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ON THE COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF DEPENDENT RANDOM VARIABLES UNDER CONDITION OF WEIGHTED INTEGRABILITY

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Abstract. Under the condition of h-integrability and appropriate conditions on the array of weights, we establish complete convergence and strong law of large numbers for weighted sums of an array of dependent random variables.

1. Introduction

Complete convergence and strong law of large numbers for sequences of random variables play a central role in the area of limit theorems in probability theory and mathematical statistics. Conditions of independence and identical distribution of random variables are basic in historic results due to Bernoulli, Borel or Kolmogorov. Since then, serious attempts have been made to relax these strong conditions. For example, independence has been relaxed to pairwise independence or pairwise negative quadrant dependence or, even replaced by conditions of dependence such as mixing or martingale. In order to relax the identical distribution, several other conditions have been considered, such as stochastic domination by an integrable random variable. The classical notion of uniform integrability of a sequence $\{X_n, n \geq 1\}$ of integrable random variables is defined through the condition $\lim_{a\to\infty} \sup_{n\geq 1} E|X_n|I(X_n| > a) = 0$. Landers and Rogge [6] prove that the uniform integrability condition is sufficient in order that a sequence of pairwise independent random variables verifies the law of large numbers. Chandra [2] introduces a new condition which is weaker than uniform integrability : the condition of Cesàro uniform integrability. A sequence $\{X_n, n \geq 1\}$ of integrable random variables is said to be Cesàro uniformly integrable if $\lim_{a\to\infty} \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n} E|X_k|I(X_k| > a) = 0$.

Ordónez Cabrera [9] introduces the condition of uniform integrability concerning the weights, which is weaker than uniform integrability, and leads to Cesàro uniform integrability as a special case.

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Definition 1.1 ([9, Ordónez Cabrera]). Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of random variables and $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ an array of constants with $\sum_{k=1}^{n} |a_{nk}| \leq C$ for all $n \geq 1$ and some constant $C > 0$. The array $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is $\{a_{nk}\}$-uniformly integrable if

$$\lim_{a \to \infty} \sup_{n \geq 1} \sum_{k=1}^{n} |a_{nk}| E|X_{nk}| I(|X_{nk}| > a) = 0.$$

Under the condition of $\{a_{nk}\}$-uniform integrability, Ordónez Cabrera [9] obtains law of large numbers for weighted sums of pairwise independent random variables; the condition of pairwise independence can be even dropped, at the price of slightly strengthening the conditions on the weights.

Ordónez Cabrera and Volodin [10] introduce the notion of h-integrability for an array of random variables concerning an array of constant weights and prove that this concept is weaker than Cesàro uniform integrability and $\{a_{nk}\}$-uniform integrability.

Definition 1.2 ([10, Ordóñez Cabrera and Volodin]). Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of random variables and $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ an array of constants with $\sum_{k=1}^{n} |a_{nk}| < C$ for all $n \geq 1$ and some constant $C > 0$. Let moreover $\{h(n), n \geq 1\}$ be an nondecreasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \to \infty$. The array $\{X_{nk}\}$ is said to be h-integrable with respect to the array of constants $\{a_{nk}\}$ if the conditions hold:

$$\sup_{n \geq 1} \sum_{k=1}^{n} |a_{nk}| E\,|X_{nk}| < \infty \text{ and } \lim_{n \to \infty} \sum_{k=1}^{n} |a_{nk}| E\,|X_{nk}|\, I\,[\,|X_{nk}| > h(n)] = 0.$$

Under appropriate conditions on the weights and h-integrability concerning the weights we will derive complete convergence and strong law of large numbers for weighted sums of an array of random variables, when these random variables are subject to some special kind of rowwise dependence: asymptotically almost negative association, pairwise negative quadrant dependence and φ-mixing.

2. Statements of results

In this section we obtain some complete convergence and strong law of large numbers for weighted sums of array of h-integrable random variables under some conditions of dependence. Namely, we consider the following rowwise dependence structures for an array: asymptotically almost negative association (AANA), pairwise negative quadrant dependence (NQD) and φ-mixing.

2.1. AANA random variables

In the first theorem of this sub-section, we are going to show that, for an array of rowwise asymptotically almost negatively associated random variables, a certain technique of truncation which preserves the dependence can be used to obtain a complete convergence and a strong law of large numbers for the weighted sums under the condition of h-integrability concerning the array of constants $\{a_{nk}\}$.

Recall that a finite family $\{X_1, \ldots, X_n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on R^A, g on R^B,

$$\mathrm{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$$

and that an infinite family of random variables is NA if every finite subfamily is NA. This concept was introduced by Joag-Dev and Proschan [4].

By inspecting the proof of Matula's [8] maximal inequality, we see that one can also allow positive correlations provided they are small. Primarily motivated by this, we introduce the following dependence condition :

Definition 2.1 ([3, Chandra and Ghosal]). A sequence $\{X_n, n \geq 1\}$ of random variables is said to be asymptotically almost negative associated (AANA) if there exists a nonnegative sequence $q(m) \to 0$ such that

$$\begin{aligned} (2.1.1) \qquad & \mathrm{Cov}(f(X_m), g(X_{m+1}, \ldots, X_{m+k})) \\ \leq \ & q(m)(\mathrm{Var}(f(X_m))\mathrm{Var}(g(X_{m+1}, \ldots, X_{m+k})))^{\frac{1}{2}} \end{aligned}$$

for all $k, m \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the righthand side of (2.1.1) is finite.

Remark. The family of AANA sequences contains NA (in particular, independent) sequences and some more sequences of random variables which are not much deviated from being negatively associated.

The following lemma gives a certain technique of truncation that preserves the dependence property.

Lemma 2.2 ([5, Kim, Ko, and Lee]). *Let $\{X_n, n \geq 1\}$ be a sequence of* AANA *random variables. Then, for any sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ of constants such that $a_n < b_n$ for all $n \geq 1$, the sequence $\{Y_n, n \geq 1\}$ is still a sequence of* AANA *random variables, where*

$$(2.1.2) \qquad Y_n = X_n I[a_n \leq X_n \leq b_n] + a_n I[X_n < a_n] + b_n I[X_n > b_n].$$

Lemma 2.3 ([3, Chandra and Ghosal]). *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero, square integrable random variables such that* (2.1.1) *holds for $1 \leq m < k + m \leq n$ and for all coordinatewise nondecreasing continuous functions f and g whenever the righthand side of* (2.1.1) *is finite. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and ${\sigma_k}^2 = E{X_k}^2$, $k \geq 1$. Then, for $\varepsilon > 0$,*

$$(2.1.3) \qquad P\{\max_{1 \leq k \leq n} |\sum_{i=1}^{k} X_i| \geq \varepsilon\} \leq 2\varepsilon^{-2}\left(A + (1 + A^2)^{\frac{1}{2}}\right)^2 \sum_{k=1}^{n} {\sigma_k}^2.$$

Remark. Lemma 2.3 extends Lemma 4 of Matula [8].

Theorem 2.4. *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise* AANA *random variables with $EX_{nk} = 0$ and $EX_{nk}^2 < \infty$ for all $1 \le k \le n$ and $n \ge 1$ and $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of positive constants with $a_{nk} \le 1$, $\sum_{k=1}^n a_{nk} \le C$ for all $n \ge 1$ and some constant $C > 0$. Let moreover $\{h(n), n \ge 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. Suppose that*

(a) *$\{X_{nk}\}$ is h-integrable concerning the array of constants $\{a_{nk}\}$,*
(b) *$h^2(n) \sum_{k=1}^n {a_{nk}}^2 = O((\log n)^{-1-\delta})$ for some $\delta (0 < \delta < 1)$,*
(c) *$h(n) \ge Cn^\alpha$ for some $\alpha > \frac{1}{2}$,*
(d) *$\sum_{m=1}^\infty q^2(m) < \infty$.*

Then, for all $\varepsilon > 0$,

$$\sum_{n=1}^\infty n^{-1} P(\max_{1 \le j \le n} |\sum_{k=1}^j a_{nk} X_{nk}| > \varepsilon) < \infty. \tag{2.1.4}$$

Proof. For each $1 \le k \le n$, $n \ge 1$, truncate at the level $h(n)$ and put
(2.1.5)
$$Y_{nk} = X_{nk} I(|X_{nk}| \le h(n)) - h(n) I(X_{nk} < -h(n)) + h(n) I(X_{nk} > h(n)).$$
Noting that $EX_{nk} I(|X_{nk}| \le h(n)) = -EX_{nk} I(|X_{nk}| > h(n))$ in view of the fact that $EX_{nk} = 0$, we have

$$\begin{aligned}
& P(\max_{1 \le j \le n} |\sum_{k=1}^j a_{nk} X_{nk}| > \varepsilon) \\
\le\ & P(\max_{1 \le j \le n} |X_{nj}| > h(n)) + P(\max_{1 \le j \le n} |\sum_{k=1}^j a_{nk} Y_{nk}| > \varepsilon) \\
\le\ & P(\max_{1 \le j \le n} |X_{nj}| > h(n)) \\
& + P(\max_{1 \le j \le n} |\sum_{k=1}^j a_{nk}(Y_{nk} - EY_{nk})| > \varepsilon - \max_{1 \le j \le n} |\sum_{k=1}^j a_{nk} EY_{nk}|) \\
\le\ & \sum_{k=1}^n P(|X_{nk}| > h(n)) \\
& + P(\max_{1 \le j \le n} |\sum_{k=1}^j a_{nk}(Y_{nk} - EY_{nk})| > \varepsilon - \max_{1 \le j \le n} |\sum_{k=1}^j a_{nk} EY_{nk}|). \qquad (2.1.6)
\end{aligned}$$

By assumption (a) we also have

$$\begin{aligned}
& \max_{1 \le j \le n} |\sum_{k=1}^j a_{nk} EY_{nk}| \\
=\ & \max_{1 \le j \le n} |\sum_{k=1}^j a_{nk} E\{X_{nk} I(|X_{nk}| \le h(n))
\end{aligned}$$

$$
\begin{aligned}
&\quad -h(n)I(X_{nk} < -h(n)) + h(n)I(X_{nk} > h(n))\}| \\
&\leq \sum_{k=1}^{n} a_{nk}\{E|X_{nk}|I(|X_{nk}| > h(n))\} \\
&\quad + \sum_{k=1}^{n} a_{nk}h(n)EI(|X_{nk}| > h(n)) \\
&\leq 2\sum_{k=1}^{n} a_{nk}E|X_{nk}|I(|X_{nk}| > h(n)) \\
&\leq 2\sum_{k=u_n}^{v_n} a_{nk}E|X_{nk}|I(|X_{nk}| > h(n)) \to 0, \text{ as } n \to \infty. \qquad (2.1.7)
\end{aligned}
$$

Hence, it follows from (2.1.6) and (2.1.7) that for n large enough

$$
\begin{aligned}
&P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}X_{nk}| > \varepsilon) \\
&\leq \sum_{k=1}^{n} P(|X_{nk}| > h(n)) + P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \frac{\varepsilon}{2}). \qquad (2.1.8)
\end{aligned}
$$

It therefore remains to show that

$$\sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} P(|X_{nk}| > h(n)) < \infty$$

and

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \frac{\varepsilon}{2}) < \infty.$$

It follows from Chebyshev inequality and assumptions $EX_{nk}^2 < \infty$ for all $1 \leq k \leq n,\ n \geq 1$ and (c) that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} P(|X_{nk}| > h(n)) \\
&\leq \sum_{n=1}^{\infty} n^{-1} \frac{\sum_{k=1}^{n} E|X_{nk}|^2}{h^2(n)} \\
&\leq C\sum_{n=1}^{\infty} n^{-2\alpha} < \infty. \qquad (2.1.9)
\end{aligned}
$$

It also follows from Lemma 2.3, assumptions (b) and (d) that

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \frac{\varepsilon}{2}) \qquad (2.1.10)$$

$$
\begin{aligned}
&\le 8\sum_{n=1}^{\infty} n^{-1}\varepsilon^{-2}(A+(1+A^2)^{\frac{1}{2}})^2 \sum_{k=1}^{n}({a_{nk}}^2 E{Y_{nk}}^2)\\
&\le C\sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{n} {a_{nk}}^2 E[{X_{nk}}^2 I(|X_{nk}|\le h(n)) + h^2(n) I(|X_{nk}|>h(n))]\\
&\le C\sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{n} {a_{nk}}^2 h^2(n) \le C\sum_{n=1}^{\infty} n^{-1}(\log n)^{-1-\delta} < \infty.
\end{aligned}
$$

Thus by (2.1.9) and (2.1.10) the proof is complete. □

Corollary 2.5. *Under the conditions of Theorem 2.4 we have*

$$
\sum_{k=1}^{n} a_{nk}X_{nk} \to 0 \text{ a.s. as } n\to\infty. \tag{2.1.11}
$$

Proof. By (2.1.4) we have

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P(\max_{1\le j\le n} |\sum_{k=1}^{j} a_{nk}X_{nk}| > \varepsilon)\\
&= \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} n^{-1} P(\max_{1\le j\le n} |\sum_{k=1}^{j} a_{nk}X_{nk}| > \varepsilon) \qquad (2.1.12)\\
&\ge \frac{1}{2}\sum_{i=1}^{\infty} P(\max_{1\le j\le 2^i} |\sum_{k=1}^{j} a_{2^i,k}X_{2^i,k}| > \varepsilon).
\end{aligned}
$$

By Borel-Cantelli Lemma and (2.1.12) we have

$$
P(\max_{1\le j\le 2^i} |\sum_{k=1}^{j} a_{2^i,k}X_{2^i,k}| > \varepsilon \text{ i.o.}) = 0
$$

and hence,

$$
\max_{1\le j\le 2^i} |\sum_{k=1}^{j} a_{2^i,k}X_{2^i,k}| \to 0 \text{ a.s. as } i\to\infty. \tag{2.1.13}
$$

From (2.1.13) and the fact that

$$
\begin{aligned}
&\lim_{n\to\infty} |\sum_{k=1}^{n} a_{nk}X_{nk}|\\
&\le \lim_{i\to\infty}\max_{2^{i-1}\le n\le 2^i} |\sum_{k=1}^{n} a_{nk}X_{nk}| \le \lim_{i\to\infty}\max_{1\le j\le 2^i} |\sum_{k=1}^{j} a_{2^i,k}X_{2^i,k}|
\end{aligned}
$$

the desired result (2.1.11) follows. □

2.2. Pairwise NQD random variables

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise negative quadrant dependent(NQD) if for all $i \neq j$ and all x_i, x_j, $P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i)P(X_j > x_j)$(see Lehmann [7]).

The following lemma is an extension of the well-known Rademacher-Mension inequality.

Lemma 2.6 ([3, Chandra and Ghosal]). *Let $Y_1, \ldots, Y_n$ be square integrable random variables and let there exist $a_1^2, \ldots, a_n^2$ satisfying*

$$E(Y_{m+1} + \cdots + Y_{m+p})^2 \leq a_{m+1}^2 + \cdots + a_{m+p}^2$$

for all $m, p \geq 1$, $m + p \leq n$. Then we have

$$E(\max_{1 \leq k \leq n} (\sum_{i=1}^{k} Y_i)^2) \leq ((\log n / \log 3) + 2)^2 \sum_{i=1}^{n} a_i^2. \tag{2.2.1}$$

Theorem 2.7. *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise pairwise NQD random variables with $EX_{nk} = 0$, $EX_{nk}^2 < \infty$ for all $1 \leq k \leq n$ and $n \geq 1$ and $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of positive constants with $a_{nk} \leq 1$, $\sum_{k=1}^{n} a_{nk} \leq C$ for all $n \geq 1$ and some constant $C > 0$. Let moreover $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. Suppose that*

(a) *$\{X_n\}$ is h-integrable concerning the array of constants $\{a_{nk}\}$,*
(b) *$h^2(n) \sum_{k=1}^{n} {a_{nk}}^2 = O((\log n)^{-3-\delta})$ for some $\delta (0 < \delta < 1)$,*
(c) *$h(n) \geq Cn^{\alpha}$ for some $\alpha > \frac{1}{2}$.*

Then for all $\varepsilon > 0$, (2.1.4) holds.

Proof. The proof is similar to that of Theorem 2.4. Define Y_{nk} as in (2.1.5). Then $\{Y_{nk}\}$ is still pairwise NQD. Hence by assumption (a), (2.1.6) and (2.1.7) we obtain

$$\begin{aligned} & P(\max_{1 \leq j \leq n} |\sum_{k=1}^{j} a_{nk} X_{nk}| > \varepsilon) \\ \leq \; & \sum_{k=1}^{n} P(|X_{nk}| > h(n)) + P(\max_{1 \leq j \leq n} |\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \frac{\varepsilon}{2}). \end{aligned}$$

Note that $a_n(Y_{nk} - EY_{nk})$'s satisfy the conditions of Lemma 2.6. As in the proof of (2.1.9) it follows from Markov inequality and assumption (c) that

$$\sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} P(|X_{nk}| > h(n)) \leq C \sum_{n=1}^{\infty} n^{-2\alpha} < \infty \tag{2.2.2}$$

and as in the proof of (2.1.10) it also follows from Lemma 2.6 and assumption (b) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\le j\le n} \Big|\sum_{k=1}^{j} a_{nk}(Y_{nk}-EY_{nk})\Big| > \frac{\varepsilon}{2}\Big) \\
(2.2.3)\qquad &\le \Big(\frac{\varepsilon}{2}\Big)^{-2} \sum_{n=1}^{\infty} n^{-1}((\log n/\log 3)+2)^2 \sum_{k=1}^{n} a_{nk}{}^2 EY_{nk}{}^2 \\
&\le \Big(\frac{\varepsilon}{2}\Big)^{-2} \sum_{n=1}^{\infty} n^{-1}((\log n/\log 3)+2)^2 \sum_{k=1}^{n} a_{nk}{}^2 h^2(n) \\
&\le C\sum_{n=1}^{\infty} n^{-1}(\log n)^{-1-\delta} < \infty.
\end{aligned}$$

Hence by (2.2.2) and (2.2.3) the proof is complete. □

Corollary 2.8. *Under conditions of Theorem 2.7,* (2.1.11) *holds.*

Proof. See the proof of Corollary 2.5. □

2.3. φ-mixing random variables

Definition 2.9. Let $\{X_n, -\infty < n < \infty\}$ be a sequence of random variables. Let $\mathcal{B}^k$ be the σ-algebra generated by $\{X_n, n \le k\}$, and $\mathcal{B}_k$ the σ-algebra generated by $\{X_n, n \ge k\}$. We say that $\{X_n, -\infty < n < \infty\}$ is φ-mixing if there exists a non-negative sequence $\{\varphi(i), i \ge 1\}$ with $\lim_{i\to\infty} \varphi(i) = 0$, such that, for each $-\infty < k < \infty$ and for each $i \ge 1$,

$$(2.3.1)\qquad |P(E_2 \,|\, E_1) - P(E_2)| \le \varphi(i) \quad \text{for } E_1 \in \mathcal{B}^k, E_2 \in \mathcal{B}_{k+i}.$$

Lemma 2.10 ([1, Billingsley]). *Let ζ be a $\mathcal{B}^k$-measurable random variable, and η be a $\mathcal{B}_{k+i}$-measurable random variable, with $|\zeta| \le C_1$ and $|\eta| \le C_2$. Then*

$$(2.3.2)\qquad |\operatorname{Cov}(\zeta,\eta)| \le 2C_1C_2\varphi(i).$$

Theorem 2.11. *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of mean zero and square integrable random variables such that for each $n \ge 1$, $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ is a φ_n-mixing sequence of random variables satisfying*

$$(2.3.3)\qquad \limsup_{n\to\infty} \sum_{i=1}^{n} \varphi_n(i) < \infty.$$

Let $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of non-negative constants such that $a_{nk} \le 1$, $\sum_{k=1}^{n} a_{nk} \le C$ for all $n \ge 1$ and some constant $C > 0$, and $a_{nj} \le a_{ni}$ if $i < j$ for all $n \ge 1$. Let moreover $\{h(n), n \ge 1\}$ be a sequence of increasing to infinity positive constant. Suppose that

(a) *$\{X_{nk}\}$ is h-integrable concerning the array of constants $\{a_{nk}\}$,*
(b) *$h^2(n)\sum_{k=1}^{n} a_{nk}{}^2 = O((\log n)^{-3-\delta})$ for some $\delta(0 < \delta < 1)$,*
(c) *$h(n) \ge Cn^{\alpha}$ for some $\alpha > \frac{1}{2}$.*

Then (2.1.4) *holds.*

Proof. The proof is similar to that of Theorem 2.7 and only we can use usual truncation technique. Hence, for each $n \geq 1$, $1 \leq k \leq n$, let

$$Y_{nk} = X_{nk} I(|X_{nk}| \leq h(n)). \tag{2.3.4}$$

Then

$$\begin{aligned}
& P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} X_{nk}| > \varepsilon) \\
\leq\ & P(\max_{1\leq j\leq n} |X_{nj}| > h(n)) + P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} Y_{nk}| > \varepsilon) \\
\leq\ & \sum_{k=1}^{n} P(|X_{nk}| > h(n)) \\
& + P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \varepsilon - \max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} EY_{nk}|).
\end{aligned}$$

Note that $EX_{nk}I(|X_{nk}| \leq h(n)) = -EX_{nk}I(|X_{nk}| > h(n))$ in view of the fact $EX_{nk} = 0$ and that

$$\begin{aligned}
\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} EY_{nk}| &= \max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} EX_{nk} I(|X_{nk}| \leq h(n))| \\
&\leq \sum_{k=1}^{n} a_{nk} E|X_{nk}| I(|X_{nk}| > h(n)) \\
&\leq \sum_{k=u_n}^{v_n} a_{nk} E|X_{nk}| I(|X_{nk}| > h(n)) \to 0 \text{ as } n \to \infty
\end{aligned}$$

by assumption (a). Hence we have

$$\begin{aligned}
& P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} X_{nk}| > \varepsilon) \\
\leq\ & \sum_{k=1}^{n} P(|X_{nk}| > h(n)) + P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \frac{\varepsilon}{2}).
\end{aligned}$$

It remains to show that

$$\sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} P(|X_{nk}| > h(n)) < \infty \tag{2.3.5}$$

and

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \frac{\varepsilon}{2}) < \infty. \tag{2.3.6}$$

By Chebyshev inequality and assumption (c), (2.3.5) follows. To apply Lemma 2.6 we need to show that

$$\limsup_{n\to\infty} \sum_{\substack{k,j=1\\k<j}}^{n} a_{nk}a_{nj}\,\mathrm{Cov}(Y_{nk}, Y_{nj}) \le 0. \tag{2.3.7}$$

By Lemma 2.10 and assumption (b) we have

$$\begin{aligned}\sum_{\substack{k,j=1\\k<j}}^{n} a_{nk}a_{nj}\,\mathrm{Cov}(Y_{nk}, Y_{nj}) &= \sum_{i=1}^{n}\sum_{k=1}^{n-i} a_{nk}a_{n(k+i)}\,\mathrm{Cov}(Y_{nk}, Y_{n(k+i)})\\ &\le 2h^2(n)\sum_{i=1}^{n}\sum_{k=1}^{n-i} {a_{nk}}^2\varphi_n(i)\\ &\le 2h^2(n)\sum_{k=1}^{n} {a_{nk}}^2\sum_{i=1}^{n}\varphi_n(i) \to 0,\end{aligned}$$

which yields (2.3.7).

Hence, by Lemma 2.6 and assumption (b), (2.3.6) follows, that is, the proof is complete. □

Corollary 2.12. *Let* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be an array of random variables such that for each* $n \ge 1$, $\{X_{nk}, 1 \le k \le n\}$ *is a* $m(n)$*-dependent sequence of random variables with* $\limsup_{n\to\infty} m(n) < \infty$. *Let the other conditions of Theorem 2.11 be satisfied. Then,* (2.1.4) *holds.*

Proof. We only have to note that we can consider $\varphi_n(i) = 0$ for $i > m(n)$ and $\varphi_n(i) = 1$ for $i \le m(n)$, and so $\sum_{i=1}^{n}\varphi_n(i) \le m(n)$ for all $n \ge 1$. □

Corollary 2.13. (1) *Under conditions of Theorem 2.11,* (2.1.11) *holds, that is,*

$$\sum_{k=1}^{\infty} a_{nk}X_{nk} \to 0 \ \textit{a.s. as}\ n \to \infty. \tag{2.3.8}$$

(2) *Under conditions of Corollary 2.12,* (2.3.8) *holds.*

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