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ON FACTORIZATIONS OF THE SUBGROUPS OF SELF-HOMOTOPY EQUIVALENCES

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ABSTRACT. For a pointed space X, the subgroups of self-homotopy equivalences $\mathrm{Aut}_{\sharp N}(X)$, $\mathrm{Aut}_{\Omega}(X)$, $\mathrm{Aut}_*(X)$ and $\mathrm{Aut}_{\Sigma}(X)$ are considered, where $\mathrm{Aut}_{\sharp N}(X)$ is the group of all self-homotopy classes f of X such that $f_\sharp = id : \pi_i(X) \to \pi_i(X)$ for all $i \leq N \leq \infty$, $\mathrm{Aut}_{\Omega}(X)$ is the group of all the above f such that $\Omega f = id$; $\mathrm{Aut}_*(X)$ is the group of all self-homotopy classes g of X such that $g_* = id : H_i(X) \to H_i(X)$ for all $i \leq \infty$, $\mathrm{Aut}_{\Sigma}(X)$ is the group of all the above g such that $\Sigma g = id$. We will prove that $\mathrm{Aut}_{\Omega}(X_1 \times \cdots \times X_n)$ has two factorizations similar to those of $\mathrm{Aut}_{\sharp N}(X_1 \times \cdots \times X_n)$ in reference [10], and that $\mathrm{Aut}_{\Sigma}(X_1 \vee \cdots \vee X_n)$, $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$ also have factorizations being dual to the former two cases respectively.

1. Introduction

For a pointed space X, let $\mathrm{Aut}(X)$ denote the set of homotopy classes of pointed self-maps of X that are homotopy equivalences. This set is a group, called the group of self-homotopy equivalences, with respect to the operation induced by the composition of maps. For a survey of the literature about $\mathrm{Aut}(X)$ and related concepts, see [1] or [13]. In this paper, we consider the subgroups of the group of self-homotopy equivalences.

For a pointed space X and a integer N with $\dim X \leq N \leq \infty$, we define the subgroups $\mathrm{Aut}_{\sharp N}(X)$ and $\mathrm{Aut}_{\Omega}(X)$ of $\mathrm{Aut}(X)$ by

$$\mathrm{Aut}_{\sharp N}(X) = \{f \in \mathrm{Aut}(X) | f_\sharp = id : \pi_i(X) \to \pi_i(X) \text{ for all } i \leq N\}$$

and

$$\mathrm{Aut}_{\Omega}(X) = \{f \in \mathrm{Aut}(X) | \Omega f = id\},$$

where $f_\sharp$ is the homomorphism on homotopy group induced by f and Ω is the loop functor. Since the homomorphisms induced by Ωf on the homotopy groups of ΩX are the same (after a shift in dimension) as those induced by f on the homotopy groups of X, $\mathrm{Aut}_{\Omega}(X)$ is a subgroup of $\mathrm{Aut}_{\sharp N}(X)$. The

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group $\mathrm{Aut}_{\sharp N}(X)$ has been studied by many authors, see [2, 3, 4, 5, 8, 10]. For example, in [5], Farjoun and Zabrodsky proved that the group $\mathrm{Aut}_{\sharp N}(X)$ is nilpotent whenever X is a finite-dimensional CW-complex; in [8], Maruyama proved that, under the same assumption and given a set of primes P, the natural map $\mathrm{Aut}_{\sharp N}(X) \to \mathrm{Aut}_{\sharp N}(X_P)$ is the P-localization homomorphism of the nilpotent group $\mathrm{Aut}_{\sharp N}(X)$. In particular, in [10], Pavešić proved that, for two pointed CW-complexes X and Y, the self-equivalences in $\mathrm{Aut}_{\sharp N}(X \times Y)$ are always reducible (see Section 2), hence $\mathrm{Aut}_{\sharp N}(X \times Y)$ can be decomposed as the product of its two natural subgroups. This paves the way to decompose $\mathrm{Aut}_{\sharp N}(X_1 \times \cdots \times X_n)$ as the n-fold product of its certain subgroups. Furthermore, Pavešić developed another method named as LU factorization by which $\mathrm{Aut}_{\sharp N}(X_1 \times \cdots \times X_n)$ can be decomposed as the product of its only two subgroups. In Section 2, we will prove that the self-equivalences in $\mathrm{Aut}_{\Omega}(X \times Y)$ are also always reducible, so that $\mathrm{Aut}_{\Omega}(X_1 \times \cdots \times X_n)$ has factorizations similar to those of $\mathrm{Aut}_{\sharp N}(X_1 \times \cdots \times X_n)$.

For a pointed space X, we can also define the subgroups $\mathrm{Aut}_*(X)$ and $\mathrm{Aut}_{\Sigma}(X)$ of $\mathrm{Aut}(X)$ by

$$\mathrm{Aut}_*(X) = \{g \in \mathrm{Aut}(X) | g_* = id : H_i(X) \to H_i(X) \text{ for all } i \geq 0\}$$

and

$$\mathrm{Aut}_{\Sigma}(X) = \{g \in \mathrm{Aut}(X) | \Sigma g = id\},$$

where g_* is the homomorphism on homology group induced by g and Σ is the suspension functor. Similar to $\mathrm{Aut}_{\sharp N}(X)$ and $\mathrm{Aut}_{\Omega}(X)$, we can see that $\mathrm{Aut}_{\Sigma}(X)$ is a subgroup of $\mathrm{Aut}_*(X)$. In [5], Farjoun and Zabrodsky also proved that $\mathrm{Aut}_*(X)$ is nilpotent whenever X is a finite-dimensional CW-complex; in [9], Maruyama proved that the natural map $\mathrm{Aut}_*(X) \to \mathrm{Aut}_*(X_P)$ is a P-localization homomorphism. In Section 3, we will prove that, for any pointed simply-connected CW-complexes X and Y, the self-equivalences in $\mathrm{Aut}_*(X \vee Y)$ are always reducible (see Section 3), and for pointed simply-connected CW-complexes $X_1, \ldots, X_n$, $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$ has factorizations dual to those of $\mathrm{Aut}_{\sharp N}(X_1 \times \cdots \times X_n)$. In Section 4, we consider the group $\mathrm{Aut}_{\Sigma}(X)$ and show that $\mathrm{Aut}_{\Sigma}(X)$ acts dually to $\mathrm{Aut}_{\Omega}(X)$.

2. $\mathbf{Aut}_{\Omega}(X_1 \times \cdots \times X_n)$

The group $\mathrm{Aut}_{\Omega}(X)$ was first introduced by Felix and Murillo in [6], where they showed that $\mathrm{Aut}_{\Omega}(X)$ and $\mathrm{Aut}_{\sharp N}(X)$ are generally different. In [11], Pavešić constructed a spectral sequence converging to $\mathrm{Aut}_{\Omega}(X)$ by Cartan-Eilenberg system. He proved that, if X is a Co-H-space, then $\mathrm{Aut}_{\Omega}(X)$ is trivial, and that if X is a Postnikov piece, then for any set of primes P, the natural map $\mathrm{Aut}_{\Omega}(X) \to \mathrm{Aut}_{\Omega}(X_P)$ is the P-localization.

In order to state our results in this section, the following notations and notions are needed. In this section, all the spaces are pointed connected CW-complexes. Let i_X and i_Y denote the inclusions of X and Y in $X \times Y$, and let p_X and p_Y be the projections of $X \times Y$ on X and Y. Given a self-map $f : X \times Y \to X \times Y$ and $I, J \in \{X, Y\}$, write $f_I : X \times Y \to I$ for the composition $f_I = p_I f$ (so that f is represented component-wise as $f = (f_X, f_Y)$ by the universal property of product spaces), and write $f_{IJ} : J \to I$ for the composition $f_{IJ} = p_I f i_J$. A self-homotopy equivalence f of $X \times Y$ is said to be reducible if f_{XX} and f_{YY} are self-homotopy equivalences of X and Y respectively.

Let $\mathrm{Aut}_X(X \times Y) = \{f \in \mathrm{Aut}(X \times Y) | p_X f = p_X\}$ and $\mathrm{Aut}_Y(X \times Y) = \{g \in \mathrm{Aut}(X \times Y) | p_Y g = p_Y\}$. In [12], Pavešić proved that $\mathrm{Aut}_X(X \times Y)$ and $\mathrm{Aut}_Y(X \times Y)$ are subgroups of $\mathrm{Aut}(X \times Y)$, and that if all the self-equivalences of $X \times Y$ are reducible, then $\mathrm{Aut}(X \times Y)$ is the product of $\mathrm{Aut}_X(X \times Y)$ and $\mathrm{Aut}_Y(X \times Y)$, i.e.,

$$\mathrm{Aut}(X \times Y) = \mathrm{Aut}_X(X \times Y)\, \mathrm{Aut}_Y(X \times Y).$$

However, when considered the group $\mathrm{Aut}_{\sharp N}(X \times Y)$ in [10], Pavešić found that all the self-equivalences in $\mathrm{Aut}_{\sharp N}(X \times Y)$ are always reducible without any restriction on X and Y, hence there is a corresponding decomposition of $\mathrm{Aut}_{\sharp N}(X \times Y)$, i.e.,

$$\mathrm{Aut}_{\sharp N}(X \times Y) = \mathrm{Aut}_{X \sharp N}(X \times Y)\, \mathrm{Aut}_{Y \sharp N}(X \times Y),$$

where $\mathrm{Aut}_{X\sharp N}(X \times Y) = \mathrm{Aut}_X(X \times Y) \cap \mathrm{Aut}_{\sharp N}(X \times Y)$ and $\mathrm{Aut}_{Y\sharp N}(X \times Y) = \mathrm{Aut}_Y(X \times Y) \cap \mathrm{Aut}_{\sharp N}(X \times Y)$. This paves the way for a generation of our approach to factorization of self-equivalences of products of more than two CW-complexes, i.e.,

$$\mathrm{Aut}_{\sharp N}(X_1 \times \cdots \times X_n) = \prod_{i=1}^{n} \mathrm{Aut}_{\prod_i \sharp N}(X_1 \times \cdots \times X_n),$$

where $\prod_i$ denotes the subproduct of $X_1 \times \cdots \times X_n$ obtained by omitting X_i, i.e., $\prod_i = X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n$ (refer to [10]).

For the group $\mathrm{Aut}_\Omega(X \times Y)$, we will also prove that all its self-equivalences are always reducible, that is,

Lemma 2.1. *For any $f \in \mathrm{Aut}_\Omega(X \times Y)$, we have $f_{XX} \in \mathrm{Aut}_\Omega(X)$ and $f_{YY} \in \mathrm{Aut}_\Omega(Y)$.*

Proof. Since $\mathrm{Aut}_\Omega(X \times Y) \subseteq \mathrm{Aut}_{\sharp N}(X \times Y)$ and all the self-equivalences in $\mathrm{Aut}_{\sharp N}(X \times Y)$ are reducible by Lemma 2.1 of [10], we have $f_{XX} \in \mathrm{Aut}_{\sharp N}(X)$ and $f_{YY} \in \mathrm{Aut}_{\sharp N}(Y)$. Since

$$\Omega f_{XX} = \Omega(p_X f i_X) = (\Omega p_X)(\Omega f)(\Omega i_X) = \Omega(p_X i_X) = id,$$

we can get $f_{XX} \in \mathrm{Aut}_\Omega(X)$ and similarly $f_{YY} \in \mathrm{Aut}_\Omega(Y)$. This shows the result. □

Now we can derive the following factorization from Theorem 2.5 of [12].

Theorem 2.2.

$$\mathrm{Aut}_\Omega(X \times Y) = \mathrm{Aut}_{X\Omega}(X \times Y)\,\mathrm{Aut}_{Y\Omega}(X \times Y),$$

where $\mathrm{Aut}_{X\Omega}(X \times Y) = \mathrm{Aut}_X(X \times Y) \cap \mathrm{Aut}_\Omega(X \times Y)$ *and* $\mathrm{Aut}_{Y\Omega}(X \times Y) = \mathrm{Aut}_Y(X \times Y) \cap \mathrm{Aut}_\Omega(X \times Y)$ *are the subgroups of* $\mathrm{Aut}_\Omega(X \times Y)$.

Proof. Given any $h \in \mathrm{Aut}_\Omega(X \times Y) \subseteq \mathrm{Aut}(X \times Y)$, since h is reducible, then by Theorem 2.5 of [12], h can be decomposed as

$$h = (p_X, f)(g, p_Y) = (g, f(g, p_Y)),$$

where $(p_X, f) \in \mathrm{Aut}_X(X \times Y)$ and $(g, p_Y) \in \mathrm{Aut}_Y(X \times Y)$. And since

$$\begin{aligned}\Omega(g, p_Y) &= \Omega(h_X, p_Y) = (\Omega(p_X h), \Omega p_Y)\\ &= ((\Omega p_X)(\Omega h), \Omega p_Y) = \Omega(p_X, p_Y) = id\end{aligned}$$

by $\Omega h = id$, this implies that $\Omega(g, p_Y) \in \mathrm{Aut}_{Y\Omega}(X \times Y)$ and then follows $(p_X, f) \in \mathrm{Aut}_{\Omega X}(X \times Y)$. As $\mathrm{Aut}_{X\Omega}(X \times Y)$ and $\mathrm{Aut}_{Y\Omega}(X \times Y)$ have trivial intersection, the above factorization is unique and then follows the desired theorem. □

By applying Theorem 2.2 and a completely same proof as that of Theorem 2.4 in [10], we can get the generalization of Theorem 2.2 as follows.

Theorem 2.3. $\mathrm{Aut}_\Omega(X_1 \times \cdots \times X_n) = \prod_{i=1}^n \mathrm{Aut}_{\prod_i \Omega}(X_1 \times \cdots \times X_n)$.

Also we can decompose $\mathrm{Aut}_\Omega(X_1 \times \cdots \times X_n)$ as the product of its only two subgroups as follows.

Theorem 2.4.

$$\mathrm{Aut}_\Omega(X_1 \times \cdots \times X_n) = L(X_1 \times \cdots \times X_n)U(X_1 \times \cdots \times X_n),$$

where $L(X_1 \times \cdots \times X_n) = \{f \in \mathrm{Aut}_\Omega(X_1 \times \cdots \times X_n) | f_{X_k} = f_{X_k} l_k, k = 1, \ldots, n\}$, $U(X_1 \times \cdots \times X_n) = \{f \in \mathrm{Aut}_\Omega(X_1 \times \cdots \times X_n) | f_{X_k} = f_{X_k} u_k, f_{X_k X_k} = id, k = 1, \ldots, n\}$; l_k *and* u_k *are self-maps of* $X_1 \times \cdots \times X_n$ *defined by* $l_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k, *, \ldots, *)$ *and* $u_k(x_1, \ldots, x_n) = (*, \ldots, *, x_k, \ldots, x_n)$ *respectively.*

The above factorization is called LU factorization as in Theorem 3.2 of [10] and their proofs are similar.

3. $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$

In this section, we will prove that for any pointed simply-connected CW-complexes $X_1, \ldots, X_n$, $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$ has two factorizations dual to those of $\mathrm{Aut}_{\sharp N}(X_1 \times \cdots \times X_n)$ as in [10].

Before stating our main results, we first fix some notions and notations. In this section, all the spaces are pointed simply-connected CW-complexes. For two spaces X and Y, let $i_X : X \to X \vee Y$ and $i_Y : Y \to X \vee Y$ be inclusions; $p_X : X \vee Y \to X$ and $p_Y : X \vee Y \to Y$ be projections. Given a self-map $f : X \vee Y \to X \vee Y$ and $I, J \in \{X, Y\}$, we denote $f i_I$ by f_I (so that we have

$f = (f_X, f_Y)$ by the universal property of wedge spaces) and $p_J f i_I$ by f_{IJ}. We say $f \in \mathrm{Aut}(X \vee Y)$ is reducible if $f_{XX} \in \mathrm{Aut}(X)$ and $f_{YY} \in \mathrm{Aut}(Y)$.

We must mention that the above notations have the same forms as those defined in Section 2, but they have different senses in this section.

Let $\mathrm{Aut}^X(X \vee Y) = \{f \in \mathrm{Aut}(X \vee Y) | f i_X = i_X\}$ and $\mathrm{Aut}^Y(X \vee Y) = \{g \in \mathrm{Aut}(X \vee Y) | g i_Y = i_Y\}$. In [14], H. B. Yu and W. H. Shen proved the following theorem:

Theorem 3.1 ([14]). $\mathrm{Aut}^X(X \vee Y)$ *and* $\mathrm{Aut}^Y(X \vee Y)$ *are subgroups of* $\mathrm{Aut}(X \vee Y)$*, and that if all the self-equivalences in* $\mathrm{Aut}(X \vee Y)$ *are reducible, then* $\mathrm{Aut}(X \vee Y) = \mathrm{Aut}^X(X \vee Y)\,\mathrm{Aut}^Y(X \vee Y)$.

In the follows, we will prove that all the self-equivalences in $\mathrm{Aut}_*(X \vee Y)$ are always reducible, and then we use this result to derive factorizations of $\mathrm{Aut}_*(X \vee Y)$ and their generalizations to $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$.

Lemma 3.2. *For any* $f \in \mathrm{Aut}_*(X \vee Y)$*, we have* $f_{XX} \in \mathrm{Aut}_*(X)$ *and* $f_{YY} \in \mathrm{Aut}_*(Y)$*, which implies that all the self-equivalences in* $\mathrm{Aut}_*(X \vee Y)$ *are reducible.*

Proof. Given any $f \in \mathrm{Aut}_*(X \vee Y)$, its induced endomorphism on $H_i(X) \oplus H_i(Y)$ is $H_i(f) : H_i(X) \oplus H_i(Y) \to H_i(X) \oplus H_i(Y)$ which can be represented by the following 2×2-matrix according to [14]:

$$H_i(f) = \begin{pmatrix} H_i(f_{XX}) & H_i(f_{XY}) \\ H_i(f_{YX}) & H_i(f_{YY}) \end{pmatrix}.$$

Since f is in $\mathrm{Aut}_*(X \vee Y)$, then we have $H_i(f) = 1_{H_i(X) \oplus H_i(Y)}$. It follows that $H_i(f_{XX}) = 1_{H_i(X)}$ and $H_i(f_{YY}) = 1_{H_i(Y)}$. By Whitehead theorem, we know that $f_{XX} \in \mathrm{Aut}_*(X)$ and $f_{YY} \in \mathrm{Aut}_*(Y)$. □

Let $\mathrm{Aut}_*^X(X \vee Y) = \mathrm{Aut}_*(X \vee Y) \cap \mathrm{Aut}^X(X \vee Y)$. Then we have

Lemma 3.3. *For any* $f \in \mathrm{Aut}_*(X \vee Y)$*, we have* $(i_X, f_Y) \in \mathrm{Aut}_*^X(X \vee Y)$.

Proof. The endomorphism on $H_i(X) \oplus H_i(Y)$ induced by (i_X, f_Y) is $H_i(i_X, f_Y)$: $H_i(X) \oplus H_i(Y) \to H_i(X) \oplus H_i(Y)$ which can be represented by the following 2×2-matrix according to [14]:

$$H_i(i_X, f_Y) = \begin{pmatrix} 1_{H_i(X)} & H_i(f_{XY}) \\ 0 & H_i(f_{YY}) \end{pmatrix}.$$

Since $f \in \mathrm{Aut}_*(X \vee Y)$ is reducible by Lemma 3.2, we have $H_i(f_{YX}) = 0$ and $H_i(f_{YY}) = 1_{H_i(Y)}$ which means that $H_i(i_X, f_Y) = 1_{H_i(X) \oplus H_i(Y)}$. By Whitehead theorem, we know that $(i_X, f_Y) \in \mathrm{Aut}_*^X(X \vee Y)$. □

Similarly we let $\mathrm{Aut}_*^Y(X \vee Y) = \mathrm{Aut}_*(X \vee Y) \cap \mathrm{Aut}^Y(X \vee Y)$, then we can derive a factorization of $\mathrm{Aut}_*(X \vee Y)$ dual to that of $\mathrm{Aut}_{\sharp N}(X \vee Y)$.

Theorem 3.4. $\mathrm{Aut}_*(X \vee Y) = \mathrm{Aut}_*^X(X \vee Y)\,\mathrm{Aut}_*^Y(X \vee Y)$.

Proof. For any $h \in \mathrm{Aut}_*(X \vee Y)$, h is reducible according to Lemma 3.2. By the Theorem 3.1, h has a unique factorization $h = (i_X, f)(g, i_Y) = ((i_X, f)g, f)$, where $(i_X, f) \in \mathrm{Aut}^X(X \vee Y)$ and $(g, i_Y) \in \mathrm{Aut}^Y(X \vee Y)$. Then by Lemma 3.3, we have $(i_X, h_Y) = (i_X, f) \in \mathrm{Aut}_*^X(X \vee Y)$. Since h and (i_X, f) are both in $\mathrm{Aut}_*(X \vee Y)$, we have $(g, i_Y) \in \mathrm{Aut}_*(X \vee Y)$, i.e., $(g, i_Y) \in \mathrm{Aut}_*^Y(X \vee Y)$. This shows the result. □

Actually, the above $\mathrm{Aut}_*^X(X \vee Y)$ and $\mathrm{Aut}_*^Y(X \vee Y)$ can be further decomposed. If we let $\mathrm{Aut}_{Y*}^X(X \vee Y) = \{f \in \mathrm{Aut}_*^X(X \vee Y) | p_Y f = p_Y\}$ and $\mathrm{Aut}_{X*}^Y(X \vee Y) = \{f \in \mathrm{Aut}_*^Y(X \vee Y) | p_X f = p_X\}$, then we have the following result:

Proposition 3.5. $\mathrm{Aut}_*^X(X \vee Y)$ *is the semi-direct product of* $\mathrm{Aut}_*(Y)$ *and* $\mathrm{Aut}_{Y*}^X(X \vee Y)$.

Proof. Define a map $\phi_Y : \mathrm{Aut}_*^X(X \vee Y) \to \mathrm{Aut}(Y)$ by $\phi_Y(i_X, f_Y) = p_Y f_Y = f_{YY}$, it is easy to verify that $\mathrm{Ker}\, \phi_Y = \mathrm{Aut}_{Y*}^X(X \vee Y)$. We begin to prove that ϕ_Y is a homomorphism.

Since

$$\begin{aligned} p_Y f_Y p_Y &= (p_Y f_Y p_Y i_X, p_Y f_Y p_Y i_Y) \\ &= (p_Y i_X, p_Y f_Y) = p_Y(i_X, f_Y), \end{aligned}$$

then for any $(i_X, f_Y), (i_X, g_Y) \in \mathrm{Aut}_*^X(X \vee Y)$, we have

$$\begin{aligned} \phi_Y((i_X, f_Y)(i_X, g_Y)) &= p_Y(i_X, f_Y)(i_X, g_Y) i_Y \\ &= p_Y f_Y p_Y g_Y \\ &= \phi_Y(i_X, f_Y)\phi_Y(i_X, g_Y). \end{aligned}$$

It is follows that ϕ_Y is a homomorphism which has a right inverse

$$\psi_Y : \mathrm{Aut}(Y) \to \mathrm{Aut}_*^X(X \vee Y)$$

given by $\psi_Y(f) = (i_X, i_Y f)$, where $f \in \mathrm{Aut}(Y)$. Then we have the following split short exact sequence:

$$0 \longrightarrow \mathrm{Aut}_{Y*}^X(X \vee Y) \xrightarrow{\phi_Y} \mathrm{Aut}_*^X(X \vee Y) \longrightarrow \mathrm{Aut}_*(Y) \longrightarrow 0.$$

It means that $\mathrm{Aut}_*^X(X \vee Y)$ is the semi-direct product of $\mathrm{Aut}_*(Y)$ and $\mathrm{Aut}_{Y*}^X(X \vee Y)$. □

Similarly, we can prove that $\mathrm{Aut}_*^Y(X \vee Y)$ is also the semi-direct product of $\mathrm{Aut}_*(X)$ and $\mathrm{Aut}_{X*}^Y(X \vee Y)$.

We can now apply inductively Theorem 3.4 to obtain a factorization of $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$. First, we need the following lemma:

Lemma 3.6. $\mathrm{Aut}_*^X(X \vee Y \vee Z) = \mathrm{Aut}_*^{X \vee Y}(X \vee Y \vee Z)\, \mathrm{Aut}_*^{X \vee Z}(X \vee Y \vee Z)$.

Proof. Since $\mathrm{Aut}_*^{X\vee Y}(X\vee Y\vee Z)\cap \mathrm{Aut}_*^{X\vee Z}(X\vee Y\vee Z)=id$, it is sufficient to prove that for any $f\in \mathrm{Aut}_*^{X}(X\vee Y\vee Z)$, we have $f=gh$, where $g\in \mathrm{Aut}_*^{X\vee Y}(X\vee Y\vee Z)$ and $h\in \mathrm{Aut}_*^{X\vee Z}(X\vee Y\vee Z)$.

Since $f\in \mathrm{Aut}_*(X\vee Y\vee Z)$ can be represented by (i_X,f_Y,f_Z), by Lemma 3.3 we have (i_X,i_Y,f_Z), $(i_X,f_Y,i_Z)\in \mathrm{Aut}_*(X\vee Y\vee Z)$, i.e., $(i_X,i_Y,f_Z)\in \mathrm{Aut}_*^{X\vee Y}(X\vee Y\vee Z)$ and $(i_X,f_Y,i_Z)\in \mathrm{Aut}_*^{X\vee Z}(X\vee Y\vee Z)$. It follows that

$$(i_X,f_Y,f_Z)=(i_X,i_Y,f_Z)(i_X,(i_X,i_Y,f_Z)^{-1}f_Y,i_Z).$$

Since $(i_X,i_Y,f_Z)^{-1}\,(i_X,f_Y,i_Z)=(i_X,(i_X,i_Y,f_Z)^{-1}f_Y,(i_X,i_Y,f_Z)^{-1}i_Z)$ is also in $\mathrm{Aut}_*(X\vee Y\vee Z)$, also by Lemma 3.3 we get that $(i_X,(i_X,i_Y,f_Z)^{-1}f_Y,i_Z)\in \mathrm{Aut}_*(X\vee Y\vee Z)$, i.e.,

$$(i_X,(i_X,i_Y,f_Z)^{-1}f_Y,i_Z)\in \mathrm{Aut}_*^{X\vee Z}(X\vee Y\vee Z).$$

This shows the result. □

For pointed simply-connected CW complexes $X_1,\dots,X_n$, let $\vee_i$ denote $X_1\vee\cdots\vee\hat{X}_i\vee\cdots\vee X_n$, where $\hat{X}_i$ means that X_i is omitted. Then we have the generalization of Theorem 3.4 as follows.

Theorem 3.7. $\mathrm{Aut}_*(X_1\vee\cdots\vee X_n)=\prod_{i=1}^{n}\mathrm{Aut}_*^{\vee_i}(X_1\vee\cdots\vee X_n)$.

Proof. By Lemma 3.6, for $k=2,3,\dots,n$, we have

$$\mathrm{Aut}_*^{X_1\vee\cdots\vee X_{k-1}}(X_1\vee\cdots\vee X_n)=\mathrm{Aut}_*^{X_1\vee\cdots\vee X_k}(X_1\vee\cdots\vee X_n)\,\mathrm{Aut}_*^{\vee_k}(X_1\vee\cdots\vee X_n).$$

Then by Theorem 3.4, we get

$$\begin{aligned}
&\mathrm{Aut}_*(X_1\vee\cdots\vee X_n)\\
&=\mathrm{Aut}_*^{X_1}(X_1\vee\cdots\vee X_n)\,\mathrm{Aut}_*^{\vee_1}(X_1\vee\cdots\vee X_n)\\
&=\mathrm{Aut}_*^{X_1\vee X_2}(X_1\vee\cdots\vee X_n)\,\mathrm{Aut}_*^{\vee_2}(X_2\vee\cdots\vee X_n)\,\mathrm{Aut}_*^{\vee_1}(X_1\vee\cdots\vee X_n)\\
&=\cdots\\
&=\mathrm{Aut}_*^{\vee_n}(X_1\vee\cdots\vee X_n)\cdots\mathrm{Aut}_*^{\vee_1}(X_1\vee\cdots\vee X_n)\\
&=\prod_{i=1}^{n}\mathrm{Aut}_*^{\vee_i}(X_1\vee\cdots\vee X_n).
\end{aligned}$$

This shows our result. □

Similar to Proposition 3.5, every $\mathrm{Aut}_*^{\vee_i}(X_1\vee\cdots\vee X_n)$ can also be further decomposed as follows.

Proposition 3.8. $\mathrm{Aut}_*^{\vee_i}(X_1\vee\cdots\vee X_n)$ *is the semi-direct product of* $\mathrm{Aut}_*(X_i)$ *and* $\mathrm{Aut}_{X_i*}^{\vee_i}(X_1\vee\cdots\vee X_n)$.

We now turn to give another factorization of $\mathrm{Aut}_*(X_1\vee\cdots\vee X_n)$ which is dual to that of $\mathrm{Aut}_{\sharp N}(X_1\times\cdots\times X_n)$ named as LU factorization.

We first fix some notations. For a self-map $f: X_1\vee\cdots\vee X_n\to X_1\vee\cdots\vee X_n$, let $f_k=fi_{X_k}$ and $f_{kl}=p_{X_l}fi_{X_k}$. Moreover, let ϕ_k and ψ_k be two self-maps of $X_1\vee\cdots\vee X_n$ defined by

$$\phi_k(*,\dots,*,x_i,*,\dots,*)=\begin{cases}(*,\dots,*,x_i,*,\dots,*), & i\le k\\ *, & i>k\end{cases}$$

and

$$\psi_k(*,\dots,*,x_i,*,\dots,*)=\begin{cases}(*,\dots,*,x_i,*,\dots,*), & i\ge k\\ *, & i<k\end{cases}$$

respectively, where $x_i \in X_i$. It is easy to verify that $\phi_k\phi_j = \phi_j\phi_k = \phi_{\min\{k,j\}}$ and $\psi_k\psi_j = \psi_j\psi_k = \psi_{\max\{k,j\}}$.

We now define the factors of our new factorization. For any $X_1,\dots,X_n$, let

$$\Phi(X_1,\dots,X_n) = \{f \in \mathrm{Aut}_*(X_1 \vee \cdots \vee X_n) \mid f_k = \phi_k f_k, k = 1,\dots,n\}$$

and

$$\Psi(X_1,\dots,X_n) = \{f \in \mathrm{Aut}_*(X_1\vee\cdots\vee X_n) \mid f_k = \psi_k f_k, f_{kk} = id, k = 1,\dots,n\}.$$

Defining relations for $\Phi(X_1,\dots,X_n)$ and $\Psi(X_1,\dots,X_n)$ are non-additive analogues of relations that define upper-triangular matrices and lower-triangular matrices respectively. Indeed, if we identify formally an element $f \in \mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$ with a $n \times n$-matrix (f_{jk}) with entries f_{jk}, then the elements of $\Phi(X_1,\dots,X_n)$ yield upper-triangular matrices and those of $\Psi(X_1,\dots,X_n)$ yield lower-triangular matrices with identities on the diagonal entries.

Proposition 3.9. $\Phi(X_1,\dots,X_n) \subseteq \Phi(X_1 \vee X_2,\dots,X_n) \subseteq \cdots \subseteq \Phi(X_1 \vee\cdots\vee X_n) = \mathrm{Aut}_*(X_1 \vee\cdots\vee X_n)$ *and* $\Psi(X_1,\dots,X_n) \supseteq \Psi(X_1 \vee X_2,\dots,X_n) \supseteq\cdots\supseteq \Psi(X_1 \vee \cdots \vee X_n) = \{id\}$.

Proof. For any $f \in \Phi(X_1 \vee \cdots \vee X_{k-1}, X_k,\dots,X_n)$, $k = 2,\dots,n$, in order to prove that $f \in \Phi(X_1 \vee \cdots \vee X_k, X_{k+1},\dots,X_n)$, it is obviously sufficient to prove that $\phi_{X_1\vee\cdots\vee X_k} f i_{X_1\vee\cdots\vee X_k} = f i_{X_1\vee\cdots\vee X_k}$, i.e., $\phi_{X_1\vee\cdots\vee X_k}(f_1,\dots,f_k) = (f_1,\dots,f_k)$, where $f_j = f i_{X_j}$.

For $j < k$, we have

$$\begin{aligned}&\phi_{X_1\vee\cdots\vee X_k}(f_1,\dots,f_k)i_{X_j}\\ &= \phi_{X_1\vee\cdots\vee X_k} f_j = \phi_{X_1\vee\cdots\vee X_k}\phi_{X_1\vee\cdots\vee X_{k-1}}(f_1,\dots,f_{k-1})i_{X_j}\\ &= \phi_{X_1\vee\cdots\vee X_{k-1}}(f_1,\dots,f_{k-1})i_{X_j}\\ &= (f_1,\dots,f_{k-1})i_{X_j} = (f_1,\dots,f_k)i_{X_j}\end{aligned}$$

and for $j = k$, we have $\phi_{X_1\vee\cdots\vee X_k}(f_1,\dots,f_k)i_{X_k} = (f_1,\dots,f_k)i_{X_k}$ for $f \in \Phi(X_1 \vee \cdots \vee X_{k-1}, X_k,\dots,X_n)$. By the universal property of wedge spaces, we see that $\phi_{X_1\vee\cdots\vee X_k}(f_1,\dots,f_k) = (f_1,\dots,f_k)$. This finishes the proof of the first inclusions. The second inclusions can be proved similarly. □

Proposition 3.10. $\Phi(X_1,\dots,X_n)$ *and* $\Psi(X_1,\dots,X_n)$ *are both subgroups of* $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$.

Proof. Let $f, g \in \Phi(X_1,\dots,X_n)$, then $fi_j = f_j = \phi_j f_j$, where $i_j = i_{X_j}$. Therefore for $j \le k$, we have

$$fi_j = \phi_j f_j = \phi_k\phi_j f_j = \phi_k f i_j.$$

Since $\phi_k i_j = i_j$, we have $f\phi_k i_j = \phi_k f \phi_k i_j$ according to the above result. Then by the universal property of wedge spaces, we have $f\phi_k = \phi_k f \phi_k$ and then

$$fgi_k = f\phi_k gi_k = \phi_k f\phi_k gi_k = \phi_k fgi_k.$$

It follows that $fg \in \Phi(X_1, \dots, X_n)$, hence $\Phi(X_1, \dots, X_n)$ is closed under multiplication.

In the follows, we prove inductively that for any $f \in \Phi(X_1, \dots, X_n)$, we have $f^{-1} \in \Phi(X_1, \dots, X_n)$.

When $n = 2$, $f \in \Phi(X_1, X_2)$ can be decomposed as

$$f = [f(i_1 f_{11}, i_2)^{-1}](i_1 f_{11}, i_2) = [f(i_1 f_{11}^{-1}, i_2)](i_1 f_{11}, i_2).$$

Since

$$f(i_1 f_{11}^{-1}, i_2)i_1 = fi_1 f_{11}^{-1} = \phi_1 f_1 f_{11}^{-1} = i_1 p_1 f_1 f_{11}^{-1} = i_1 f_{11} f_{11}^{-1} = i_1,$$

we have $f(i_1 f_{11}^{-1}, i_2) \in \mathrm{Aut}_*^{X_1}(X_1 \vee X_2) \subseteq \Phi(X_1, X_2)$. Since $\mathrm{Aut}_*^{X_1}(X_1 \vee X_2)$ is a group, we have

$$[f(i_1 f_{11}^{-1}, i_2)]^{-1} = (i_1 f_{11}, i_2) f^{-1} \in \mathrm{Aut}_*^{X_1}(X_1 \vee X_2) \subseteq \Phi(X_1, X_2).$$

It is easy to verify that $(i_1 f_{11}^{-1}, i_2) \in \Phi(X_1, X_2)$, so we have

$$f^{-1} = (i_1 f_{11}^{-1}, i_2)(i_1 f_{11}, i_2) f^{-1} \in \Phi(X_1, X_2).$$

This means that $\Phi(X_1, X_2)$ is closed under formation of inverses.

Suppose that $\Phi(X_1', \dots, X_{n-1}')$ is closed under formation of inverses for any simply connected CW-complexes $X_1', \dots, X_{n-1}'$ with $n > 2$. Then given any $f \in \Phi(X_1, \dots, X_n) \subseteq \Phi(X_1 \vee X_2, X_3, \dots, X_n)$ (by Proposition 3.9), we get $f^{-1} \in \Phi(X_1 \vee X_2, X_3, \dots, X_n)$ according to the inductive step. It remains to prove that $f^{-1} i_1 = \phi_1 f^{-1} i_1$, or equivalently that $p_{X_1 \vee X_2} f^{-1} i_{X_1 \vee X_2} \in \Phi(X_1, X_2)$. Since $p_{X_1 \vee X_2} f i_{X_1 \vee X_2} \in \Phi(X_1, X_2)$ which is a group, it follows that

$$p_{X_1 \vee X_2} f^{-1} i_{X_1 \vee X_2} = (p_{X_1 \vee X_2} f i_{X_1 \vee X_2})^{-1} \in \Phi(X_1, X_2).$$

According to the above results, we see that $\Phi(X_1, \dots, X_n)$ is a group.

Similarly, we can prove that $\Psi(X_1, \dots, X_n)$ is also a group. □

Now we can prove our main result as follows.

Theorem 3.11. $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n) = \Psi(X_1, \dots, X_n)\Phi(X_1, \dots, X_n)$.

Proof. Since $\Psi(X_1, \dots, X_n) \cap \Phi(X_1, \dots, X_n) = id$, it is sufficient to prove inductively the existence of the factorization of the self-equivalence in $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$.

When $n = 2$, any $f \in \mathrm{Aut}_*(X_1 \vee X_2)$ can be decomposed as

$$f = [(f_1, i_2)(i_1 f_{11}^{-1}, i_2)][(i_1 f_{11}, i_2)(f_1, i_2)^{-1} f].$$

For $(f_1, i_2)(i_1 f_{11}^{-1}, i_2) \in \mathrm{Aut}_*^{X_2}(X_1 \vee X_2)$, since

$$p_1(f_1, i_2)(i_1 f_{11}^{-1}, i_2)i_1 = p_1(f_1, i_2)i_1 f_{11}^{-1} = f_{11} f_{11}^{-1} = id_{X_1},$$

then $(f_1, i_2)(i_1 f_{11}^{-1}, i_2) \in \Psi(X_1, X_2)$.

From $(f_1,i_2)^{-1}(f_1,i_2) = ((f_1,i_2)^{-1}f_1,i_2) = id_{X_1\vee X_2} = (i_1,i_2)$, we get $(f_1,i_2)^{-1}f_1 = i_1$. It follows that

$$(f_1,i_2)^{-1}f = (f_1,i_2)^{-1}(f_1,f_2) = ((f_1,i_2)^{-1}f_1,(f_1,i_2)^{-1}f_2) = (i_1,(f_1,i_2)^{-1}f_2).$$

Then we have $(f_1,i_2)^{-1}f \in \mathrm{Aut}_*^{X_1}(X_1\vee X_2) \subseteq \Phi(X_1,X_2)$. Since obviously $(i_1f_{11},i_2) \in \Phi(X_1,X_2)$, we have $(i_1f_{11},i_2)(f_1,i_2)^{-1}f \in \Phi(X_1,X_2)$.

For any $f \in \mathrm{Aut}_*(X_1\vee\cdots\vee X_n) = \mathrm{Aut}_*((X_1\vee X_2)\vee X_3\vee\cdots\vee X_n)$, we can assume inductively that $f = f'f''$, where $f' \in \Psi(X_1\vee X_2,X_3,\ldots,X_n)$ and $f'' \in \Phi(X_1\vee X_2,X_3,\ldots,X_n)$.

Let $p_{12} = p_{X_1\vee X_2}$, $i_{12} = i_{X_1\vee X_2}$ and define $\bar{f} := p_{12}f''i_{12}$. Since f'' is reducible, we have $\bar{f} \in \mathrm{Aut}_\sharp(X_1\vee X_2)$ and then $\bar{f} = \psi\phi$, where $\psi \in \Psi(X_1,X_2)$ and $\phi \in \Phi(X_1,X_2)$. Define $\bar{\psi} := (i_{12}\psi,i_3,\ldots,i_n)$, then $\bar{\psi} \in \Psi(X_1,\ldots,X_n) \cap \Phi(X_1\vee X_2,X_3,\ldots,X_n)$. Since $f'' \in \Phi(X_1\vee X_2,X_3,\ldots,X_n)$, we have $f''i_{12} = \phi_{X_1\vee X_2}f''i_{12} = i_{12}p_{12}f''i_{12}$. Then for $\bar{\psi}^{-1}f'' \in \Phi(X_1\vee X_2,X_3,\ldots,X_n)$, we have

$$p_{12}\bar{\psi}^{-1}f''i_{12} = (p_{12}\bar{\psi}^{-1}i_{12})(p_{12}f''i_{12}) = \psi^{-1}\bar{f} = \phi \in \Phi(X_1,X_2).$$

This implies that $\bar{\psi}^{-1}f'' \in \Phi(X_1,\ldots,X_n)$. It follows that $f = (f'\bar{\psi})(\bar{\psi}^{-1}f'')$, where $f'\bar{\psi} \in \Psi(X_1,\ldots,X_n)$ and $\bar{\psi}^{-1}f'' \in \Phi(X_1,\ldots,X_n)$. This finishes the proof of the theorem. □

4. $\mathrm{Aut}_\Sigma(X)$

As a subgroup of $\mathrm{Aut}_*(X)$, $\mathrm{Aut}_\Sigma(X)$ is dual to $\mathrm{Aut}_\Omega(X)$. However, we can not find $\mathrm{Aut}_\Sigma(X)$ appears in any other reference. So in the follows, we will simply describe the general property of $\mathrm{Aut}_\Sigma(X)$ and also list some problems related to it.

First we give a characterization of $\mathrm{Aut}_\Sigma(X)$ as follows.

Proposition 4.1. *For any pointed space X, $f \in \mathrm{Aut}_\Sigma(X)$ if and only if $f \in \mathrm{Aut}(X)$ and $f^* = id : [X,\Omega Y] \to [X,\Omega Y]$ for every pointed space Y.*

Proof. ($\Longrightarrow$) Suppose that $f \in \mathrm{Aut}_\Sigma(X)$, then for any pointed space Y and $g \in [X,\Omega Y]$, we have $\hat{g}(\Sigma f) = \hat{g}$ for $\Sigma f = id$, where $\hat{g} : \Sigma X \to Y$ is the adjoint of g. Take the adjoint of the equation, we have $gf = g$, i.e., $f^*(g) = g$. This implies that $f^* = id$.

($\Longleftarrow$) Given any $f \in \mathrm{Aut}(X)$ such that $f^* = id : [X,\Omega Y] \to [X,\Omega Y]$ for every pointed space, we take $Y = \Sigma X$ and $\alpha : X \to \Omega\Sigma X$ be the adjoint of $id : \Sigma X \to \Sigma X$, then we have $f^*(\alpha) = \alpha f = \alpha$. By taking adjoint, we get $\Sigma f = id$ which implies that $f \in \mathrm{Aut}_\Sigma(X)$. □

In [5], Pavešić proved that if X is a Co-H-space, then the group $\mathrm{Aut}_\Omega(X)$ is trivial. Dually, we have the following result:

Corollary 4.2. *If X is a H-space, then $\mathrm{Aut}_\Sigma(X)$ is trivial.*

Proof. Since X is a H-space, X is a retract of ΩY for some space Y (see p.201 of [7]). Then there exist maps $r : \Omega Y \to X$ and $i : X \to \Omega Y$ such that $ri = id$. Given any $f \in \mathrm{Aut}_\Sigma(X)$, we have $f^*(i) = if = i$ by Proposition 4.1. By applying r to both sides, we get $f = id$ which shows that $\mathrm{Aut}_\Sigma(X)$ is trivial. □

In [11], Pavešić asked that if there is a finite CW-complex X such that $\mathrm{Aut}_\Omega(X) \neq \mathrm{Aut}_{\sharp\infty}(X)$. Since $\mathrm{Aut}_\Omega(X)$ is trivial when X is a Co-H-space, we may find a finite Co-H-space X such that $\mathrm{Aut}_{\sharp\infty}(X) \neq \{id\}$. Dually for $\mathrm{Aut}_\Sigma(X)$, we have a conjecture as follows.

Conjecture 4.3. *There is a finite CW complex X such that $\mathrm{Aut}_\Sigma(X) \neq \mathrm{Aut}_*(X)$.*

By Corollary 4.2, a possible approach to Conjecture 4.3 is to find a finite H-space X such that $\mathrm{Aut}_*(X) \neq \{id\}$.

In [6], Felix and Murillo showed that for pointed CW-complex X, $\mathrm{Aut}_\Omega(X)$ is a nilpotent group and its order of nilpotency is bounded by the Ljusternik-Schnirelman category of X. Naturally we have the following conjecture:

Conjecture 4.4. *For any pointed CW-complex X, $\mathrm{Aut}_\Sigma(X)$ is a nilpotent group, and its order of nilpotency is bounded by the Ljusternik-Schnirelman cocategory of X.*

According to the theorem of Maruyama [9], if the above conjecture is correct, then we will ask whether the natural map $\mathrm{Aut}_\Sigma(X) \to \mathrm{Aut}_\Sigma(X_P)$ is a P-localization for any set of primes P.

Now we turn to the factorization of $\mathrm{Aut}_\Sigma(X_1 \vee \cdots \vee X_n)$ for any pointed simply-connected CW-complexes $X_1, \ldots, X_n$. Since we have already proved that, for pointed simply-connected CW-complexes X and Y, all the self-equivalences in $\mathrm{Aut}_*(X \vee Y)$ are always reducible (see Lemma 3.2), so by a similar proof to that of Lemma 2.1, we have

Proposition 4.5. *For pointed simply-connected CW-complexes X and Y, given any $f \in \mathrm{Aut}_\Sigma(X \vee Y)$, we have $f_{XX} \in \mathrm{Aut}_\Sigma(X)$ and $f_{YY} \in \mathrm{Aut}_\Sigma(Y)$.*

This enables us to get the following theorem by a proof similar to that of Theorem 3.7.

Theorem 4.6. $\mathrm{Aut}_\Sigma(X_1 \vee \cdots \vee X_n) = \prod_{i=1}^n \mathrm{Aut}_\Sigma^{\vee_i}(X_1 \vee \cdots \vee X_n)$.

Also we can decompose $\mathrm{Aut}_\Sigma(X_1 \vee \cdots \vee X_n)$ as the product of its only two subgroups similarly to Theorem 3.11.

Theorem 4.7. $\mathrm{Aut}_\Sigma(X_1 \vee \cdots \vee X_n) = \Psi'(X_1, \ldots, X_n)\Phi'(X_1, \ldots, X_n)$, *where $\Psi'(X_1, \ldots, X_n)$ and $\Phi'(X_1, \ldots, X_n)$ are defined similarly to $\Psi(X_1, \ldots, X_n)$ and $\Phi(X_1, \ldots, X_n)$ in Section 3 respectively.*

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