} M_p^q\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1+r_2}{2}\Big) dr_2\bigg). \end{aligned}$$

It is clear that there is a constant C independent of f such that

(20)

$$\left|\int_0^{3/4} (1-r_2)^{\alpha_2+q} M_p^q\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1+r_2}{2}\Big) dr_2\right| \leq C \max_{z \in \overline{D^2(0,7/8)}} \Big|\frac{\partial f}{\partial z_2}(z_1, z_2)\Big|^q.$$

Similar to Lemma 4 we can prove the following inequality

(21)

$$\max_{z \in \overline{D^2(0,7/8)}} \Big|\frac{\partial f}{\partial z_2}(z_1, z_2)\Big|^q \leq C \int_{[0,1)^2} M_p^q\Big(\frac{\partial f}{\partial z_2}, r_1, r_2\Big)(1-r_1^2)^{\alpha_1}(1-r_2^2)^{\alpha_2+q} dr_1 dr_2.$$

On the other hand, using the change $(1 + r_2)/2 \to r_2$ we obtain

$$\begin{aligned} &\int_{3/4}^1 (1-r_2)^{\alpha_2+q} M_p^q\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1+r_2}{2}\Big) dr_2 \\ =\ & C_1 \int_{7/8}^1 (1-r_2)^{\alpha_2+q} M_p^q\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, r_2\Big) dr_2 = J_1 \end{aligned}$$

for some $C_1 > 0$.

Using again the monotonicity of the integral means, we obtain that there is a constant C independent of f such that

$$\begin{aligned} J_1 &= \frac{1}{C_1(\alpha_1+1)2^{\alpha_1+1}}\int_{7/8}^{1}(1-r_2)^{\alpha_2+q}M_p^q\Big(\frac{\partial f}{\partial z_2},\frac{1}{2},r_2\Big)dr_2 \\ &= \frac{1}{C_1}\int_{1/2}^{1}(1-r_1)^{\alpha_1}dr_1\int_{7/8}^{1}(1-r_2)^{\alpha_2+q}M_p^q\Big(\frac{\partial f}{\partial z_2},\frac{1}{2},r_2\Big)dr_2 \\ &\le \frac{1}{C_1}\int_{1/2}^{1}(1-r_1)^{\alpha_1}\int_{7/8}^{1}(1-r_2)^{\alpha_2+p}M_p^q\Big(\frac{\partial f}{\partial z_2},r_1,r_2\Big)dr_2dr_1 \\ &\le C\int_0^1\int_0^1(1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2+q}M_p^q\Big(\frac{\partial f}{\partial z_2},r_1,r_2\Big)dr_2dr_1. \qquad (22) \end{aligned}$$

From (17)-(22) the result and the asymptotics in Theorem 2 follow, for $m=1$. Using induction we obtain the result for $m \ge 2$.

(c)⇒(a). As in the previous case, we may assume that $n=2$ and $f(0,0)=0$. From (17)-(19), it follows that we should estimate the following quantities

$$I_1 = \int_0^1\int_0^1 M_p^q\Big(\frac{\partial f}{\partial z_1},r_1,r_2\Big)(1-r_1^2)^{\alpha_1+q}r_1dr_1(1-r_2^2)^{\alpha_2}r_2dr_2,$$

$$I_2 = \max_{|z_1|\le 1/2,|z_2|\le 1/2}|f(z_1,z_2)|^q$$

and

$$I_3 = \int_0^1(1-r_2)^{\alpha_2+q}M_p^q\Big(\frac{\partial f}{\partial z_2},\frac{1}{2},\frac{1+r_2}{2}\Big)dr_2.$$

Using the inequality

$$\max_{z\in \overline{D^2(0,1/2)}}\Big|\frac{\partial f}{\partial z_j}(z_1,z_2)\Big|^q \le CM_p^q\Big(\frac{\partial f}{\partial z_j},r_1,r_2\Big), \quad j\in\{1,2\} \quad r_1,r_2\in[3/4,1),$$

which can be proved similar to (8), taking $r_2=3/4$ for $j=1$ and $r_1=3/4$ for $j=2$, it follows that

$$\max_{z\in \overline{D^2(0,1/2)}}\Big|\frac{\partial f}{\partial z_1}(z_1,z_2)\Big|^q \le CM_p^q\Big(\frac{\partial f}{\partial z_1},r_1,\frac{3}{4}\Big) \qquad (23)$$

when $r_1\in[3/4,1)$, and

$$\max_{z\in \overline{D^2(0,1/2)}}\Big|\frac{\partial f}{\partial z_2}(z_1,z_2)\Big|^q \le CM_p^q\Big(\frac{\partial f}{\partial z_2},\frac{3}{4},r_2\Big), \qquad (24)$$

when $r_2\in[3/4,1)$.

Multiplying (23) by $(1-r_1)^{\alpha_1+q}$, then integrating obtained inequality from 3/4 to 1 with respect to r_1, and multiplying (24) by $(1-r_2)^{\alpha_2+q}$, then integrating from 3/4 to 1 with respect to r_2, and using inequality (7) with $p=q$, we get

$$I_2 \le C\Big(\int_{3/4}^{1}(1-r_1)^{\alpha_1+q}M_p^q\Big(\frac{\partial f}{\partial z_1},r_1,\frac{3}{4}\Big)dr_1 + \int_{3/4}^{1}(1-r_2)^{\alpha_2+q}M_p^q\Big(\frac{\partial f}{\partial z_2},\frac{3}{4},r_2\Big)dr_2\Big). \tag{25}$$

By Lemma 3 and the inequality $(x+y)^p \le c_p(x^p+y^p)$, $x,y\ge 0$, where $c_p=1$ when $p\in(0,1)$, and $c_p=2^{p-1}$ when $p\ge 1$, we obtain that there is a positive constant C such that

$$M_p^q\Big(\frac{\partial f}{\partial z_2},\frac{3}{4},r_2\Big) \le C\Big(M_p^q\Big(\frac{\partial f}{\partial z_2},0,r_2\Big)+M_p^q\Big(\frac{\partial^2 f}{\partial z_1\partial z_2},\frac{3}{4},r_2\Big)\Big) \tag{26}$$

and

$$M_p^q\Big(\frac{\partial f}{\partial z_1},r_1,\frac{3}{4}\Big) \le C\Big(M_p^q\Big(\frac{\partial f}{\partial z_1},r_1,0\Big)+M_p^q\Big(\frac{\partial^2 f}{\partial z_1\partial z_2},r_1,\frac{3}{4}\Big)\Big). \tag{27}$$

Multiplying (26) by $(1-r_2)^{\alpha_2+q}$, then integrating obtained inequality from 0 to 1 with respect to r_2, and multiplying (27) by $(1-r_1)^{\alpha_1+q}$, then integrating obtained inequality from 0 to 1 with respect to r_1, using the monotonicity of the integral means $M_p(\cdot,r_1,r_2)$ in each variable and (25), we get

(28)

$$I_2 \le C\Big(\int_0^1(1-r_2^2)^{\alpha_2+q}M_p^q\Big(\frac{\partial f}{\partial z_2},0,r_2\Big)dr_2 + \int_0^1(1-r_1^2)^{\alpha_1+q}M_p^q\Big(\frac{\partial f}{\partial z_1},r_1,0\Big)dr_1 + \int_0^1\int_0^1(1-r_1^2)^{\alpha_1+q}(1-r_2^2)^{\alpha_2+q}M_p^q\Big(\frac{\partial^2 f}{\partial z_1\partial z_2},r_1,r_2\Big)dr_1dr_2\Big) = CI_4.$$

Further, using the change $\frac{1+r_2}{2}\to r_2$, the monotonicity of $M_p(\cdot,r_1,r_2)$, (27) and (28), we have that

$$I_3 = C\int_{1/2}^{1}(1-r_2)^{\alpha_2+q}M_p^q\Big(\frac{\partial f}{\partial z_2},\frac{1}{2},r_2\Big)dr_2 \le CI_4. \tag{29}$$

By Fubini's theorem, Lemma 5, Lemma 3 and the monotonicity of $M_p(\cdot,r_1,r_2)$, it follows that

$$I_1 \le C\Big(\int_0^1 M_p^q\Big(\frac{\partial f}{\partial z_1},r_1,\frac{3}{4}\Big)(1-r_1)^{\alpha_1+q}dr_1 + \int_0^1\int_0^1 M_p^q\Big(\frac{\partial^2 f}{\partial z_1\partial z_2},r_1,r_2\Big)(1-r_2^2)^{\alpha_2+q}dr_2(1-r_1^2)^{\alpha_1+q}dr_1\Big). \tag{30}$$

From (23)-(30) we see that for the case $n = 2$, the quantities I_1, I_2 and I_3 are estimated by a linear combination of the terms

$$\int_{[0,1)^2} M_p^q\Big(\frac{D^{|S|}f}{\prod_{j\in S}\partial z_j}, \chi_S(1)r_1, \chi_S(2)r_2\Big)\prod_{j\in S}(1-r_j^2)^{\alpha_j+q}dr_j, \quad S\subseteq\{1,2\},$$

from which the implication follows. For the case $n \geq 3$ we can use the induction. □

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