

STRONG CONVERGENCE FOR THREE CLASSES OF UNIFORMLY EQUI-CONTINUOUS AND ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we introduce a modified three-step iteration scheme with errors for three classes of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings in the framework of uniformly convex Banach spaces. We then use this scheme to approximate a common fixed point of these mappings. The results obtained in this paper extend and improve the recent ones announced by Khan, Fukhar-ud-di, Zhou, Cho, Noor and some others.

1. Introduction and preliminaries

Let E be a real Banach space, C a nonempty subset of E . Throughout the paper, \mathbb{N} denotes the set of positive integers and $F(T) := \{x : Tx = x\}$ the set of fixed points of a mapping T . A mapping $T : C \rightarrow C$ is said to be *asymptotically nonexpansive* if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$, we have

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for $x, y \in C$ and for all $n \in \mathbb{N}$.

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] in 1972. They proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E , then every asymptotically nonexpansive self-mapping T of C has a fixed point. Moreover, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, many authors have studied weak and strong convergence problems of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see [1, 6, 7, 11, 12, 17] and references therein).

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The mapping T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|T^n x - p\| \leq k_n \|x - p\|$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

The mapping T is said to be *uniformly L-lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$.

The mapping T is said to be *uniformly Hölder continuous* if there exists a positive constants k and α such that

$$\|T^n x - T^n y\| \leq k \|x - y\|^\alpha$$

for all $x, y \in C$ and $n \geq 1$.

The mapping T is said to be *uniformly equi-continuous* if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|T^n x - T^n y\| \leq \epsilon$$

whenever $\|x - y\| \leq \delta$ for all $x, y \in C$ and $n \geq 1$ or, equivalently, T is uniformly equi-continuous if and only if $\|T^n x_n - T^n y_n\| \rightarrow 0$ whenever $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Remark 1.1. (1) It is easy to see that, if T is asymptotically nonexpansive, then it is uniformly L-lipschitzian.

(2) If T is uniformly L-lipschitzian, then it is uniformly Hölder continuous.

(3) If T is uniformly Hölder continuous, then it is uniformly equi-continuous.

In recent years, Mann iterative scheme [10], Ishikawa iterative scheme [5] and Noor iterative scheme [11] have been studied extensively by many authors. In 1995, Liu [8] introduced iterative schemes with errors as follows.

The sequence $\{x_n\}$ defined by

$$(1.1) \quad \begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n + u_n, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{u_n\}$ a sequence in E satisfying

$$\sum_{n=1}^{\infty} \|u_n\| < \infty$$

is known as Mann iterative scheme with errors.

The sequence $\{x_n\}$ defined by

$$(1.2) \quad \begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n + v_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, $\{u_n\}$ and $\{v_n\}$ are sequences in E satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$ is known as Ishikawa iterative scheme with errors.

While it is clear that consideration of errors terms in iterative schemes is an important part of the theory, it is also clear that the iterative scheme with errors introduced by Liu [8], as in (1.1), (1.2) above, are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms in (1.1), (1.2) which say that they tend to zero as n tends to infinity are, therefore, unreasonable. Xu [16] introduced a more satisfactory error term in the following iterative schemes.

The sequence $\{x_n\}$ defined by

$$(1.3) \quad \begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n T x_n + \beta_n x_n + \gamma_n u_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in C , is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme if $\gamma_n = 0$.

The sequence $\{x_n\}$ defined by

$$(1.4) \quad \begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n T y_n + \beta_n x_n + \gamma_n u_n, \\ y_n = \alpha'_n T x_n + \beta'_n x_n + \gamma'_n v_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C , is known as Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative schemes if $\gamma_n = \gamma'_n = 0$. Chidume and Moore [2] and Takahashi and Tamura [14] studied the above schemes, respectively.

The sequence $\{x_n\}$ defined by

$$(1.5) \quad \begin{cases} z_n = \alpha''_n T x_n + \beta''_n x_n + \gamma''_n w_n, \\ y_n = \alpha'_n T z_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n T y_n + \beta_n x_n + \gamma_n u_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C , is known as Noor iterative scheme with errors. This scheme reduces to Noor iterative schemes if $\gamma_n = \gamma'_n = \gamma''_n = 0$.

Recently Khan and Fukhar-ud-din [4] generalized iterative scheme (1.4) to the one with errors as follows

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n S y_n + \beta_n x_n + \gamma_n u_n, \\ y_n = \alpha'_n T x_n + \beta'_n x_n + \gamma'_n v_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0, 1]$ with $0 < \delta \leq \alpha_n$, $\alpha'_n \leq 1 - \delta < 1$, $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C .

Many authors starting from Das and Debata [3] and including Takahashi and Tamura [14] and Khan and Takahashi [7] have studied the two mappings case of iterative schemes for different types of mappings. We now suggest an iterative scheme with errors for three classes of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings. It worth mentioning that our scheme can be viewed as an extension of all above schemes.

In this paper, we generalize scheme (1.5) to the one with errors as follows.

$$(1.6) \quad \begin{cases} z_n = \alpha''_n T_1^n x_n + \beta''_n x_n + \gamma''_n w_n, \\ y_n = \alpha'_n T_2^n z_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n T_3^n y_n + \beta_n x_n + \gamma_n u_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C .

The purpose of this paper is to construct this iterative scheme for approximating a common fixed point of three classes of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings and to prove strong convergence result. Our results improve and extend Cho et al. [1], Khan, Fukhar-ud-di [4], Zhou et al. [17], Noor [11] and some others.

Before we proceed further, we give the following definitions and lemmas which we shall need in the sequel.

Lemma 1.1 (Schu [12]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n \rightarrow \infty} \|x_n\| \leq r, \quad \limsup_{n \rightarrow \infty} \|y_n\| \leq r$$

and

$$\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$$

hold for some $r \geq 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 1.2 (Tan and Xu [15]). *Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \leq (1 + s_n)r_n + t_n \quad \text{for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \rightarrow \infty} r_n$ exists.

Recall that a mapping $T : C \rightarrow C$ where C is a subset of E , is said to satisfy Condition (A) [13] if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Senter and Dotson [13] approximated fixed points of a nonexpansive mapping T by Mann iterates, Later on, Maiti and Ghosh [9] and Tan and Xu [14] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same Condition (A) which is weaker than the requirement that T is demicompact. we modify this condition for three mappings T_1, T_2 and $T_3 : C \rightarrow C$ as follows.

Three mappings T_1, T_2 and $T_3 : C \rightarrow C$ where C a subset of E , are said to satisfy Condition (A') if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $a\|x - T_1x\| + b\|x - T_2x\| + c\|x - T_3x\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T_1) \cap F(T_2) \cap F(T_3)\}$ and a, b and c are three nonnegative real numbers such that $a + b + c = 1$.

Remark 1.2. Condition (A') reduces to Condition (A) when $T_1 = T_2 = T_3$. Besides, in the sequel, we shall denote $F(T_1) \cap F(T_2) \cap F(T_3)$ by F ; that is, $F = F(T_1) \cap F(T_2) \cap F(T_3)$.

2. Main results

In this section, we shall prove the strong convergence of the iterative scheme (1.6) to approximate a common fixed point of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings T_1, T_2 and T_3 . We first prove the following lemmas.

Lemma 2.1. *Let E be a normed space and C its nonempty convex subset. Let T_1, T_2 and $T_3 : C \rightarrow C$ be three mappings satisfying $\|T_1^n x - p\| \leq k_n \|x - p\|$, $\|T_2^n x - p\| \leq l_n \|x - p\|$ and $\|T_3^n x - p\| \leq j_n \|x - p\|$ for all $n \in \mathbb{N}$, where $\{k_n\}, \{l_n\}$ and $\{j_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$ and $\sum_{n=1}^{\infty} (j_n - 1) < \infty$, respectively. Let $\{x_n\}$ be the sequence as defined in (1.6) with $\sum_{n=1}^{\infty} \gamma_n'' < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. If $F \neq \emptyset$, then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F$. Note that $\{w_n\}, \{v_n\}$ and $\{u_n\}$ are bounded sequences in C . We set $M_1 = \sup\{\|u_n - p\| : n \geq 1\}$, $M_2 = \sup\{\|v_n - p\| : n \geq 1\}$, $M_3 = \sup\{\|w_n - p\| : n \geq 1\}$, $M = \max\{M_i : i = 1, 2, 3\}$.

It follows from (1.6) that

$$\begin{aligned} \|z_n - p\| &= \|\alpha_n'' T_1^n x_n + \beta_n'' x_n + \gamma_n'' w_n - p\| \\ &\leq \alpha_n'' \|T_1^n x_n - p\| + \beta_n'' \|x_n - p\| + \gamma_n'' \|w_n - p\| \\ &\leq \alpha_n'' k_n \|x_n - p\| + \beta_n'' \|x_n - p\| + \gamma_n'' \|w_n - p\| \\ &\leq k_n \|x_n - p\| + \gamma_n'' M. \end{aligned}$$

That is,

$$(2.1) \quad \|z_n - p\| \leq k_n \|x_n - p\| + \gamma_n'' M.$$

From (1.6) and (2.1), we get

$$\begin{aligned}
\|y_n - p\| &= \|\alpha'_n T_2^n z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\
&\leq \alpha'_n \|T_2^n z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leq \alpha'_n l_n \|z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leq \alpha'_n l_n \|z_n - p\| + (1 - \alpha'_n) \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leq \alpha'_n l_n (k_n \|x_n - p\| + \gamma''_n M) + (1 - \alpha'_n) \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leq k_n l_n \|x_n - p\| + l_n \gamma''_n M + \gamma'_n M.
\end{aligned}$$

That is,

$$(2.2) \quad \|y_n - p\| \leq k_n l_n \|x_n - p\| + l_n \gamma''_n M + \gamma'_n M.$$

Again, from (1.6) and (2.2) we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n T_3^n y_n + \beta_n x_n + \gamma_n u_n - p\| \\
&\leq \alpha_n \|T_3^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n j_n \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n j_n \|y_n - p\| + (1 - \alpha_n) \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n j_n (k_n l_n \|x_n - p\| + l_n \gamma''_n M + \gamma'_n M) \\
&\quad + (1 - \alpha_n) \|x_n - p\| + \gamma_n M \\
&\leq k_n l_n j_n \|x_n - p\| + j_n l_n \gamma''_n M + j_n \gamma'_n M + \gamma_n M.
\end{aligned}$$

That is,

$$\begin{aligned}
(2.3) \quad \|x_{n+1} - p\| &\leq [1 + (k_n - 1)(l_n - 1)(j_n - 1) + (k_n - 1)(l_n - 1) \\
&\quad + (l_n - 1)(j_n - 1) + (j_n - 1)(k_n - 1) + (k_n - 1) \\
&\quad + (l_n - 1) + (j_n - 1)] \|x_n - p\| + (j_n l_n \gamma''_n + j_n \gamma'_n + \gamma_n) M.
\end{aligned}$$

Note that since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$ and $\sum_{n=1}^{\infty} (j_n - 1) < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$, where

$$\begin{aligned}
A_n &= (k_n - 1)(l_n - 1)(j_n - 1) + (k_n - 1)(l_n - 1) + (l_n - 1)(j_n - 1) \\
&\quad + (j_n - 1)(k_n - 1) + (k_n - 1) + (l_n - 1) + (j_n - 1).
\end{aligned}$$

Therefore, it is easy to get $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$ by using Lemma 1.2. This completes the proof. \square

Lemma 2.2. *Let E be a uniformly convex Banach space and C its nonempty convex closed subset. Let T_1, T_2 and $T_3 : C \rightarrow C$ be three mappings satisfying $\|T_1^n x - p\| \leq k_n \|x - p\|$, $\|T_2^n x - p\| \leq l_n \|x - p\|$ and $\|T_3^n x - p\| \leq j_n \|x - p\|$ for all $n \in \mathbb{N}$, where $\{k_n\}$, $\{l_n\}$ and $\{j_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$ and $\sum_{n=1}^{\infty} (j_n - 1) < \infty$, respectively. Let T_1, T_2 and T_3 also be uniformly equi-continuous. Let $\{x_n\}$ be the sequence as defined in (1.6)*

with $\sum_{n=1}^{\infty} \gamma_n'' < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\alpha_n\}$, $\{\alpha_n'\}$ and $\{\alpha_n''\} \in [q, p]$, for some $q, p \in (0, 1)$. If $F \neq \emptyset$, then

$$\lim_{n \rightarrow \infty} \|T_1 x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_2 x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_3 x_n - x_n\| = 0.$$

Proof. Take $p \in F$. It follows from Lemma 2.1 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = c$ for some $c \geq 0$. Since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded, it follows that $\{u_n - x_n\}$, $\{v_n - x_n\}$ and $\{w_n - x_n\}$ are all bounded. Now, we set

$$\begin{aligned} r_1 &= \sup\{\|u_n - x_n\| : n \geq 1\}, \quad r_2 = \sup\{\|v_n - x_n\| : n \geq 1\}, \\ r_3 &= \sup\{\|w_n - x_n\| : n \geq 1\}, \quad r = \max\{r_i : 1 \leq i \leq 3\}. \end{aligned}$$

Taking limsup on both the sides in the inequality (2.1), we have

$$(2.4) \quad \limsup_{n \rightarrow \infty} \|z_n - p\| \leq c.$$

Similarly, taking limsup on both the sides in the inequality (2.2), we have

$$(2.5) \quad \limsup_{n \rightarrow \infty} \|y_n - p\| \leq c.$$

Observe that

$$\begin{aligned} \|T_3^n y_n - p + \gamma_n(u_n - x_n)\| &\leq \|T_3^n y_n - p\| + \gamma_n \|u_n - x_n\| \\ &\leq j_n \|y_n - p\| + \gamma_n r. \end{aligned}$$

Taking limsup on both the sides in the above inequality and using (2.5) we get that

$$\limsup_{n \rightarrow \infty} \|T_3^n y_n - p + \gamma_n(u_n - x_n)\| \leq c.$$

Note that

$$\begin{aligned} \|x_n - p + \gamma_n(u_n - x_n)\| &\leq \|x_n - p\| + \gamma_n \|u_n - x_n\| \\ &\leq \|x_n - p\| + \gamma_n r, \end{aligned}$$

which implies that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n(u_n - x_n)\| \leq c.$$

Again, $\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c$ means that

$$\lim_{n \rightarrow \infty} \|\alpha_n(T_3^n y_n - p + \gamma_n(u_n - x_n)) + (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| = c.$$

An application of Lemma 1.1, we have

$$(2.6) \quad \lim_{n \rightarrow \infty} \|T_3^n y_n - x_n\| = 0.$$

On the other hand,

$$\begin{aligned} \|x_n - p\| &\leq \|T_3^n y_n - x_n\| + \|T_3^n y_n - p\| \\ &\leq \|T_3^n y_n - x_n\| + j_n \|y_n - p\|, \end{aligned}$$

which gives that

$$c \leq \liminf_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq c.$$

That is,

$$\lim_{n \rightarrow \infty} \|y_n - p\| = c.$$

Observe that $\lim_{n \rightarrow \infty} \|y_n - p\| = c$ is equivalent to

$$(2.7) \quad \lim_{n \rightarrow \infty} \|\alpha'_n(T_2^n z_n - p + \gamma'_n(v_n - x_n)) + (1 - \alpha'_n)(x_n - p + \gamma'_n(v_n - x_n))\| = c.$$

On the other hand, we have

$$\begin{aligned} \|T_2^n z_n - p + \gamma'_n(v_n - x_n)\| &\leq \|T_2^n z_n - p\| + \gamma'_n \|v_n - x_n\| \\ &\leq l_n \|z_n - p\| + \gamma'_n r. \end{aligned}$$

Taking limsup on both the sides in the above inequality and using (2.4), we have

$$(2.8) \quad \limsup_{n \rightarrow \infty} \|T_2^n z_n - p + \gamma'_n(v_n - x_n)\| \leq c.$$

Notice that

$$\begin{aligned} \|x_n - p + \gamma'_n(v_n - x_n)\| &\leq \|x_n - p\| + \gamma'_n \|v_n - x_n\| \\ &\leq \|x_n - p\| + \gamma'_n r, \end{aligned}$$

which yields that

$$(2.9) \quad \limsup_{n \rightarrow \infty} \|x_n - p + \gamma'_n(v_n - x_n)\| \leq c.$$

Applying Lemma 1.1, it follows from (2.7), (2.8) and (2.9) that

$$(2.10) \quad \lim_{n \rightarrow \infty} \|T_2^n z_n - x_n\| = 0.$$

Observe that

$$\begin{aligned} \|x_n - p\| &\leq \|T_2^n z_n - x_n\| + \|T_2^n z_n - p\| \\ &\leq \|T_2^n z_n - x_n\| + l_n \|z_n - p\|, \end{aligned}$$

which yields that

$$c \leq \liminf_{n \rightarrow \infty} \|z_n - p\| \leq \limsup_{n \rightarrow \infty} \|z_n - p\| \leq c.$$

That is,

$$\lim_{n \rightarrow \infty} \|z_n - p\| = c.$$

Again, $\lim_{n \rightarrow \infty} \|z_n - p\| = c$ can be expressible as

$$(2.11) \quad \lim_{n \rightarrow \infty} \|\alpha''_n(T_1^n x_n - p + \gamma''_n(w_n - x_n)) + (1 - \alpha''_n)(x_n - p + \gamma''_n(w_n - x_n)) - p\| = c.$$

Moreover,

$$\begin{aligned} \|T_1^n x_n - p + \gamma''_n(w_n - x_n)\| &\leq \|T_1^n x_n - p\| + \gamma''_n \|w_n - x_n\| \\ &\leq k_n \|x_n - p\| + \gamma''_n r, \end{aligned}$$

which implies that

$$(2.12) \quad \limsup_{n \rightarrow \infty} \|T_1^n x_n - p + \gamma''_n(w_n - x_n)\| \leq c.$$

It follows from

$$\begin{aligned} \|x_n - p + \gamma_n''(w_n - x_n)\| &\leq \|x_n - p\| + \gamma_n''\|w_n - x_n\| \\ &\leq \|x_n - p\| + \gamma_n''r \end{aligned}$$

that

$$(2.13) \quad \limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n''(w_n - x_n)\| \leq c.$$

Combine (2.11), (2.12) with (2.13) yields

$$(2.14) \quad \lim_{n \rightarrow \infty} \|T_1^n x_n - x_n\| = 0.$$

On the other hand, we also have

$$\begin{aligned} \|z_n - x_n\| &= \|\alpha_n'' T_1^n x_n + \beta_n'' x_n + \gamma_n'' w_n - x_n\| \\ &= \|\alpha_n''(T_1^n x_n - x_n) + \gamma_n''(w_n - x_n)\| \\ &\leq \alpha_n'' \|T_1^n x_n - x_n\| + \gamma_n'' r. \end{aligned}$$

Therefore, it follows from (2.14) and condition $\sum_{n=0}^{\infty} \gamma_n'' < \infty$ that

$$\|z_n - x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

It follows from the uniform equi-continuity of T_2 that

$$\|T_2^n z_n - T_2^n x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Now observe that

$$(2.15) \quad \|x_n - T_2^n x_n\| \leq \|T_2^n x_n - T_2^n z_n\| + \|T_2^n z_n - x_n\|.$$

From (2.10) and (2.15), we can obtain

$$(2.16) \quad \lim_{n \rightarrow \infty} \|T_2^n x_n - x_n\| = 0.$$

Similarly, we get

$$\begin{aligned} \|y_n - x_n\| &= \|\alpha_n' T_2^n x_n + \beta_n' x_n + \gamma_n' v_n - x_n\| \\ &= \|\alpha_n'(T_2^n x_n - x_n) + \gamma_n'(v_n - x_n)\| \\ &\leq \alpha_n' \|T_2^n x_n - x_n\| + \gamma_n' r. \end{aligned}$$

Therefore, it follows from (2.16) and the condition $\sum_{n=0}^{\infty} \gamma_n' < \infty$ that

$$\|y_n - x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

It follows from the uniform equi-continuity of T_3 that

$$(2.17) \quad \|T_3^n y_n - T_3^n x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Now we consider that

$$\|x_n - T_3^n x_n\| \leq \|T_3^n x_n - T_3^n y_n\| + \|T_3^n y_n - x_n\|.$$

From (2.6) and (2.17), we get that

$$(2.18) \quad \lim_{n \rightarrow \infty} \|T_3^n x_n - x_n\| = 0.$$

Hence, we obtain that

$$\lim_{n \rightarrow \infty} \|T_1^n x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_2^n x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_3^n x_n - x_n\| = 0.$$

Next, we show

$$\lim_{n \rightarrow \infty} \|T_1 x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_2 x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_3 x_n - x_n\| = 0.$$

It follows from (2.6) and $\sum_{n=0}^{\infty} \gamma_n < \infty$ that

$$\|x_{n+1} - x_n\| \leq \alpha_n \|T_3^n y_n - x_n\| + \gamma_n \|u_n - x_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

It follows from the uniform equi-continuity of T_1 that $\|T_1^{n+1} x_{n+1} - T_1^{n+1} x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Again it follows from (2.14) that

$$\|T_1^{n+1} x_n - T_1 x_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Therefore, we have

$$\begin{aligned} \|T_1 x_n - x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| + \|T_1^{n+1} x_{n+1} - T_1^{n+1} x_n\| \\ &\quad + \|T_1^{n+1} x_n - T_1 x_n\|. \end{aligned}$$

That is,

$$\|T_1 x_n - x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Similarly, we can prove

$$\lim_{n \rightarrow \infty} \|T_2^n x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_3^n x_n - x_n\| = 0.$$

This completes the proof of the Lemma. \square

Next we prove the strong convergence theorem.

Theorem 2.3. *Let E be a uniformly convex Banach space and C, T_1, T_2, T_3 and $\{x_n\}$ are as in Lemma 2.2. Further T_1, T_2 and T_3 satisfy Condition (A'). If $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3 .*

Proof. Since Lemma 2.1, we know $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$. Let it be c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Next, Suppose $c > 0$. By Lemma 2.2, we know $\lim_{n \rightarrow \infty} \|T_1 x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_2 x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_3 x_n - x_n\| = 0$, and (2.3) gives that

$$\begin{aligned} &\inf_{p \in F} \|x_{n+1} - p\| \\ &\leq \inf_{p \in F} [1 + (k_n - 1)(l_n - 1)(j_n - 1) + (k_n - 1)(l_n - 1) + (l_n - 1)(j_n - 1) \\ &\quad + (j_n - 1)(k_n - 1) + (k_n - 1) + (l_n - 1) + (j_n - 1)] \|x_n - p\| \\ &\quad + (j_n l_n \gamma_n'' + j_n \gamma_n' + \gamma_n) M. \end{aligned}$$

That is,

$$\begin{aligned} & d(x_{n+1}, F) \\ & \leq [1 + (k_n - 1)(l_n - 1)(j_n - 1) + (k_n - 1)(l_n - 1) + (l_n - 1)(j_n - 1) \\ & \quad + (j_n - 1)(k_n - 1) + (k_n - 1) + (l_n - 1) + (j_n - 1)]d(x_n, F) \\ & \quad + (j_n l_n \gamma_n'' + j_n \gamma_n' + \gamma_n)M. \end{aligned}$$

gives that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists by virtue of Lemma 1.2. Now by Condition (A'), we obtain $\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0$. It follows from the property of f that $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Indeed, since $\lim_{n \rightarrow \infty} d(x_n, F)$ exists, we can suppose $\lim_{n \rightarrow \infty} d(x_n, F) = r_0 > 0$. Thus, there exists a natural number N , $\forall n > N$, we have $d(x_n, F) > \frac{r_0}{2}$. Since f is a nondecreasing function, $f(0) = 0$ and $f(r) > 0$ for $\forall r \in (0, \infty)$, we can obtain $f(d(x_n, F)) \geq f(\frac{r_0}{2}) > 0$ for $\forall n > N$. On the other hand, we have $\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0$. This derives a contradiction. Now we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and sequence $\{y_j\} \subset F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. Then following the method of Tan and Xu [15], we get that y_j is a Cauchy sequence in F and so it converges. Let $y_j \rightarrow y$. Since F is closed, therefore $y \in F$ and then $x_{n_j} \rightarrow y$. As $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, $x_n \rightarrow y \in F$ thereby completing the proof. \square

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