

# LORENTZIAN SURFACES WITH CONSTANT CURVATURES AND TRANSFORMATIONS IN THE 3-DIMENSIONAL LORENTZIAN SPACE

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ABSTRACT. We study Lorentzian surfaces with the constant Gaussian curvatures or the constant mean curvatures in the 3-dimensional Lorentzian space and their transformations. Such surfaces are associated to the Lorentzian Grassmannian systems and some transformations on such surfaces are given by dressing actions on those systems.

## 1. Introduction

Recently, the theory of surfaces and their transformations has been studied extensively in differential geometry. Classically, many mathematicians were interested in the surfaces in the 3-dimensional Euclidean space  $\mathbb{R}^3$  which have nice coordinate systems and have special kinds of the Gauss-Codazzi equations. Nowadays, the Gauss equations of such surfaces are known as soliton equations and their transformations can be explained by the dressing actions on the space of the solutions of the soliton equations. The sine-Gordon equations for surfaces with the negative constant curvatures and the sinh-Gordon equations for the constant mean curvature surfaces in  $\mathbb{R}^3$  are well-known examples.

These ideas on the surfaces in  $\mathbb{R}^3$  can be applied to the surfaces in the 3-dimensional Lorentzian space  $\mathbb{R}^{2,1}$  and many results have been obtained for the surfaces with the constant Gaussian curvatures or the constant mean curvatures in  $\mathbb{R}^{2,1}$  [2], [4], [5], [6], [8], [10]. It is well-known that some surfaces in  $\mathbb{R}^{2,1}$  with the constant curvatures correspond to the solutions of the sine-Gordon, the elliptic sine-Gordon, the sinh-Gordon or the hyperbolic sinh-Gordon equations.

In this paper, we investigate on the relation between Lorentzian Grassmannian systems and the Gauss equations of the constant curvature surfaces in  $\mathbb{R}^{2,1}$ , and obtain a special kind of transformations on them using the dressing actions on the solutions of the systems. Lorentzian Grassmannian systems are special cases of the so-called the  $n$ -dimensional system or  $G/K$ -system defined by Terng [9]. Some Grassmannian systems are associated to special surfaces

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in  $\mathbb{R}^3$  [1] and this idea can be applied to the Lorentzian cases, too. We also give examples of new solutions of the sine-Gordon, the sinh-Gordon, the elliptic sine-Gordon and the hyperbolic sinh-Gordon equations from the vacuum solution  $u = 0$  and the isometric immersions of the surfaces corresponding to them.

## 2. Preliminaries

The 3-dimensional Lorentzian space  $\mathbb{R}^{2,1}$  is the set  $\mathbb{R}^3$  with the nondegenerate metric  $\langle x, y \rangle = x_1y_1 + x_2y_2 - x_3y_3$ . First, we introduce basic knowledge and notations about the geometry of Lorentzian surfaces [7], [11].

Suppose  $X : M \rightarrow \mathbb{R}^{2,1}$  is an isometric immersion of a Lorentzian surface  $M$  into  $\mathbb{R}^{2,1}$ . Throughout this paper, we denote by  $e_1, e_2, e_3$  a local orthonormal frame on  $M$  such that  $e_3$  is normal to  $M$ . Put  $\epsilon_i = \langle e_i, e_j \rangle$ . Hence  $M$  is spacelike if  $\epsilon_3 = -1$  and timelike if  $\epsilon_3 = 1$ .

Take  $\omega_1, \omega_2$  the dual coframe to  $e_1, e_2$  on  $M$ , that is,  $\omega_i(e_j) = \langle e_i, e_j \rangle = \epsilon_i \delta_{ij}$ . Then

$$dX = \epsilon_1 e_1 \otimes \omega_1 + \epsilon_2 e_2 \otimes \omega_2.$$

The connection 1-forms  $\omega_{ij}$  are defined by

$$de_j = \sum_{i=1}^3 e_i \otimes \omega_{ij}, \quad 1 \leq j \leq 3,$$

which satisfies

$$\epsilon_i \omega_{ij} + \epsilon_j \omega_{ji} = 0.$$

Then the structure, the Gauss and the Codazzi equations become

$$\begin{aligned} (2.1) \quad & d\omega_1 + \omega_2 \wedge \omega_{21} = 0, & d\omega_2 + \omega_1 \wedge \omega_{12} = 0, \\ & d\omega_{12} + \omega_{13} \wedge \omega_{32} = 0, \\ & d\omega_{13} + \omega_{12} \wedge \omega_{23} = 0, & d\omega_{23} + \omega_{21} \wedge \omega_{13} = 0. \end{aligned}$$

The Gaussian curvature  $K$  of  $M$  is defined by

$$(2.2) \quad d\omega_{12} = K \epsilon_1 \omega_1 \wedge \omega_2.$$

From the Gauss equation (2.1) and (2.2), we obtain

$$K = \epsilon_3 \det A.$$

The mean curvature  $H$  of  $M$  is

$$H = \text{trace}(-de_3) = k_1 + k_2,$$

where  $k_1$  and  $k_2$  are the principal curvatures of  $M$ .

The first and the second fundamental forms on  $M$  are defined by  $I = \langle dX, dX \rangle$  and  $II = \langle dX, -de_3 \rangle$ . In particular, if  $e_1$  and  $e_2$  are principal directions, that is,  $-de_3(e_1) = k_1 e_1$  and  $-de_3(e_2) = k_2 e_2$ , then they become

$$I = \epsilon_1 \omega_1 \otimes \omega_1 + \epsilon_2 \omega_2 \otimes \omega_2, \quad II = \epsilon_1 k_1 \omega_1 \otimes \omega_1 + \epsilon_2 k_2 \omega_2 \otimes \omega_2.$$

It is well-known that a surface  $M$  with  $K = \pm 1$  or  $H = 1$  has a special coordinate system  $(x, y)$  and a function  $u(x, y)$  which satisfies a special kind of partial differential equation [6], [11]. More precisely,

(1) *When  $M^2$  is a spacelike surface  $\mathbb{R}^{2,1}$  with  $K = 1$ ,*

$$\begin{aligned} I &= \cos^2 u \, dx^2 + \sin^2 u \, dy^2, \\ II &= \sin u \cos u (dx^2 - dy^2), \end{aligned}$$

*and  $u(x, y)$  satisfies the sine-Gordon equation*

$$(2.3) \quad u_{xx} - u_{yy} = -\sin u \cos u.$$

(2) *When  $M^{1,1}$  is a timelike surface with  $K = -1$ ,*

$$\begin{aligned} I &= \cos^2 u \, dx^2 - \sin^2 u \, dy^2, \\ II &= \sin u \cos u (dx^2 + dy^2), \end{aligned}$$

*and  $u(x, y)$  satisfies the elliptic sine-Gordon equation*

$$(2.4) \quad u_{xx} + u_{yy} = \sin u \cos u.$$

(3) *When  $M^2$  is a spacelike surface with  $K = -1$ ,*

$$\begin{aligned} I &= \cosh^2 u \, dx^2 + \sinh^2 u \, dy^2, \\ II &= \sinh u \cosh u (dx^2 + dy^2), \end{aligned}$$

*and  $u(x, y)$  satisfies the sinh-Gordon equation*

$$(2.5) \quad u_{xx} + u_{yy} = \sinh u \cosh u.$$

(4) *When  $M^{1,1}$  is a timelike surface with  $K = 1$  which has the diagonalizable shape operator and has no umbilic points,*

$$\begin{aligned} I &= \cosh^2 u \, dx^2 - \sinh^2 u \, dy^2, \\ II &= \sinh u \cosh u (dx^2 - dy^2), \end{aligned}$$

*if the timelike principal curvature is bigger than that of the spacelike one, and*

$$\begin{aligned} I &= -\cosh^2 u \, dx^2 + \sinh^2 u \, dy^2, \\ II &= -\sinh u \cosh u (dx^2 - dy^2), \end{aligned}$$

*if the spacelike principal curvature is bigger.  $u(x, y)$  satisfies the hyperbolic sinh-Gordon equation*

$$(2.6) \quad u_{xx} - u_{yy} = \sinh u \cosh u.$$

Now, we introduce the  $G/K$ -system for a symmetric space  $G/K$ , which was defined by Terng [9]. Later, we will associate to Lorentzian surfaces with constant curvatures the solutions of appropriate Lorentzian Grassmannian  $G/K$ -systems.

Let  $G/K$  be a rank  $n$  symmetric space,  $\sigma : \mathcal{G} \rightarrow \mathcal{G}$  the corresponding involution on the Lie algebra  $\mathcal{G}$  of  $G$ ,  $\mathcal{G} = \mathcal{K} + \mathcal{P}$  the Cartan decomposition, and  $\mathcal{A} \subset \mathcal{P}$  a maximal abelian subalgebra. Let  $a_1, \dots, a_n$  be a basis for  $\mathcal{A}$ . Denote by  $\mathcal{A}^\perp$  the orthogonal complement of  $\mathcal{A}$  in  $\mathcal{G}$  with respect to the Killing form. Then  $G/K$  system for  $v : \mathbb{R}^n \rightarrow \mathcal{P} \cap \mathcal{A}^\perp$  is

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \quad 1 \leq i \neq j \leq n,$$

where,  $v_{x_i} = \frac{\partial v}{\partial x_i}$ .

It is easy to see that  $v$  is a solution for  $G/K$  system if and only if the following one-parameter family of  $\mathcal{G} \otimes \mathbb{C}$ -valued connections on  $\mathbb{R}^n$  is flat;

$$(2.7) \quad \theta_\lambda = \sum_{i=1}^n (a_i \lambda + [a_i, v]) dx_i.$$

A map  $E : \mathbb{R}^n \rightarrow G$  such that  $E^{-1}dE = \theta_\lambda$  is called a *trivialization* for  $v$ .

For later use, we recall the definitions of the following transformations [3].

**Definition 2.1.** Let  $M$  and  $\widetilde{M}$  be surfaces in  $\mathbb{R}^{2,1}$ .

(1) A *sphere congruence* is a diffeomorphism  $l : M \rightarrow \widetilde{M}$  such that the lines normal to  $M$  at  $p$  and  $\widetilde{M}$  at  $l(p)$  intersect at a point equidistant to  $p$  and  $l(p)$ .

(2) A sphere congruence  $l : M \rightarrow \widetilde{M}$  is called a *Ribaucour transformation* if any principal vector  $e_p$  to  $M$  is sent to a principal vector  $l_*(e_p)$  to  $\widetilde{M}$  and the lines in these directions at  $p$  and at  $l(p)$  intersect at a point equidistant to  $p$  and  $l(p)$ .

(3) A Ribaucour transformation  $l : M \rightarrow \widetilde{M}$  is called a *Darboux transformation* if  $l$  is a conformal diffeomorphism.

### 3. $O(4, 1)/O(2, 1) \times O(2)$ systems

Denote by  $\mathcal{M}_{m \times n}$  the set of  $m \times n$  matrices,  $\mathcal{O}$  a simply connected open subset in  $\mathbb{R}^2$  containing  $(0, 0)$ . Put  $J_1 = \text{diag}(1, 1, -1, 1, 1) \in \mathcal{M}_{5 \times 5}$ ,  $I_n$  the  $n \times n$  identity matrix and  $I_{m,n} = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$ .

Let  $O(4, 1) = \{A \in GL(5, \mathbb{R}) \mid A^t J_1 A = J_1\}$ . Define an involution  $\sigma$  by

$$\sigma(A) = \begin{pmatrix} I_{2,1} & 0 \\ 0 & -I_2 \end{pmatrix} A \begin{pmatrix} I_{2,1} & 0 \\ 0 & -I_2 \end{pmatrix}, \quad A \in O(4, 1).$$

Then the Cartan decomposition becomes  $\mathfrak{o}(4, 1) = \mathcal{K} + \mathcal{P}$ , where

$$\mathcal{K} = \mathfrak{o}(2, 1) \times \mathfrak{o}(2), \quad \mathcal{P} = \left\{ \begin{pmatrix} 0 & C \\ -C^t I_{2,1} & 0 \end{pmatrix} \mid C \in \mathcal{M}_{3 \times 2} \right\}.$$

Take a maximal abelian subalgebra  $\mathcal{A}_S \subset \mathcal{P}$  spanned by  $e_{41} - e_{14}$  and  $e_{52} - e_{25}$ , where  $e_{ij} \in \mathcal{M}_{5 \times 5}$  is defined by  $(e_{ij})_{ij} = 1$  and  $(e_{ij})_{kl} = 0$  for  $(k, l) \neq (i, j)$ . Put  $C = \begin{pmatrix} F \\ G \end{pmatrix}$ , where  $F \in \mathcal{M}_{2 \times 2}$  and  $G \in \mathcal{M}_{1 \times 2}$ . Then

the corresponding flat connection 1-form  $\theta_\lambda$  in (2.7) for  $O(4, 1)/O(2, 1) \times O(2)$  system becomes

$$(3.1) \quad \theta_\lambda = \begin{pmatrix} \delta F^t - F\delta & -\delta G^t & -\lambda\delta \\ -G\delta & 0 & 0 \\ \lambda\delta & 0 & \delta F - F^t\delta \end{pmatrix},$$

where  $\delta = \text{diag}(dx, dy)$ ,  $F \in \mathcal{M}_{2 \times 2}$  with  $f_{ii} = 0$ . Put

$$(3.2) \quad \omega = \begin{pmatrix} \delta F^t - F\delta & -\delta G^t \\ -G\delta & 0 \end{pmatrix} \in o(2, 1), \quad \eta = \delta F - F^t\delta \in o(2).$$

Since  $\theta_\lambda$  in (3.1) is flat for any  $\lambda$ , so are  $\omega$  and  $\eta$ . Thus there exist maps  $A : \mathcal{O} \rightarrow O(2, 1)$  and  $B : \mathcal{O} \rightarrow O(2)$  such that

$$(3.3) \quad A^{-1}dA = \omega, \quad B^{-1}dB = \eta.$$

Since  $B$  is  $O(2)$ -valued, we may assume

$$B = \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix}.$$

Then by (3.2) and (3.3), we have

$$\eta = \begin{pmatrix} 0 & f_{12}dx - f_{21}dy \\ -f_{12}dx + f_{21}dy & 0 \end{pmatrix} = \begin{pmatrix} 0 & du \\ -du & 0 \end{pmatrix}$$

and thus  $f_{12} = u_x$  and  $f_{21} = -u_y$ . Hence we will say  $(u, g_1, g_2)$  is a *solution for  $O(4, 1)/O(2, 1) \times O(2)$  system of the spacelike type* when  $\theta_\lambda$  in (3.1) is flat, where  $f_{12} = u_x$ ,  $f_{21} = -u_y$  and  $G = (g_1, g_2)$ .

**Proposition 3.1.** *Suppose  $(u, g_1, g_2)$  is a solution for  $O(4, 1)/O(2, 1) \times O(2)$  system of the spacelike type. Then*

(i)

$$\begin{cases} (g_1)_y = -u_y g_2, \\ (g_2)_x = u_x g_1, \\ u_{xx} - u_{yy} = g_1 g_2. \end{cases}$$

(ii) *There exists a map  $X : \mathcal{O} \rightarrow \mathcal{M}_{3 \times 2}$  such that  $dX = A \begin{pmatrix} \delta \\ 0 \end{pmatrix} B^t$ , where  $A$  and  $B$  are given by (3.3).*

(iii)  *$X$  can be obtained by the Sym's formula*

$$\frac{\partial E}{\partial \lambda} \cdot E^{-1} \Big|_{\lambda=0} = \begin{pmatrix} 0 & -X \\ X^t I_{2,1} & 0 \end{pmatrix},$$

where  $E$  is a trivialization for  $\theta_\lambda$  in (3.1).

(iv) *Put  $A = (e_1, e_2, e_3) \in \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \times \mathbb{R}^{2,1}$  and  $X = (X_1, X_2) \in \mathbb{R}^{2,1} \times \mathbb{R}^{2,1}$ . Then  $e_1, e_2, e_3$  are a local orthonormal frame for spacelike immersions  $X_1$  and*

$X_2$ , and

$$\begin{cases} dX_1 = e_1 \otimes \cos u \, dx + e_2 \otimes \sin u \, dy, \\ dX_2 = -e_1 \otimes \sin u \, dx + e_2 \otimes \cos u \, dy, \\ -de_3 = e_1 \otimes g_1 \, dx + e_2 \otimes g_2 \, dy. \end{cases}$$

(v)  $X_1$  and  $X_2$  have the Gaussian curvatures

$$K_1 = -K_2 = -\frac{g_1 g_2}{\sin u \cos u}.$$

*Proof.* (i) follows from the flatness of  $\omega$ . Taking a gauge transformation on  $\theta_\lambda$  by  $h = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ , we obtain a flat connection

$$h * \theta_\lambda = \begin{pmatrix} 0 & -\lambda A \begin{pmatrix} \delta \\ 0 \end{pmatrix} B^t \\ \lambda B (\delta \ 0) A^{-1} & 0 \end{pmatrix}.$$

Since  $h * \theta_\lambda$  is flat, the (1,2) block is closed so that  $dX = A \begin{pmatrix} \delta \\ 0 \end{pmatrix} B^t$  for some  $X$ . The left ones can be proved by direct calculations.  $\square$

We can associate a spacelike  $K = 1$  surface in  $\mathbb{R}^{2,1}$  to some solution for this system of the spacelike type.

**Theorem 3.2.** *( $u, \pm \sin u, \mp \cos u$ ) is a solution for  $O(4, 1)/O(2, 1) \times O(2)$  system of the spacelike type if and only if  $u$  is a solution of the sine-Gordon equation (2.3) and  $X_1$  defined in Proposition 3.1 (iv) gives an isometric immersion of a spacelike surface  $M^2$  with  $K = 1$  into  $\mathbb{R}^{2,1}$ . In this case,  $X_1$  has the fundamental forms as*

$$(3.4) \quad \begin{aligned} I &= \cos^2 u \, dx^2 + \sin^2 u \, dy^2, \\ II &= \pm \sin u \cos u (dx^2 - dy^2). \end{aligned}$$

*Proof.* Put  $(g_1, g_2) = (\pm \sin u, \mp \cos u)$ . Then  $X_1$  has  $K = 1$  by Proposition 3.1. Conversely, suppose  $X_1 : M^2 \rightarrow \mathbb{R}^{2,1}$  is an isometric immersion of a spacelike surface  $M$  with  $K = 1$  into  $\mathbb{R}^{2,1}$ . Then there exist a coordinate system  $(x, y)$  and a function  $u$  satisfying (3.4) by Proposition 2.1, and it is easy to show that  $(u, \pm \sin u, \mp \cos u)$  is a solution for this system of the spacelike type.  $\square$

**Remark 3.3.** In Theorem 3.2, the map  $X_2$  associated to  $(u, \sin u, -\cos u)$  has fundamental forms

$$\begin{aligned} I &= \sin^2 u \, dx^2 + \cos^2 u \, dy^2, \\ II &= -\sin^2 u \, dx^2 - \cos u \, dy^2, \end{aligned}$$

so that it gives a totally umbilic spacelike surface with  $K = -1$ .

Now, we consider the timelike surface in  $\mathbb{R}^{2,1}$  with  $K = -1$ . In the Cartan decomposition of  $\mathfrak{o}(4, 1) = \mathcal{K} + \mathcal{P}$ , take a maximal subalgebra  $\mathcal{A}_T$  of  $\mathcal{P}$  spanned

by  $e_{42} - e_{24}$  and  $e_{53} + e_{35}$ , and put  $C = \begin{pmatrix} G \\ F \end{pmatrix}$ , where  $G \in \mathcal{M}_{1 \times 2}$  and  $F \in \mathcal{M}_{2 \times 2}$ . Then the corresponding flat connection for  $O(4, 1)/O(2, 1) \times O(2)$  is of the form

$$(3.5) \quad \theta_\lambda = \begin{pmatrix} 0 & -G\delta & 0 \\ I_{1,1}\delta G^t & I_{1,1}\delta F^t I_{1,1} - F\delta & -\lambda\delta I_{1,1} \\ 0 & \lambda\delta & \delta F - F^t\delta \end{pmatrix},$$

where  $\delta = \text{diag}(dx, dy)$ ,  $F \in \mathcal{M}_{2 \times 2}$  with  $f_{ii} = 0$  and  $G = (g_1, g_2)$ . A similar argument as before shows that  $f_{12} = u_x$  and  $f_{21} = -u_y$  for some function  $u$ . Again, we will call  $(u, g_1, g_2)$  is a *solution for  $O(4, 1)/O(2, 1) \times O(2)$  system of the timelike type* when  $\theta_\lambda$  in (3.5) is flat. As does in the case of the spacelike type, the following holds.

**Proposition 3.4.** *Suppose  $(u, g_1, g_2)$  is a solution for system*

$$O(4, 1)/O(2, 1) \times O(2)$$

*of the timelike type. Then*

(i)

$$\begin{cases} (g_1)_y = -u_y g_2, \\ (g_2)_x = u_x g_1, \\ u_{xx} + u_{yy} + g_1 g_2 = 0. \end{cases}$$

(ii) *There exists a map  $X : \mathcal{O} \rightarrow \mathcal{M}_{3 \times 2}$  such that  $dX = A \begin{pmatrix} 0 \\ 0 \end{pmatrix} I_{1,1} B^t$ , where  $A \in O(2, 1)$  and  $B = \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix}$  satisfy*

$$A^{-1}dA = \begin{pmatrix} 0 & -G\delta \\ I_{1,1}\delta G^t & I_{1,1}\delta F^t I_{1,1} - F\delta \end{pmatrix} \in o(2, 1), \quad B^{-1}dB = \delta F - F^t\delta \in o(2).$$

(iii)

$$\left. \frac{\partial E}{\partial \lambda} \cdot E^{-1} \right|_{\lambda=0} = \begin{pmatrix} 0 & -X \\ X^t I_{2,1} & 0 \end{pmatrix},$$

where  $E$  is a trivialization for  $\theta_\lambda$  in (3.5).

(iv) *Put  $A = (e_3, e_1, e_2) \in \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \times \mathbb{R}^{2,1}$  and  $X = (X_1, X_2) \in \mathbb{R}^{2,1} \times \mathbb{R}^{2,1}$ . Then  $e_1, e_2, e_3$  are a local orthonormal frame for  $X_1$  and  $X_2$ , where  $\epsilon_2 = -1$  and*

$$\begin{cases} dX_1 = e_1 \otimes \cos u \, dx - e_2 \otimes \sin u \, dy, \\ dX_2 = -e_1 \otimes \sin u \, dx - e_2 \otimes \cos u \, dy, \\ -de_3 = -e_1 \otimes g_1 \, dx + e_2 \otimes g_2 \, dy. \end{cases}$$

(v)  $X_1$  and  $X_2$  have the Gaussian curvatures

$$K_1 = -K_2 = \frac{g_1 g_2}{\sin u \cos u}.$$

We now associate a timelike  $K = -1$  surface in  $\mathbb{R}^{2,1}$  to some solution for this system of the timelike type.

**Theorem 3.5.**  $(u, \sin u, -\cos u)$  is a solution for  $O(4, 1)/O(2, 1) \times O(2)$  system of the timelike type if and only if  $u$  is a solution of the elliptic sine-Gordon equation (2.4) and  $X_1$  defined in Proposition 3.4 (iv) gives an isometric immersion of a timelike surface  $M^{1,1}$  with  $K = -1$  into  $R^{2,1}$ . In this case,  $X_1$  has the fundamental forms as

$$(3.6) \quad \begin{aligned} I &= \cos^2 u \, dx^2 - \sin^2 u \, dy^2, \\ II &= -\sin u \cos u (dx^2 + dy^2). \end{aligned}$$

*Proof.* It can be proved by a similar way as Theorem 3.2.  $\square$

*Remark 3.6.* In Theorem 3.5, the map  $X_2$  associated to  $(u, \sin u, -\cos u)$  has fundamental forms

$$\begin{aligned} I &= \sin^2 u \, dx^2 - \cos^2 u \, dy^2, \\ II &= \sin^2 u \, dx^2 - \cos^2 u \, dy^2, \end{aligned}$$

so that it gives a totally umbilic timelike surface with  $K = 1$ .

From now on, we investigate on the construction of an action on the solutions for  $O(4, 1)/O(2, 1) \times O(2)$  system (cf. [1]).

Consider the bilinear form  $\langle \cdot, \cdot \rangle_1$  on  $\mathbb{C}^5$  given by  $\langle U, V \rangle_1 = \bar{U}^t J_1 V$ . Let  $W = (w_1, w_2, w_3)^t$  and  $Z = (z_1, z_2)^t$  be unit vectors in  $\mathbb{R}^{2,1}$  and in  $\mathbb{R}^2$ , respectively, and let  $\pi$  be the orthogonal projection of  $\mathbb{C}^5$  onto  $\mathbb{C} \begin{pmatrix} W \\ iZ \end{pmatrix}$  with respect to  $\langle \cdot, \cdot \rangle_1$ . So,  $\pi = \frac{1}{2} \begin{pmatrix} WW^t I_{2,1} & -iWZ^t \\ iZ\bar{W}^t & ZZ^t \end{pmatrix}$ . For  $0 \neq s \in \mathbb{R}$ , define

$$(3.7) \quad q_{s,\pi}(\lambda) = \left( \pi + \frac{\lambda - is}{\lambda + is} (I - \pi) \right) \left( \bar{\pi} + \frac{\lambda + is}{\lambda - is} (I - \bar{\pi}) \right).$$

Let  $(F, G)$  be the solution for  $O(4, 1)/O(2, 1) \times O(2)$  system of the spacelike type,  $\theta_\lambda$  the corresponding flat connection as in (3.1), and  $E$  be a trivialization of  $(F, G)$ , i.e.,  $E^{-1}dE = \theta_\lambda$ .

Put  $\begin{pmatrix} \bar{W} \\ i\bar{Z} \end{pmatrix}(x, y) = E(x, y, -is)^{-1} \begin{pmatrix} W \\ iZ \end{pmatrix}$  and let  $\tilde{\pi}(x, y)$  be the orthogonal projection onto  $\mathbb{C} \begin{pmatrix} \bar{W} \\ i\bar{Z} \end{pmatrix}$ . Denote by

$$(3.8) \quad \tilde{E}(x, y, \lambda) = E(x, y, \lambda) q_{s,\tilde{\pi}}(\lambda)^{-1}, \quad \widehat{W} = \tilde{W} / \|\tilde{W}\|_{2,1}, \quad \widehat{Z} = \tilde{Z} / \|\tilde{Z}\|_2.$$

**Lemma 3.6.** *The following holds.*

(i) For  $\tilde{E}$  given by (3.8),  $\tilde{\theta}_\lambda = \tilde{E}^{-1}d\tilde{E}$  is of the form (3.1) and gives a new solution  $(\tilde{F}, \tilde{G})$  for  $O(4, 1)/O(2, 1) \times O(2)$  system of the spacelike type.

(ii)

$$E(x, y, 0) = \begin{pmatrix} A(x, y) & 0 \\ 0 & B(x, y) \end{pmatrix}, \quad \tilde{E}(x, y, 0) = \begin{pmatrix} \tilde{A}(x, y) & 0 \\ 0 & \tilde{B}(x, y) \end{pmatrix}$$

for some  $A, \tilde{A} \in O(2, 1)$  and  $B, \tilde{B} \in O(2)$ , and

$$(3.9) \quad \tilde{A} = A(I - 2\widehat{W}\widehat{W}^t I_{2,1}), \quad \tilde{B} = B(I - 2\widehat{Z}\widehat{Z}^t).$$



(iii)

$$(3.10) \quad \begin{pmatrix} \widetilde{F} \\ \widetilde{G} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} - 2s(\widehat{W}\widehat{Z}^t)_*,$$

where  $(c_{ij})_*$  means  $c_{ii} = 0$ .

 (iv)  $(\widetilde{W}, \widetilde{Z})$  is a solution of

$$(3.11) \quad d \begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix} = -\theta_{-is} \begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix},$$

 (v)  $X, \widetilde{X} : \mathcal{O} \rightarrow \mathcal{M}_{3 \times 2}$  given by

$$\frac{\partial E}{\partial \lambda} \cdot E^{-1} \Big|_{\lambda=0} = \begin{pmatrix} 0 & -X \\ X^t I_{2,1} & 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \widetilde{E}}{\partial \lambda} \cdot \widetilde{E}^{-1} \Big|_{\lambda=0} = \begin{pmatrix} 0 & -\widetilde{X} \\ \widetilde{X}^t I_{2,1} & 0 \end{pmatrix}$$

satisfy

$$(3.12) \quad \widetilde{X} = X - \frac{2}{s} A \widehat{W} \widehat{Z}^t B^t.$$

(vi)

$$d\widetilde{X} = \widetilde{A} \begin{pmatrix} \delta \\ 0 \end{pmatrix} \widetilde{B}^{-1}.$$

*Proof.* see [1]. □

This action by  $q_{s,\pi}(\lambda)$  of the form (3.7) on the immersions  $X = (X_1, X_2)$  can be interpreted as a geometric transformation.

**Proposition 3.7.** *Let  $X = (X_1, X_2)$  and  $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)$  be given by (v) in the Lemma 3.6. Then  $X_i$  and  $\widetilde{X}_i$  are in a Ribaucour transformation. In particular,*

$$X_j + \psi_{ij} e_i = \widetilde{X}_j + \psi_{ij} \widetilde{e}_i,$$

where  $\psi_{ij} = -\frac{\epsilon_i}{s\widehat{w}_i} (\widehat{Z}^t B^t)_{1j}$ ,  $\epsilon_1 = \epsilon_2 = -\epsilon_3 = 1$ ,  $A = (e_1, e_2, e_3)$ ,  $\widetilde{A} = (\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3)$ .

*Proof.* From (3.9), we have  $\widetilde{e}_i = e_i - 2\epsilon_i \widehat{w}_i A \widehat{W}$ . Comparing this fact with (3.12), we get the desired result. □

From the above argument, we can obtain the following.

**Theorem 3.8.** *Suppose  $(u, \sin u, -\cos u)$  is a solution corresponding to  $\theta_\lambda$  in (3.1). Let  $X = (X_1, X_2)$ ,  $q_{s,\pi}(\lambda)$ ,  $E, W, Z, \widetilde{E}, \widetilde{W}, \widetilde{Z}, \widetilde{\theta}_\lambda$  and  $\widetilde{X}$  be defined as in Lemma 3.6. If  $s w_3 = z_1 \sin u(0, 0) - z_2 \cos u(0, 0)$ , then  $(\widetilde{u}, -\sin \widetilde{u}, \cos \widetilde{u})$  is a solution corresponding to  $\widetilde{\theta}_\lambda$ , where  $\widetilde{u} = 2\alpha - u$  and  $(\widehat{z}_1, \widehat{z}_2) = (-\sin \alpha, \cos \alpha)$ . In this case,*

$$\widetilde{X}_1 = X_1 - \frac{2}{s} (\widehat{z}_1 \cos u + \widehat{z}_2 \sin u) \sum_{i=1}^3 \widehat{w}_i e_i.$$

The fundamental forms of  $\tilde{X}_1$  are

$$\begin{aligned}\tilde{I} &= \cos^2 \tilde{u} dx^2 + \sin^2 \tilde{u} dy^2, \\ \tilde{II} &= -\sin \tilde{u} \cos \tilde{u} (dx^2 - dy^2),\end{aligned}$$

and the immersions  $X_1$  and  $\tilde{X}_1$  are  $K = 1$  spacelike surfaces and they are in a Ribaucour transformation.

*Proof.* From (3.11), we have

$$\begin{cases} d\tilde{w}_1 = -\tilde{w}_2 \omega_{12} + \tilde{w}_3 \sin u dx + s\tilde{z}_1 dx, \\ d\tilde{w}_2 = -\tilde{w}_1 \omega_{21} - \tilde{w}_3 \cos u dy - s\tilde{z}_2 dy, \\ d\tilde{w}_3 = \tilde{w}_1 \sin u dx - \tilde{w}_2 \cos u dy, \\ d\tilde{z}_1 = s\tilde{w}_1 dx - \tilde{z}_2 du, \\ d\tilde{z}_2 = s\tilde{w}_2 dy - \tilde{z}_1 du. \end{cases}$$

Thus  $d(s\tilde{w}_3 - \tilde{z}_1 \sin u + \tilde{z}_2 \cos u) = 0$  and by the initial condition, we have

$$s\tilde{w}_3 = \sin u \tilde{z}_1 - \cos u \tilde{z}_2,$$

or,  $s\hat{w}_3 = \sin u \hat{z}_1 - \cos u \hat{z}_2 = -\cos(u - \alpha)$ . Hence by (3.11), we get

$$\begin{cases} \tilde{g}_1 = \sin u - 2s\hat{w}_3 \hat{z}_1 = \sin u - 2\cos(u - \alpha) \sin \alpha = -\sin \tilde{u}, \\ \tilde{g}_2 = -\cos u - 2s\hat{w}_3 \hat{z}_2 = -\cos u + 2\cos(u - \alpha) \cos \alpha = \cos \tilde{u}, \end{cases}$$

and thus  $(\tilde{u}, -\sin \tilde{u}, \cos \tilde{u})$  is a new solution of the spacelike type, which implies that  $\tilde{X}_2$  have  $K = 1$ . From (3.9),

$$\begin{aligned}\tilde{B} &= B(I - \hat{Z}\hat{Z}^t) \\ &= \begin{pmatrix} \cos(2\alpha - u) & \sin(2\alpha - u) \\ \sin(2\alpha - u) & -\cos(2\alpha - u) \end{pmatrix} = \begin{pmatrix} \cos \tilde{u} & \sin \tilde{u} \\ \sin \tilde{u} & -\cos \tilde{u} \end{pmatrix},\end{aligned}$$

and thus  $\tilde{u} = 2\alpha - u$ . The formula for  $\tilde{X}_1$  comes from (3.12) and the fact that  $X_1$  and  $\tilde{X}_1$  are in a Ribaucour transformation follows from the Proposition 3.7.  $\square$

We now consider an action for the timelike type. In this case, all the properties in Lemma 3.6 hold except the following;

**Lemma 3.9.** *Let  $(G, F)$  be the solution for  $O(4, 1)/O(2, 1) \times O(2)$  system of the timelike type. Then the action  $q_{s, \pi}(\lambda)$  defined by (3.7) on  $(G, F)$  gives a new solution for  $O(4, 1)/O(2, 1) \times O(2)$  system of the timelike type*

$$(3.13) \quad \begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} = \begin{pmatrix} G \\ F \end{pmatrix} - 2s(\widehat{W}\widehat{Z}^t)_{*'} ,$$

where  $(c_{ij})_{*'}_1$  means  $c_{21} = c_{32} = 0$ . Moreover,  $d\tilde{X} = \tilde{A} \begin{pmatrix} 0 \\ \delta \end{pmatrix} I_{1,1} \tilde{B}^{-1}$ .

**Proposition 3.10.** *For the action  $q_{s,\pi}(\lambda)$  on the solution  $(G, F)$  of the time-like type for  $O(4, 1)/O(2, 1) \times O(2)$  system, let  $X = (X_1, X_2)$  and  $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$  be given as in the Lemma 3.6 (v). Then  $X_i$  and  $\tilde{X}_i$  are in a Ribaucour transformation. In particular,*

$$X_j + \psi_{2j} e_1 + \psi_{3j} e_2 + \psi_{1j} e_3 = \tilde{X}_j + \psi_{2j} \tilde{e}_1 + \psi_{3j} \tilde{e}_2 + \psi_{1j} \tilde{e}_3,$$

where  $\psi_{ij} = -\frac{\epsilon_i}{s\tilde{w}_i} (\tilde{Z}^t B^t)_{1j}$ ,  $\epsilon_1 = -\epsilon_2 = \epsilon_3 = 1$ ,  $A = (e_3, e_1, e_2)$ ,  $\tilde{A} = (\tilde{e}_3, \tilde{e}_1, \tilde{e}_2)$ .

*Proof.* In this case, (3.9) and (3.12) still hold. Now the results follow from these fact.  $\square$

**Theorem 3.11.** *Suppose  $(u, \sin u, -\cos u)$  is a solution corresponding to  $\theta_\lambda$  in (3.5). Let  $X = (X_1, X_2)$ ,  $q_{s,\pi}(\lambda)$ ,  $E, W, Z, \tilde{E}, \tilde{W}, \tilde{Z}, \tilde{\theta}_\lambda$  and  $\tilde{X}$  be defined as in the Lemma 3.6. If  $s w_3 = z_1 \sin u(0, 0) - z_2 \cos u(0, 0)$ , then  $(\tilde{u}, -\sin \tilde{u}, \cos \tilde{u})$  is a solution corresponding to  $\tilde{\theta}_\lambda$ , where  $\tilde{u} = 2\alpha - u$  and  $(\tilde{z}_1, \tilde{z}_2) = (-\sin \alpha, \cos \alpha)$ . In this case,*

$$\tilde{X}_1 = X_1 - \frac{2}{s} (\tilde{z}_1 \cos u + \tilde{z}_2 \sin u) (\hat{w}_2 e_1 + \hat{w}_3 e_2 + \hat{w}_1 e_3).$$

The fundamental forms of  $\tilde{X}_1$  are

$$\begin{aligned} \tilde{I} &= \cos^2 \tilde{u} dx^2 - \sin^2 \tilde{u} dy^2, \\ \tilde{II} &= \sin \tilde{u} \cos \tilde{u} (dx^2 + dy^2). \end{aligned}$$

The immersions  $X_1$  and  $\tilde{X}_1$  are  $K = -1$  timelike surfaces and they are in a Ribaucour transformation.

*Proof.* It follows from Proposition 3.10 by putting  $g_1 = \sin u$  and  $g_2 = \cos u$ .  $\square$

#### 4. $O(3, 2)/O(2, 1) \times O(1, 1)$ systems

We now associate another kind of constant curvature surfaces in  $\mathbb{R}^{2,1}$  to some Lorentzian Grassmannian systems. Put  $J_{1,1} = \text{diag}(1, 1, -1, 1, -1) \in \mathcal{M}_{5 \times 5}$ , and let  $O(3, 2) = \{A \in GL(5, \mathbb{R}) \mid A^t J_{1,1} A = J_{1,1}\}$ . Define an involution  $\sigma$  by

$$\sigma(A) = \begin{pmatrix} I_{2,1} & 0 \\ 0 & -I_2 \end{pmatrix} A \begin{pmatrix} I_{2,1} & 0 \\ 0 & -I_2 \end{pmatrix}, \quad A \in O(3, 2).$$

Then the Cartan decomposition becomes  $o(3, 2) = \mathcal{K} + \mathcal{P}$ , where

$$\mathcal{K} = o(2, 1) \times o(1, 1), \quad \mathcal{P} = \left\{ \begin{pmatrix} 0 & C \\ -I_{1,1} C^t I_{2,1} & 0 \end{pmatrix} \mid C \in \mathcal{M}_{3 \times 2} \right\}.$$

Take a maximal abelian subalgebra  $\mathcal{A}_S \subset \mathcal{P}$  spanned by  $e_{41} - e_{14}$  and  $e_{52} + e_{25}$ , and put  $C = \begin{pmatrix} F \\ G \end{pmatrix}$ , where  $F \in \mathcal{M}_{2 \times 2}$  and  $G \in \mathcal{M}_{1 \times 2}$ . Then the

corresponding flat connection 1-form  $\theta_\lambda$  in (2.7) to  $O(3,2)/O(2,1) \times O(1,1)$  system becomes

$$(4.1) \quad \theta_\lambda = \begin{pmatrix} \delta F^t - F\delta & -\delta G^t & -\lambda I_{1,1}\delta \\ -G\delta & 0 & 0 \\ \lambda\delta & 0 & \delta F + F^t\delta \end{pmatrix},$$

where  $\delta = \text{diag}(dx, dy)$ ,  $F \in \mathcal{M}_{2 \times 2}$  with  $f_{ii} = 0$ . Here,

$$(4.2) \quad \omega = \begin{pmatrix} \delta F^t - F\delta & -\delta G^t \\ -G\delta & 0 \end{pmatrix} \in o(2,1), \quad \eta = \delta F + F^t\delta \in o(1,1),$$

are flat connections so that there exist maps  $A \in O(2,1)$  and  $B = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}$  such that

$$(4.3) \quad A^{-1}dA = \omega, \quad B^{-1}dB = \eta.$$

It is easy to show that  $f_{12} = u_x$  and  $f_{21} = u_y$  and thus we will say that  $(u, g_1, g_2)$  is a solution for  $O(3,2)/O(2,1) \times O(1,1)$  system of the spacelike type when  $\theta_\lambda$  in (4.1) is flat.

**Proposition 4.1.** *Suppose  $(u, g_1, g_2)$  is a solution for  $O(3,2)/O(2,1) \times O(1,1)$  system of the spacelike type. Then*

(i)

$$\begin{cases} (g_1)_y = u_y g_2, \\ (g_2)_x = u_x g_1, \\ u_{xx} + u_{yy} = g_1 g_2. \end{cases}$$

(ii) *There exists a map  $X : \mathcal{O} \rightarrow \mathcal{M}_{3 \times 2}$  such that  $dX = A \begin{pmatrix} \delta \\ 0 \end{pmatrix} B I_{1,1}$ , where  $A$  and  $B$  are given by (4.3).*

(iii)

$$\left. \frac{\partial E}{\partial \lambda} \cdot E^{-1} \right|_{\lambda=0} = \begin{pmatrix} 0 & -X \\ I_{1,1} X^t I_{2,1} & 0 \end{pmatrix},$$

where  $E$  is a trivialization for  $\theta_\lambda$  in (4.1).

(iv) *Put  $A = (e_1, e_2, e_3) \in \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \times \mathbb{R}^{2,1}$  and  $X = (X_1, X_2) \in \mathbb{R}^{2,1} \times \mathbb{R}^{2,1}$ . Then  $e_1, e_2, e_3$  are a local orthonormal frame for spacelike immersions  $X_1$  and  $X_2$ , and*

$$\begin{cases} dX_1 = e_1 \otimes \cosh u \, dx + e_2 \otimes \sinh u \, dy, \\ dX_2 = -(e_1 \otimes \sinh u \, dx + e_2 \otimes \cosh u \, dy), \\ -de_3 = e_1 \otimes g_1 \, dx + e_2 \otimes g_2 \, dy. \end{cases}$$

(v)  $X_1$  and  $X_2$  have the Gaussian curvatures

$$K_1 = K_2 = -\frac{g_1 g_2}{\sinh u \cosh u}.$$

*Proof.* Since the proof is similar to Proposition 3.1, we omit the details.  $\square$

Now, we associate a spacelike  $K = -1$  surface in  $\mathbb{R}^{2,1}$  to some solution for this system of the spacelike type. The following theorems are obtained easily by Proposition 4.1 and we just state without proofs.

**Theorem 4.2.**  *$(u, \pm \cosh u, \pm \sinh u)$  is a solution for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system of the spacelike type if and only if  $u$  is a solution of the sinh-Gordon equation (2.5) and  $X_2$  defined in Proposition 4.1 (iv) gives an isometric immersion of a spacelike surface  $M^2$  with  $K = -1$  into  $R^{2,1}$  of which has no umbilic points. In this case,  $X_2$  has the fundamental forms as*

$$\begin{aligned} I &= \sinh^2 u \, dx^2 + \cosh^2 u \, dy^2, \\ II &= \pm \sinh u \cosh u (dx^2 + dy^2). \end{aligned}$$

Moreover, the maps  $Y_1 = X_1 - X_2$  and  $Y_2 = X_1 + X_2$  give spacelike surfaces with the constant mean curvatures  $H = \pm 1$  whose fundamental forms are

$$\begin{aligned} I_1 &= e^{2u}(dx^2 + dy^2), & II_1 &= \pm e^u(\cosh u \, dx^2 + \sinh u \, dy^2), \\ I_2 &= e^{-2u}(dx^2 + dy^2), & II_2 &= \pm e^{-u}(\cosh u \, dx^2 - \sinh u \, dy^2). \end{aligned}$$

**Theorem 4.3.**  *$(u, \sinh u, \cosh u)$  is a solution for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system of the spacelike type if and only if  $u$  is a solution of the sinh-Gordon equation (2.5) and  $X_1$  defined in Proposition 4.1 (iv) gives an isometric immersion of a spacelike surface  $M^2$  with  $K = -1$  into  $R^{2,1}$  which has no umbilic points. In this case,  $X_1$  has the fundamental forms as*

$$\begin{aligned} I &= \cosh^2 u \, dx^2 + \sinh^2 u \, dy^2, \\ II &= \sinh u \cosh u (dx^2 + dy^2). \end{aligned}$$

Moreover, the maps  $Y_1 = X_1 - X_2$  and  $Y_2 = X_1 + X_2$  give spacelike surfaces with  $H_1 = 1$  and  $H_2 = -1$  respectively, whose fundamental forms are

$$\begin{aligned} I_1 &= e^{2u}(dx^2 + dy^2), & II_1 &= e^u(\sinh u \, dx^2 + \cosh u \, dy^2), \\ I_2 &= e^{-2u}(dx^2 + dy^2), & II_2 &= e^{-u}(\sinh u \, dx^2 - \cosh u \, dy^2). \end{aligned}$$

We now deal with the timelike surface in  $\mathbb{R}^{2,1}$  with  $K = 1$  whose shape operator is diagonalizable. In the Cartan decomposition of  $o(3, 2) = \mathcal{K} + \mathcal{P}$ , take a maximal subalgebra  $\mathcal{A}_T$  of  $\mathcal{P}$  spanned by  $e_{42} - e_{24}$  and  $e_{53} - e_{35}$ , and put  $C = \begin{pmatrix} G \\ F \end{pmatrix}$ , where  $G \in \mathcal{M}_{1 \times 2}$  and  $F \in \mathcal{M}_{2 \times 2}$ . Then the corresponding flat connection for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system is of the form

$$(4.4) \quad \theta_\lambda = \begin{pmatrix} 0 & -G\delta_1 & 0 \\ I_{1,1}\delta_1 G^t & -\delta_1 F^t - F\delta_1 & -\lambda\delta_1 \\ 0 & \lambda\delta_1 & \delta_1 F + F^t\delta_1 \end{pmatrix},$$

where  $\delta_1 = \text{diag}(dy, dx)$ ,  $F \in \mathcal{M}_{2 \times 2}$  with  $f_{ii} = 0$ . Again,  $f_{12} = u_y$  and  $f_{21} = u_x$  for some function  $u$ . We will call  $(u, g_1, g_2)$  is a solution for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system of the timelike type when  $\theta_\lambda$  in (4.4) is flat.

**Proposition 4.4.** *Suppose  $(u, g_1, g_2)$  is a solution for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system of the timelike type. Then*

(i)

$$\begin{cases} (g_1)_x = u_x g_2, \\ (g_2)_y = u_y g_1, \\ u_{xx} - u_{yy} = g_1 g_2. \end{cases}$$

(ii) *There exists a map  $X : \mathcal{O} \rightarrow \mathcal{M}_{3 \times 2}$  such that  $dX = A \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} I_{1,1} B I_{1,1}$ , where  $A \in O(2, 1)$  and  $B = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}$  satisfy*

$$A^{-1}dA = \begin{pmatrix} 0 & -G\delta_1 \\ I_{1,1}\delta_1 G^t & -\delta_1 F^t - F\delta_1 \end{pmatrix} \in o(2, 1), \quad B^{-1}dB = \delta_1 F + F^t \delta_1 \in o(1, 1).$$

(iii)

$$\frac{\partial E}{\partial \lambda} \cdot E^{-1} \Big|_{\lambda=0} = \begin{pmatrix} 0 & -X \\ I_{1,1} X^t I_{2,1} & 0 \end{pmatrix},$$

where  $E$  is a trivialization for  $\theta_\lambda$  in (4.4).

(iv) *Put  $A = (e_3, e_2, e_1) \in \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \times \mathbb{R}^{2,1}$  and  $X = (X_1, X_2) \in \mathbb{R}^{2,1} \times \mathbb{R}^{2,1}$ . Then  $e_1, e_2, e_3$  are a local orthonormal frame for  $X_1$  and  $X_2$ , where  $\epsilon_1 = -1$  and*

$$\begin{cases} dX_1 = -e_1 \otimes \sinh u \, dx + e_2 \otimes \cosh u \, dy, \\ dX_2 = e_1 \otimes \cosh u \, dx - e_2 \otimes \sinh u \, dy, \\ -de_3 = e_1 \otimes g_2 \, dx - e_2 \otimes g_1 \, dy. \end{cases}$$

(v)  $X_1$  and  $X_2$  have the Gaussian curvatures

$$K_1 = K_2 = \frac{g_1 g_2}{\sinh u \cosh u}.$$

We can associate a nonumbilic timelike  $K = 1$  surface in  $\mathbb{R}^{2,1}$  which has a diagonalizable shape operator to some solution for this system of the timelike type.

**Theorem 4.5.**  *$(u, \pm \cosh u, \pm \sinh u)$  is a solution for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system of the timelike type if and only if  $u$  is a solution of the hyperbolic sinh-Gordon equation (2.6) and  $X_2$  defined in Proposition 4.4 (iv) gives an isometric immersion of a timelike surface  $M^{1,1}$  with  $K = 1$  into  $\mathbb{R}^{2,1}$  of which the space-like principal curvature is bigger. In this case,  $X_2$  has the fundamental forms as*

$$\begin{aligned} I &= -\cosh^2 u \, dx^2 + \sinh^2 u \, dy^2, \\ II &= \mp \sinh u \cosh u \, (dx^2 - dy^2). \end{aligned}$$

Moreover, the maps  $Y_1 = X_1 - X_2$  and  $Y_2 = X_1 + X_2$  give timelike surfaces with  $H = \mp 1$ , whose fundamental forms are

$$\begin{aligned} I_1 &= e^{2u}(-dx^2 + dy^2), & II_1 &= \pm e^u(\sinh u \, dx^2 - \cosh u \, dy^2), \\ I_2 &= e^{-2u}(-dx^2 + dy^2), & II_2 &= \mp e^{-u}(\sinh u \, dx^2 + \cosh u \, dy^2). \end{aligned}$$

**Theorem 4.6.**  $(u, \sinh u, \cosh u)$  is a solution for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system of the timelike type if and only if  $u$  is a solution of the hyperbolic sinh-Gordon equation (2.6) and  $X_1$  defined in Proposition 4.4 gives an isometric immersion of a timelike surface  $M^{1,1}$  with  $K = 1$  into  $R^{2,1}$  of which the timelike principal curvature is bigger. In this case,  $X_1$  has the fundamental forms as

$$\begin{aligned} I &= -\sinh^2 u \, dx^2 + \cosh^2 u \, dy^2, \\ II &= \sinh u \cosh u (dx^2 - dy^2). \end{aligned}$$

Moreover, the maps  $Y_1 = X_1 - X_2$  and  $Y_2 = X_1 + X_2$  give timelike surfaces with  $H = -1$  and  $H = 1$  respectively, whose fundamental forms are

$$\begin{aligned} I_1 &= e^{2u}(-dx^2 + dy^2), & II_1 &= e^u(\cosh u \, dx^2 - \sinh u \, dy^2), \\ I_2 &= e^{-2u}(-dx^2 + dy^2), & II_2 &= -e^{-u}(\cosh u \, dx^2 + \sinh u \, dy^2). \end{aligned}$$

From now on, we investigate on the construction of an action on the solutions for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system. Since all the arguments are similar to  $O(3, 2)/O(2, 1) \times O(2)$  system, we only sketch the process briefly without details.

Consider the bilinear form  $\langle \cdot, \cdot \rangle_2$  on  $\mathbb{C}^5$  given by  $\langle U, V \rangle_2 = \bar{U}^t J_{1,1} V$ . Let  $W = (w_1, w_2, w_3)^t$  and  $Z = (z_1, z_2)^t$  be unit vectors in  $\mathbb{R}^{2,1}$  and in  $\mathbb{R}^{1,1}$ , respectively, and let  $\pi = \frac{1}{2} \begin{pmatrix} WW^t I_{2,1} & -iWZ^t I_{1,1} \\ iZW^t I_{2,1} & ZZ^t I_{1,1} \end{pmatrix}$  be the orthogonal projection of  $\mathbb{C}^5$  onto  $\mathbb{C} \begin{pmatrix} W \\ iZ \end{pmatrix}$  with respect to  $\langle \cdot, \cdot \rangle_2$ . For  $0 \neq s \in \mathbb{R}$ , define  $q_{s,\pi}(\lambda)$  similar to (3.7). Let  $(F, G)$  be the solution for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system of the spacelike type,  $\theta_\lambda$  the corresponding flat connection as in (4.1), and  $E$  be a trivialization of  $\theta_\lambda$ .

Put  $\begin{pmatrix} \bar{W} \\ i\bar{Z} \end{pmatrix}(x, y) = E(x, y, -is)^{-1} \begin{pmatrix} W \\ iZ \end{pmatrix}$  and let  $\tilde{\pi}(x, y)$  be the orthogonal projection onto  $\mathbb{C} \begin{pmatrix} \bar{W} \\ i\bar{Z} \end{pmatrix}$ . Denote by

$$(4.5) \quad \tilde{E}(x, y, \lambda) = E(x, y, \lambda) q_{s,\tilde{\pi}}(\lambda)^{-1}, \quad \widehat{W} = \bar{W} / \|\bar{W}\|_{2,1}, \quad \widehat{Z} = \bar{Z} / \|\bar{Z}\|_{1,1}.$$

**Lemma 4.7.** (i) For  $\tilde{E}$  given by (4.6),  $\tilde{\theta}_\lambda = \tilde{E}^{-1} d\tilde{E}$  gives a new solution  $(\tilde{F}, \tilde{G})$  of the spacelike type for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system.

(ii)

$$E(x, y, 0) = \begin{pmatrix} A(x, y) & 0 \\ 0 & B(x, y) \end{pmatrix}, \quad \tilde{E}(x, y, 0) = \begin{pmatrix} \tilde{A}(x, y) & 0 \\ 0 & \tilde{B}(x, y) \end{pmatrix}$$

for some  $A, \tilde{A} \in O(2, 1)$  and  $B, \tilde{B} \in O(1, 1)$ , and

$$(4.6) \quad \tilde{A} = A(I - 2\widehat{W}\widehat{W}^t I_{2,1}), \quad \tilde{B} = B(I - 2\widehat{Z}\widehat{Z}^t I_{1,1}).$$

(iii)

$$(4.7) \quad \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} - 2s(\widehat{W}\widehat{Z}^t I_{1,1})_*.$$

(iv)  $(\widetilde{W}, \widetilde{Z})$  is a solution of

$$(4.8) \quad d \begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix} = -\theta_{-is} \begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix},$$

(v)  $X, \widetilde{X} : \mathcal{O} \rightarrow \mathcal{M}_{3 \times 2}$  given by

$$\frac{\partial E}{\partial \lambda} \cdot E^{-1} \Big|_{\lambda=0} = \begin{pmatrix} 0 & -X \\ I_{1,1} X^t I_{2,1} & 0 \end{pmatrix}, \quad \frac{\partial \widetilde{E}}{\partial \lambda} \cdot \widetilde{E}^{-1} \Big|_{\lambda=0} = \begin{pmatrix} 0 & -\widetilde{X} \\ I_{1,1} \widetilde{X}^t I_{2,1} & 0 \end{pmatrix}$$

satisfy

$$(4.9) \quad \widetilde{X} = X - \frac{2}{s} A \widehat{W} \widehat{Z}^t B^t I_{1,1}.$$

$$(vi) \quad d\widetilde{X} = \widetilde{A} \begin{pmatrix} \delta \\ 0 \end{pmatrix} I_{1,1} \widetilde{B}^{-1}.$$

**Proposition 4.8.** *Let  $X = (X_1, X_2)$  and  $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)$  be given by Lemma 4.7. Then  $X_i$  and  $\widetilde{X}_i$  are in a Ribaucour transformation. In particular,*

$$X_j + \psi_{ij} e_i = \widetilde{X}_j + \psi_{ij} \widetilde{e}_i,$$

where,  $\psi_{ij} = -\frac{\epsilon_j}{s\widehat{w}_i} (\widehat{Z}^t B^t I_{1,1})_{1j}$ ,  $\epsilon_3 = -1$ ,  $A = (e_1, e_2, e_3)$  and  $\widetilde{A} = (\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3)$ .

*Proof.* It follows from (4.6) and (4.9).  $\square$

From the above Lemma and the Proposition, we obtain the followings.

**Theorem 4.9.** *Suppose  $(u, \cosh u, \sinh u)$  is a solution corresponding to  $\theta_\lambda$  in (4.4). Let  $X = (X_1, X_2)$ ,  $q_{s,\pi}(\lambda)$ ,  $E, W, Z, \widetilde{E}, \widetilde{W}, \widetilde{Z}, \widetilde{\theta}_\lambda$  and  $\widetilde{X}$  be defined as in Lemma 4.7. If  $s w_3 = z_1 \cosh u(0,0) + z_2 \sinh u(0,0)$ , then  $(\widetilde{u}, -\cosh \widetilde{u}, -\sinh \widetilde{u})$  is a solution corresponding to  $\theta_\lambda$ , where  $\widetilde{u} = 2\alpha - u$  and  $(\widehat{z}_1, \widehat{z}_2) = (\cosh \alpha, -\sinh \alpha)$ . In this case,*

$$\widetilde{X}_2 = X_2 + \frac{2}{s} (\widehat{z}_1 \sinh u + \widehat{z}_2 \cosh u) \sum_{i=1}^3 \widehat{w}_i e_i.$$

The fundamental forms of  $\widetilde{X}_2$  are

$$\begin{aligned} \widetilde{I} &= \sinh^2 \widetilde{u} dx^2 + \cosh^2 \widetilde{u} dy^2, \\ \widetilde{II} &= \sinh \widetilde{u} \cosh \widetilde{u} (dx^2 + dy^2), \end{aligned}$$

and the immersions  $X_2$  and  $\widetilde{X}_2$  are spacelike  $K = -1$  surfaces and they are in a Ribaucour transformation.

*Proof.* From (4.8), we have  $s\widetilde{w}_3 = \cosh u \widetilde{z}_1 + \sinh u \widetilde{z}_2 = \|Z\|_{1,1} \cosh(\alpha - u)$ . Also, by (4.7), we get  $\widetilde{g}_1 = \cosh u - 2s\widehat{w}_3 \widehat{z}_1 = -\cosh \widetilde{u}$ , and  $\widetilde{g}_2 = \sinh u - 2s\widehat{w}_3 \widehat{z}_2 = -\sinh \widetilde{u}$ . Therefore,  $X_2$  and  $\widetilde{X}_2$  are spacelike  $K = -1$  surfaces, which are in a Ribaucour transformation by Proposition 4.8.  $\square$



**Corollary 4.10.** *Under the hypothesis of Theorem 4.9, let  $Y_1 = X_1 - X_2$ ,  $Y_2 = X_1 + X_2$ ,  $\tilde{Y}_1 = \tilde{X}_1 - \tilde{X}_2$ ,  $\tilde{Y}_2 = \tilde{X}_1 + \tilde{X}_2$ . Then  $\tilde{Y}_i$  have the constant mean curvatures  $\tilde{H}_i = 1$ , whose fundamental forms are*

$$\begin{aligned}\tilde{I}_1 &= e^{-2\tilde{u}}(dx^2 + dy^2), & \tilde{II}_1 &= e^{-\tilde{u}}(\cosh \tilde{u} dx^2 - \sinh \tilde{u} dy^2), \\ \tilde{I}_2 &= e^{2\tilde{u}}(dx^2 + dy^2), & \tilde{II}_2 &= e^{\tilde{u}}(\cosh \tilde{u} dx^2 + \sinh \tilde{u} dy^2).\end{aligned}$$

Moreover,  $Y_i$  and  $\tilde{Y}_i$  are in a Darboux transformation.

*Proof.*  $\tilde{H}_i = 1$  comes from Theorem 4.2. Also, it is easy to show that  $\tilde{Y}_i + \phi_{ik}\tilde{e}_k = Y_i + \phi_{ik}e_k$ , where  $\phi_{1k} = \frac{e_k}{s\tilde{w}_k}e^u(\hat{z}_1 + \hat{z}_2)$  and  $\phi_{2k} = \frac{e_k}{s\tilde{w}_k}e^{-u}(\hat{z}_1 - \hat{z}_2)$ .  $\square$

**Theorem 4.11.** *Suppose  $(u, \sinh u, \cosh u)$  is a solution corresponding to  $\theta_\lambda$  in (4.4). Let  $X = (X_1, X_2)$ ,  $q_{s,\pi}(\lambda)$ ,  $E, W, Z, \tilde{E}, \tilde{W}, \tilde{Z}, \tilde{\theta}_\lambda$  and  $\tilde{X}$  be defined as in Lemma 4.7. If  $s w_3 = z_1 \sinh u(0, 0) + z_2 \cosh u(0, 0)$ , then  $(\tilde{u}, \sinh \tilde{u}, \cosh \tilde{u})$  is a solution corresponding to  $\tilde{\theta}_\lambda$ , where  $\tilde{u} = 2\alpha - u$  and  $(\hat{z}_1, \hat{z}_2) = (\cosh \alpha, -\sinh \alpha)$ . In this case,*

$$\tilde{X}_1 = X_1 - \frac{2}{s}(\hat{z}_1 \cosh u + \hat{z}_2 \sinh u) \sum_{i=1}^3 \tilde{w}_i e_i.$$

The fundamental forms of  $\tilde{X}_1$  are

$$\begin{aligned}\tilde{I} &= \cosh^2 \tilde{u} dx^2 + \sinh^2 \tilde{u} dy^2, \\ \tilde{II} &= -\sinh \tilde{u} \cosh \tilde{u} (dx^2 + dy^2),\end{aligned}$$

and the immersions  $X_1$  and  $\tilde{X}_1$  are spacelike  $K = -1$  surfaces and they are in a Ribaucour transformation.

Moreover, for  $\tilde{Y}_1 = \tilde{X}_1 - \tilde{X}_2$  and  $\tilde{Y}_2 = \tilde{X}_1 + \tilde{X}_2$ , they have the constant mean curvatures  $\tilde{H}_1 = 1$  and  $\tilde{H}_2 = -1$  whose fundamental forms are

$$\begin{aligned}\tilde{I}_1 &= e^{-2\tilde{u}}(dx^2 + dy^2), & \tilde{II}_1 &= e^{-\tilde{u}}(-\sinh \tilde{u} dx^2 + \cosh \tilde{u} dy^2), \\ \tilde{I}_2 &= e^{2\tilde{u}}(dx^2 + dy^2), & \tilde{II}_2 &= -e^{\tilde{u}}(\sinh \tilde{u} dx^2 + \cosh \tilde{u} dy^2),\end{aligned}$$

and  $Y_i$  and  $\tilde{Y}_i$  are in a Darboux transformation.

We now consider the action  $q_{s,\pi}(\lambda)$  on the solution  $(G, F)$  for

$$O(3, 2)/O(2, 1) \times O(1, 1)$$

system of the timelike type. Then we obtain a new solution of the form  $(\tilde{G}, \tilde{F}) = (\frac{G}{F}) - 2s(\tilde{W}\tilde{Z}^t)_*$ . Put  $A = (e_3, e_2, e_1)$  and  $\tilde{A} = (\tilde{e}_3, \tilde{e}_2, \tilde{e}_1)$ .

**Proposition 4.12.** *For the action  $q_{s,\pi}(\lambda)$  on the solution  $(G, F)$  of the timelike type for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system, let  $(\tilde{G}, \tilde{F})$  be the new solution,  $X = (X_1, X_2)$  and  $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$  be maps associated to them, respectively. Then  $X_i$  and  $\tilde{X}_i$  are in a Ribaucour transformation.*

**Theorem 4.13.** *Suppose  $(u, \cosh u, \sinh u)$  is a solution of the timelike type for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system,  $X = (X_1, X_2)$ ,  $q_{s, \pi}(\lambda)$ ,  $E, W, Z, \tilde{E}, \tilde{W}, \tilde{Z}$ ,  $\tilde{\theta}_\lambda$  and  $\tilde{X}$  be as before. If  $s w_3 = z_1 \cosh u(0, 0) + z_2 \sinh u(0, 0)$ , then*

$$(\tilde{u}, -\cosh \tilde{u}, -\sinh \tilde{u})$$

*is a new solution of the same type. In this case,*

$$\tilde{X}_2 = X_2 + \frac{2}{s}(\hat{z}_1 \sinh u + \hat{z}_2 \cosh u)(\hat{w}_3 e_1 + \hat{w}_2 e_2 + \hat{w}_1 e_3).$$

*The fundamental forms of  $\tilde{X}_2$  are*

$$\begin{aligned} \tilde{I} &= -\cosh^2 \tilde{u} dx^2 + \sinh^2 \tilde{u} dy^2, \\ \tilde{II} &= \sinh \tilde{u} \cosh \tilde{u} (dx^2 - dy^2). \end{aligned}$$

*The immersions  $X_2$  and  $\tilde{X}_2$  are  $K = 1$  timelike surfaces of which the spacelike principal curvatures are bigger, and they are in a Ribaucour transformation.*

*Furthermore,  $\tilde{Y}_1 = \tilde{X}_1 - X_2$  and  $\tilde{Y}_2 = \tilde{X}_1 + \tilde{X}_2$  have  $\tilde{H}_i = -1$ , and the fundamental forms*

$$\begin{aligned} \tilde{I}_1 &= e^{-2\tilde{u}}(-dx^2 + dy^2), & \tilde{II}_1 &= -e^{-\tilde{u}}(\sinh \tilde{u} dx^2 + \cosh \tilde{u} dy^2), \\ \tilde{I}_2 &= e^{2\tilde{u}}(-dx^2 + dy^2), & \tilde{II}_2 &= e^{\tilde{u}}(\sinh \tilde{u} dx^2 - \cosh \tilde{u} dy^2). \end{aligned}$$

*$Y_i$  and  $\tilde{Y}_i$  are in a Darboux transformation.*

**Theorem 4.14.** *Suppose  $(u, \sinh u, \cosh u)$  is a solution of the timelike type for  $O(3, 2)/O(2, 1) \times O(1, 1)$  system,  $X = (X_1, X_2)$ ,  $q_{s, \pi}(\lambda)$ ,  $E, W, Z, \tilde{E}, \tilde{W}, \tilde{Z}$ ,  $\tilde{\theta}_\lambda$  and  $\tilde{X}$  be as before. If  $s w_3 = z_1 \cosh u(0, 0) + z_2 \sinh u(0, 0)$ , then*

$$(\tilde{u}, \sinh \tilde{u}, \cosh \tilde{u})$$

*is a new solution of the same type. In this case,*

$$\tilde{X}_1 = X_1 - \frac{2}{s}(\hat{z}_1 \cosh u + \hat{z}_2 \sinh u)(\hat{w}_3 e_1 + \hat{w}_2 e_2 + \hat{w}_1 e_3).$$

*The fundamental forms of  $\tilde{X}_1$  are*

$$\begin{aligned} \tilde{I} &= -\sinh^2 \tilde{u} dx^2 + \cosh^2 \tilde{u} dy^2, \\ \tilde{II} &= \sinh \tilde{u} \cosh \tilde{u} (-dx^2 + dy^2). \end{aligned}$$

*The immersions  $X_1$  and  $\tilde{X}_1$  are  $K = 1$  timelike surfaces of which the timelike principal curvatures are bigger, and they are in a Ribaucour transformation.*

*Furthermore,  $\tilde{Y}_1 = \tilde{X}_1 - X_2$  and  $\tilde{Y}_2 = \tilde{X}_1 + \tilde{X}_2$  have  $\tilde{H}_1 = -1$  and  $\tilde{H}_2 = 1$ , respectively, and the fundamental forms are*

$$\begin{aligned} \tilde{I}_1 &= e^{-2\tilde{u}}(-dx^2 + dy^2), & \tilde{II}_1 &= e^{-\tilde{u}}(\cosh \tilde{u} dx^2 + \sinh \tilde{u} dy^2), \\ \tilde{I}_2 &= e^{2\tilde{u}}(-dx^2 + dy^2), & \tilde{II}_2 &= e^{\tilde{u}}(-\cosh \tilde{u} dx^2 + \sinh \tilde{u} dy^2). \end{aligned}$$

*$Y_i$  and  $\tilde{Y}_i$  are in a Darboux transformation.*

### 5. Examples

In this section, we give examples of new solutions  $\tilde{u}$  for the equations (2.3), (2.4), (2.5) and (2.6) from the vacuum solution  $u = 0$  by the dressing action  $q_{s,\pi}(\lambda)$ . In the below,  $\tilde{X}_i$  is the corresponding immersion of the nonumbilic Lorentzian surface with  $K = 1$  or  $K = -1$ ,  $\tilde{Y}_i$  the corresponding surface with  $H = \pm 1$ , and  $c$  and  $a$  are constants.

**Example 5.1.** The sine-Gordon equation (2.3);

$$(1) \quad \tilde{u} = -\arctan\left(\frac{2 \tanh c \cosh(x \sinh c) \sinh(y \cosh c)}{\cosh^2(x \sinh c) - \tanh^2 c \sinh^2(y \cosh c)}\right).$$

$$\tilde{X}_1 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} \sinh(x \sinh c) \\ \cosh y \cosh(y \cosh c) - \operatorname{sech} c \sinh y \sinh(y \cosh c) \\ \sinh y \cosh(y \cosh c) - \operatorname{sech} c \cosh y \sinh(y \cosh c) \end{pmatrix},$$

$$\text{where } r = \frac{-2 \cosh(x \sinh c)}{\sinh c \{\cosh^2(x \sinh c) + \tanh^2 c \sinh^2(y \cosh c)\}}.$$

$$(2) \quad \tilde{u} = -\arctan\left(\frac{2 \tanh c \sinh(x \sinh c) \cosh(y \cosh c)}{\sinh^2(x \sinh c) - \tanh^2 c \cosh^2(y \cosh c)}\right).$$

$$(3) \quad \tilde{u} = -\arctan\left(\frac{2 a \tanh c e^{x \sinh c} e^{-y \cosh c}}{a^2 e^{2x \sinh c} - \tanh^2 c e^{-2y \cosh c}}\right).$$

**Example 5.2.** The elliptic sine-Gordon equation (2.4);

$$(1) \quad \tilde{u} = -\arctan\left(\frac{2 \coth c \sinh(x \cosh c) \cos(y \sinh c)}{\sinh^2(x \cosh c) - \coth^2 c \cos^2(y \sinh c)}\right).$$

$$\tilde{X}_1 = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} + r \begin{pmatrix} \operatorname{csch} c \cosh y \cos(y \sinh c) + \sinh y \sin(y \sinh c) \\ \cosh(x \cosh c) \\ \operatorname{csch} c \sinh y \cos(y \sinh c) + \cosh y \sin(y \sinh c) \end{pmatrix},$$

$$\text{where } r = \frac{-2 \sinh(x \cosh c)}{\cosh c \{\sinh^2(x \cosh c) + \coth^2 c \cos^2(y \sinh c)\}}.$$

$$(2) \quad \tilde{u} = \arctan\left(\frac{2 \tan c \sinh(x \sin c) \sinh(y \cos c)}{\sinh^2(x \sin c) - \tan^2 c \sinh^2(y \cos c)}\right).$$

$$(3) \quad \tilde{u} = \arctan\left(\frac{2 \tan c \cosh(x \sin c) \cosh(y \cos c)}{\cosh^2(x \sin c) - \tan^2 c \cosh^2(y \cos c)}\right).$$

$$(4) \quad \tilde{u} = \arctan\left(\frac{2 a \tan c e^{x \sin c} e^{-y \cos c}}{a^2 e^{2x \sin c} - \tan^2 c e^{-2y \cos c}}\right).$$

$$(5) \quad \tilde{u} = -\arctan\left(\frac{2y \sinh x}{\sinh^2 x - y^2}\right).$$

**Example 5.3.** The sinh-Gordon equation (2.5);

$$(1) \quad \tilde{u} = \log \frac{\tanh c \cosh(x \cosh c) - \sin(y \sinh c)}{\tanh c \cosh(x \cosh c) + \sin(y \sinh c)}.$$

$$\tilde{X}_2 = \begin{pmatrix} 0 \\ -y \\ 0 \end{pmatrix} + r \begin{pmatrix} \cosh x \sinh(x \cosh c) - \operatorname{sech} c \sinh x \cosh(x \cosh c) \\ \cos(y \sinh c) \\ -\sinh x \sinh(x \cosh c) + \operatorname{sech} c \cosh x \cosh(x \cosh c) \end{pmatrix},$$

$$\text{where } r = \frac{2 \sin(y \sinh c)}{\sinh c \{ \tanh^2 c \cosh^2(x \cosh c) - \sin^2(y \sinh c) \}}.$$

$$\tilde{Y}_1 = \begin{pmatrix} \sinh x \\ y \\ 1 - \cosh x \end{pmatrix} + r_1 \begin{pmatrix} \cosh x \sinh(x \cosh c) - \operatorname{sech} c \sinh x \cosh(x \cosh c) \\ \cos(y \sinh c) \\ -\sinh x \sinh(x \cosh c) + \operatorname{sech} c \cosh x \cosh(x \cosh c) \end{pmatrix},$$

$$\text{where } r_1 = \frac{-2}{\sinh c \{ \tanh c \cosh(x \cosh c) - \sin(y \sinh c) \}}.$$

$$(2) \quad \tilde{u} = \log \frac{\tanh c \cosh(x \cosh c) - \sin(y \sinh c)}{\tanh c \cosh(x \cosh c) + \sin(y \sinh c)}.$$

$$(3) \quad \tilde{u} = \log \frac{\cosh(x \sin c) - \tan c \sinh(y \cos c)}{\cosh(x \sin c) + \tan c \sinh(y \cos c)}.$$

$$(4) \quad \tilde{u} = \log \frac{\sinh(x \sin c) - \tan c \cosh(y \cos c)}{\sinh(x \sin c) + \tan c \cosh(y \cos c)}.$$

$$(5) \quad \tilde{u} = \log \frac{e^{x \sin c} - \tan c e^{y \cos c}}{e^{x \sin c} + \tan c e^{y \cos c}}.$$

$$(6) \quad \tilde{u} = \log \frac{\cosh x - y}{\cosh x + y}.$$

**Example 5.4.** The hyperbolic sinh-Gordon equation (2.6);

$$(1) \quad \tilde{u} = \log \frac{\coth c \cosh(y \sinh c) - \cosh(x \cosh c)}{\coth c \cosh(y \sinh c) + \cosh(x \cosh c)}.$$

$$\tilde{X}_2 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} + r \begin{pmatrix} \operatorname{csch} c \cos y \cosh(y \sinh c) - \sin y \sinh(y \sinh c) \\ \operatorname{csch} c \sin y \cosh(y \sinh c) + \cos y \sinh(y \sinh c) \\ \sinh(x \cosh c) \end{pmatrix},$$

$$\text{where } r = \frac{2 \cosh(x \cosh c)}{\cosh c \{ \coth^2 c \cosh^2(y \sinh c) - \cosh^2(x \cosh c) \}}.$$

$$\tilde{Y}_1 = \begin{pmatrix} \cos y - 1 \\ \sin y \\ -x \end{pmatrix} + r_1 \begin{pmatrix} \operatorname{csch} c \cos y \cosh(y \sinh c) - \sin y \sinh(y \sinh c) \\ \operatorname{csch} c \sin y \cosh(y \sinh c) + \cos y \sinh(y \sinh c) \\ \sinh(x \cosh c) \end{pmatrix},$$

$$\text{where } r_1 = \frac{-2}{\cosh c \{ \coth c \cosh(y \sinh c) - \cosh(x \cosh c) \}}.$$

$$(2) \quad \tilde{u} = \log \frac{\coth c \sinh(y \sinh c) - \sinh(x \cosh c)}{\coth c \sinh(y \sinh c) + \sinh(x \cosh c)}.$$

$$(3) \quad \tilde{u} = \log \frac{\tan c \sin(y \cos c) - \sinh(x \sin c)}{\tan c \sin(y \cos c) + \sinh(x \sin c)}.$$

$$(4) \quad \tilde{u} = \log \frac{a \coth c e^{y \sinh c} - e^{x \cosh c}}{a \coth c e^{y \sinh c} + e^{x \cosh c}}.$$

$$(5) \quad \tilde{u} = \log \frac{y - \sinh x}{y + \sinh x}.$$

$$(6) \quad \tilde{u} = \log \frac{\cosh(y \sinh c) - \tanh c \cosh(x \cosh c)}{\cosh(y \sinh c) + \tanh c \cosh(x \cosh c)}.$$

$$(7) \quad \tilde{u} = \log \frac{\sinh(y \sinh c) - \tanh c \sinh(x \cosh c)}{\sinh(y \sinh c) + \tanh c \sinh(x \cosh c)}.$$

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