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QUANTUM MARKOVIAN SEMIGROUPS ON QUANTUM SPIN SYSTEMS: GLAUBER DYNAMICS

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ABSTRACT. We study a class of KMS-symmetric quantum Markovian semigroups on a quantum spin system $(\mathcal{A}, \tau, \omega)$, where $\mathcal{A}$ is a quasi-local algebra, τ is a strongly continuous one parameter group of $*$-automorphisms of $\mathcal{A}$ and ω is a Gibbs state on $\mathcal{A}$. The semigroups can be considered as the extension of semigroups on the nontrivial abelian subalgebra. Let $\mathcal{H}$ be a Hilbert space corresponding to the GNS representation constructed from ω. Using the general construction method of Dirichlet form developed in [8], we construct the symmetric Markovian semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{H}$. The semigroup $\{T_t\}_{t\geq 0}$ acts separately on two subspaces $\mathcal{H}_d$ and $\mathcal{H}_{od}$ of $\mathcal{H}$, where $\mathcal{H}_d$ is the diagonal subspace and $\mathcal{H}_{od}$ is the off-diagonal subspace, $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_{od}$. The restriction of the semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{H}_d$ is Glauber dynamics, and for any $\eta \in \mathcal{H}_{od}$, $T_t\eta$ decays to zero exponentially fast as t approaches to the infinity.

1. Introduction

A KMS symmetric quantum Markovian semigroup $\{S_t\}_{t\geq 0}$ on a von Neumann algebra $\mathcal{M}$ is a KMS symmetric, weakly continuous, contractive and identity preserving semigroup on $\mathcal{M}$ [6]. Quantum Markovian semigroups are the natural generalization of classical Markovian semigroups and were introduced in physics to model the decay equilibrium of quantum open systems [2, 3, 6, 8, 9].

Many mathematicians and physicists are interested to the problems whether quantum Markovian semigroups on the subalgebra of a von Neumann algebra or a C^*-algebra have their extensions on the full algebra. The problem of the extension was studied in [1, 5, 7]. In [5], authors constructed a special class of generic quantum Markovian semigroups arising in the stochastic limit of a discrete system with generic free Hamiltonian interacting with a mean zero, gauge invariant, Gaussian field, and studied its properties. The semigroups are constructed on the algebra $B(h)$ of all bounded operators on a complex

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separable Hilbert space h and leave invariant not only the diagonal subalgebra but also the off diagonal subspace with respect to a fixed basis of h. The action on diagonal operators describes a classical Markov jump process. Goderis and Maes [7] studied a quantum dynamical system which is an extension of the classical system such that the property of local reversibility is preserved.

Let $\mathcal{M}$ be a von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ and ξ_0 be a fixed cyclic and separating vector for $\mathcal{M}$. Let Δ and J be the modular operator and the modular conjugation associated with the pair $(\mathcal{M}, \xi_0)$ [4]. Consider the symmetric embedding:

$$i_0 : \mathcal{M} \longrightarrow \mathcal{H}, \quad i_0(A) = \Delta^{1/4} A \xi_0.$$

For a given KMS symmetric quantum Markovian semigroup $\{S_t\}_{t\geq 0}$ on $\mathcal{M}$, the semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{H}$ defined by

$$T_t \circ i_0 = i_0 \circ S_t$$

is symmetric, strongly continuous, positive preserving, contractive and $T_t\xi_0 = \xi_0$ for all $t \geq 0$. The semigroup $\{T_t\}_{t\geq 0}$ is called a symmetric Markovian semigroup on $\mathcal{H}$. Conversely, for a given symmetric Markovian semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{H}$, the semigroup $\{S_t\}_{t\geq 0}$ on $\mathcal{M}$ defined by

$$i_0 \circ S_t = T_t \circ i_0$$

is a KMS symmetric quantum Markovian semigroup. (See Theorem 2.11 and Theorem 2.12 of [6].)

The purpose of this paper is to study a class of KMS symmetric quantum Markovian semigroups on a quantum spin system $(\mathcal{A}, \tau, \omega)$, where $\mathcal{A}$ is a quasi-local algebra, τ is a strongly continuous one parameter group of $*$-automorphisms of $\mathcal{A}$ and ω is a Gibbs state on $\mathcal{A}$. The semigroups can be considered as the extension of semigroups on the nontrivial abelian subalgebra. Let $\mathcal{H}$ be a Hilbert space corresponding to the GNS representation constructed from ω. Using the general construction method of Dirichlet forms developed in [8] (noncommutative Dirichlet form in the sense of Cipriani [6]), we construct the symmetric Markovian semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{H}$. The semigroup $\{T_t\}_{t\geq 0}$ acts separately on two subspaces $\mathcal{H}_d$ and $\mathcal{H}_{od}$ of $\mathcal{H}$, where $\mathcal{H}_d$ is the diagonal subspace and $\mathcal{H}_{od}$ is the off-diagonal subspace, $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_{od}$. The restriction of the semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{H}_d$ is Glauber dynamics, and for any $\eta \in \mathcal{H}_{od}$, $T_t\eta$ decays to zero exponentially fast as $t \to \infty$.

This paper is organized as follows. In Section 2, we introduce a quantum spin system, and construct the symmetric Markovian semigroups by employing the general construction method of Dirichlet forms developed in [8] on standard forms of von Neumann algebras. In Section 3, we give the concrete action of the semigroup constructed in Section 2 and investigate some properties of the semigroup.

2. Quantum Markovian semigroups

In this section, we introduce a quantum spin system and construct the symmetric Markovian semigroup by employing the general construction method of Dirichlet forms developed in [8] on standard forms of von Neumann algebras.

Let $\mathbb{M}_2(\mathbb{C})$ be the algebra of 2×2 matrices with complex entries. Any 2×2 matrix is decomposed as a linear combination of the Pauli matrices S^0, S^x, S^y, S^z defined by

$$S^0 = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; S^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; S^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; S^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define projection operators on $\mathbb{C}^2$ with an inner product $(\cdot, \cdot)$:

$$N = |u\rangle\langle u| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; \mathbf{1} - N = |d\rangle\langle d| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$b = |d\rangle\langle u| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; b^* = |u\rangle\langle d| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where $u = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $d = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $|\eta\rangle\langle\xi|$ denotes the one rank operator on $\mathbb{C}^2$ such that $|\eta\rangle\langle\xi|\zeta = (\xi, \zeta)\eta$.

Let $\mathcal{F}$ be the family of bounded sets in $\mathbb{Z}^\nu$:

$$\mathcal{F} = \{\Lambda \in \mathbb{Z}^\nu \; : \; \Lambda \text{ is finite}\}.$$

For each $\Lambda \in \mathcal{F}$, $\mathcal{A}_\Lambda$ is the local C^*-algebra given by

$$\mathcal{A}_\Lambda = \bigotimes_{k \in \Lambda} \mathbb{M}_k,$$

where $\mathbb{M}_k$ is an identical copy of $\mathbb{M}_2(\mathbb{C})$. We denote $\mathcal{A}_{\{k\}}$ by $\mathcal{A}_k$, $k \in \mathbb{Z}^\nu$. For $\Lambda_1, \Lambda_2 \in \mathcal{F}$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, $\mathcal{A}_{\Lambda_1}$ is isomorphic to the C^*-subalgebra $\mathcal{A}_{\Lambda_1} \otimes \mathbf{1}_{\Lambda_2}$ of $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$, where $\mathbf{1}_{\Lambda_2}$ denotes the identity operator of $\mathcal{A}_{\Lambda_2}$. We identify $\mathcal{A}_{\Lambda_1}$ and $\mathcal{A}_{\Lambda_1} \otimes \mathbf{1}_{\Lambda_2}$. Then $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$ whenever $\Lambda_1 \subset \Lambda_2$, and $[A_1, A_2] = 0$, $A_1 \in \mathcal{A}_{\Lambda_1}$, $A_2 \in \mathcal{A}_{\Lambda_2}$ whenever $\Lambda_1 \cap \Lambda_2 = \emptyset$. Here $[A, B]$ means $AB - BA$.

The quasi local algebra $\mathcal{A}$ is defined as the uniform closure of $\mathcal{A}_0$:

$$\begin{aligned} \mathcal{A}_0 &= \bigcup_{\Lambda \in \mathcal{F}} \mathcal{A}_\Lambda, \\ \mathcal{A} &= \overline{\mathcal{A}_0}. \end{aligned} \tag{2.1}$$

For each $k \in \mathbb{Z}^\nu$, the elements S_k^x, S_k^y, S_k^z, N_k, b_k^*, b_k in $\mathbb{M}_k$ are identical copies of S^x, S^y, S^z, N, b^*, b, respectively, and we write $A_k \otimes \mathbf{1}_{\{k\}^c} \in \mathcal{A}_k$ as A_k.

We consider the Ising Hamiltonian: for $\Lambda \in \mathcal{F}$,

$$\begin{aligned} H_\Lambda &= -\beta \sum_{\substack{\{k,l\} \subset \Lambda \\ |k-l|=1}} S_k^z S_l^z \\ &= -\frac{\beta}{2} \sum_{l \in \Lambda} \sum_{\substack{k \in \Lambda \\ |k-l|=1}} S_k^z S_l^z, \end{aligned} \tag{2.2}$$

where $\beta > 0$ is an inverse temperature. Clearly H_Λ is a self-adjoint element of $\mathcal{A}_\Lambda$. The time evolution τ_t is given by the strongly convergent limit of a one parameter semigroups τ_t^Λ on $\mathcal{A}_\Lambda$ such that

$$\lim_{\Lambda \to \mathbb{Z}^\nu} \|\tau_t(A) - \tau_t^\Lambda(A)\|, \quad A \in \mathcal{A}, \tag{2.3}$$

where $\tau_t^\Lambda(A) = e^{-itH_\Lambda} A e^{itH_\Lambda}$, $A \in \mathcal{A}_\Lambda$, $t \in \mathbb{R}$. (See Theorem 6.2.4 and Theorem 6.2.6 of [4].)

For $\Lambda \in \mathcal{F}$, define the local Gibbs state ω_Λ on $\mathcal{A}_\Lambda$ associated with H_Λ by

$$\omega_\Lambda(A) = \mathrm{Tr}(\rho_\Lambda A), \quad A \in \mathcal{A}_\Lambda, \tag{2.4}$$

where $\rho_\Lambda = e^{H_\Lambda}/\mathrm{Tr}(e^{H_\Lambda})$. ω_Λ has an extension $\tilde{\omega}_\Lambda$ to a state on $\mathcal{A}$ by Proposition 2.3.24 of [4]. Moreover there exist nets of extensions $\tilde{\omega}_{\Lambda_\alpha}$ of ω_{Λ_α} such that $\tilde{\omega}_{\Lambda_\alpha}$ converges weakly* to a state ω on $\mathcal{A}$:

$$\lim_{\Lambda_\alpha \to \mathbb{Z}^\nu} \tilde{\omega}_{\Lambda_\alpha}(A) = \omega(A)$$

for all $A \in \mathcal{A}$. Hence ω is a thermodynamic limit point of the local Gibbs states in the sense that

$$\omega(A) = \lim_\alpha \omega_{\Lambda_\alpha}(A)$$

for all $A \in \mathcal{A}_\Lambda$ and all $\Lambda \in \mathcal{F}$. The thermodynamic limit ω is a τ-KMS state on $\mathcal{A}$ (Proposition 6.2.15 of [4]).

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS-representation (or cyclic representation) of $(\mathcal{A}, \omega)$ [4]. Throughout this paper, we write that $\mathcal{H} = \mathcal{H}_\omega$, $\xi_0 = \Omega_\omega$, $\mathcal{M}_\Lambda = \pi_\omega(\mathcal{A}_\Lambda)$, $\mathcal{M} = \pi_\omega(\mathcal{A})''$ and $\tau_t^\omega = \pi_\omega \circ \tau_t$. To simplify the notations, we will omit π_ω such that $S_k^z := \pi_\omega(S_k^z)$, etc. Denote by $\langle \cdot, \cdot \rangle$ the inner product of $\mathcal{H}$. Let σ_t, $t \in \mathbb{R}$, be the modular automorphism with respect to ω and Δ and J be the modular operator and modular conjugation associated to the pair $(\mathcal{M}, \xi_0)$ [4], respectively. By Theorem 5.3.10 of [4], $\sigma_t = \tau_t^\omega$ and $\sigma_t(A) = \Delta^{it} A \Delta^{-it}$, $A \in \mathcal{M}$. Let $\mathcal{M}'$ be the commutant of $\mathcal{M}$. The map $j : \mathcal{M} \to \mathcal{M}'$ is the anti-linear *-isomorphism defined by $j(A) = JAJ$, $A \in \mathcal{M}$.

To construct a generator of a symmetric Markovian semigroup on $\mathcal{H}$, we introduce an (normalized) admissible function [8].

Definition 2.1. An analytic function $f : D \to \mathbb{C}$ on a domain D containing the strip $I_{1/4} = \{z \,|\, |\mathrm{Im}\, z| \le 1/4\}$ is said to be admissible if the following properties hold:

(a) $f(t) \ge 0$ for all $t \in \mathbb{R}$,
(b) $f(t + i/4) + f(t - i/4) \ge 0$ for all $t \in \mathbb{R}$,
(c) there exist $M > 0$ and $p > 1$ such that the bound

$$|f(t + is)| \le M(1 + |t|)^{-p}$$

holds uniformly in $s \in [-1/4, 1/4]$.

Moreover, if $\int f(t)dt = 1$ then it is called a normalized admissible function.

The function

$$g(t) = \frac{2}{\sqrt{2\pi}} \int (e^{k/4} + e^{-k/4})^{-1} e^{-\frac{1}{2}k^2} e^{-ikt} dk$$

is admissible. (See Lemma 3.1 of [8].)

For a fixed normalized admissible function f, we define an operator H on $\mathcal{H}$ by

$$\begin{aligned} D(H) &= \{\xi \in \mathcal{H} : \sum_{k\in\mathbb{Z}^\nu} \|H_k\xi\|^2 < \infty\}, \\ H\xi &= \sum_{k\in\mathbb{Z}^\nu} H_k\xi,\ \xi \in D(H), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} H_k &= H_{k,1} + H_{k,2} + H_{k,3}, \\ H_{k,l} &= \int [\sigma_{t+i/4}(x_{k,l}^*) - j(\sigma_{t+i/4}(x_{k,l}))] \\ &\qquad \times[\sigma_{t-i/4}(x_{k,l}) - j(\sigma_{t-i/4}(x_{k,l}^*))] f(t)dt \end{aligned} \tag{2.6}$$

for $k \in \mathbb{Z}^\nu$, $l = 1, 2, 3$ and $x_{k,1} = b_k^*$, $x_{k,2} = b_k$, $x_{k,3} = N_k$.

Theorem 2.2. *Let H be an operator defined as in* (2.5) *and* (2.6). *Then it is a generator of a strongly continuous, symmetric Markovian semigroup* $\{T_t\}_{t\geq 0}$, $T_t = e^{-tH}$ *on* $\mathcal{H}$.

Proof. By Theorem 2.1 of [3], for each $k \in \mathbb{Z}^\nu$, H_k is a (bounded) generator of a strongly continuous, symmetric Markovian semigroup on $\mathcal{H}$. See also Theorem 3.1 of [8]. Clearly $\mathcal{M}_\Lambda\xi_0 \subset D(H)$ for any $\Lambda \in \mathcal{F}$, and so H is densely defined. Since H is a symmetric operator on $\mathcal{H}$ it has a closed extension, denoted by H again. By Theorem 5.2 of [6], H is a generator of a strongly continuous, symmetric Markovian semigroup $\{T_t\}_{t\geq 0}$, $T_t = e^{-tH}$ on $\mathcal{H}$. □

3. Action of the generator H

In this section, we introduce two subspaces of $\mathcal{H}$, the diagonal subspace $\mathcal{H}_d$ and the off-diagonal subspace $\mathcal{H}_{od}$, $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_{od}$, and investigate the concrete action of the generator H of the semigroup $\{T_t\}_{t\geq 0}$ constructed in Theorem 2.2 on $\mathcal{H}_d$ and $\mathcal{H}_{od}$, respectively.

We first give elementary facts.

Lemma 3.1. (a) *The following relations hold: for any* $k \in \mathbb{Z}^\nu$

$$(S_k^z)^2 = \mathbf{1},\ b_k^2 = (b_k^*)^2 = 0, \tag{3.1}$$

$$b_k b_k^* = (\mathbf{1} - N)_k = -S_k^z(1 - N)_k,\ b_k^* b_k = N_k = S_k^z N_k, \tag{3.2}$$

$$b_k^* S_k^z = -S_k^z b_k^* = -b_k^*, \quad b_k S_k^z = -S_k^z b_k = b_k, \tag{3.3}$$

$$[S_k^z, b_k^*] = 2b_k^*,\ [S_k^z, b_k] = -2b_k. \tag{3.4}$$

(b) *The actions of modular operator* σ_t

$$\begin{aligned}\sigma_t(b_k^*) &= \exp(i\beta t \sum_{\substack{l\in\mathbb{Z}^\nu\\|l-k|=1}} S_l^z)b_k^*, \\ \sigma_t(b_k) &= \exp(-i\beta t \sum_{\substack{l\in\mathbb{Z}^\nu\\|l-k|=1}} S_l^z)b_k, \\ \sigma_t(N_k) &= N_k\end{aligned} \tag{3.5}$$

hold for any $k \in \mathbb{Z}^\nu$ *and* $t \in \mathbb{R}$.

Remark 3.2. In fact, the relations (3.5) hold for all $t \in \mathbb{C}$. (See Proposition 2.5.22 of [4].)

Proof of Lemma 3.1. (a) This directly follows from the definitions of b_k^*, b_k, N_k and S_k^z for any $k \in \mathbb{Z}^\nu$.

(b) Notice that for $A \in \mathcal{M}_\Lambda$, $\Lambda \in \mathcal{F}$

$$\begin{aligned}\delta(A) &:= \frac{d}{dt}\sigma_t(A)\,|_{t=0} \\ &= -i[H_{\Lambda'}, A],\ \Lambda \subset \Lambda'.\end{aligned} \tag{3.6}$$

Choosing $A = b_k^*$ in (3.6), we have

$$\begin{aligned}\delta(b_k^*) &= \frac{i\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu\\|l-k|=1}} S_l^z[S_k^z, b_k^*] \\ &= i\beta \sum_{\substack{l\in\mathbb{Z}^\nu\\|l-k|=1}} S_l^z b_k^*.\end{aligned}$$

Here we have used (3.4) in the second equality. Thus we get

$$\sigma_t(b_k^*) = \exp(i\beta t \sum_{\substack{l\in\mathbb{Z}^\nu\\|l-k|=1}} S_l^z)b_k^*.$$

The other relations are obtained by the similar calculation. □

We define a (diagonal) subalgebra $\mathcal{M}_d$ of $\mathcal{M}$, and two Hilbert spaces $\mathcal{H}_d$ and $\mathcal{H}_{od}$ called the diagonal subspace and the off-diagonal subspace, respectively:

$$\begin{aligned}&\mathcal{M}_d := \text{the subalgebra generated by } \{S_k^z : k \in \mathbb{Z}^\nu\}, \\ &\mathcal{H}_d := \text{the closure of } \mathcal{M}_d\xi_0 = \{A\xi_0 : A \in \mathcal{M}_d\} \text{ in } \mathcal{H}, \\ &\mathcal{H}_{od} := \text{the orthogonal complement } \mathcal{H}_d^\perp \text{ of } \mathcal{H}_d.\end{aligned}$$

Remark 3.3. (1) Since $S_k^z = N_k - (\mathbf{1} - N)_k$, $k \in \mathbb{Z}^\nu$ and $\sigma_t(N_k) = N_k$, $\mathcal{M}_d$ is the centerizer of σ_t in the sense that $\sigma_t(\mathcal{M}_d) = \mathcal{M}_d$ for all $t \in \mathbb{R}$.

(2) Since $(S_k^z)^2 = \mathbf{1}$ for $k \in \mathbb{Z}^\nu$, the subspace generated by the vectors of the form

$$\eta_\Lambda := (\prod_{k\in\Lambda} S_k^z)\xi_0,\ \Lambda \in \mathcal{F}$$

is dense in $\mathcal{H}_d$.

(3) $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_{od}$.

(4) $\mathcal{M}_d \mathcal{H}_d \subset \mathcal{H}_d$.

Theorem 3.4 (Action of the generator H on $\mathcal{H}_d$). *For each $k \in \mathbb{Z}^\nu$, let H_k be defined as in* (2.6). *Then it acts on $\mathcal{H}_d$ as follows: for any $\eta_\Lambda = (\prod_{l\in\Lambda} S_l^z)\xi_0 \in \mathcal{H}_d$, $\Lambda \in \mathcal{F}$,*

$$H_k \eta_\Lambda = \begin{cases} 4\exp\big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\big)\eta_\Lambda & \text{if } k \in \Lambda \\ 0 & \text{if } k \notin \Lambda. \end{cases} \tag{3.7}$$

Remark 3.5. (1) $T_t \mathcal{H}_d \subset \mathcal{H}_d$ for all $t \in \mathbb{R}$.

(2) For a fixed $k \in \mathbb{Z}^\nu$, let $\eta = g(S_k^z, S_l^z; l \in \mathbb{Z}^\nu \setminus \{k\})\xi_0$ and $\eta^{(k)} = g(-S_k^z, S_l^z; l \in \mathbb{Z}^\nu \setminus \{k\})\xi_0$, where g is continuous as a function on $\mathbb{R}^2$. Then

$$\eta - \eta^{(k)} = \begin{cases} 2\eta & \text{if } g \text{ is odd for } S_k^z \\ 0 & \text{if } g \text{ is even for } S_k^z. \end{cases}$$

The expression (3.7) is rewritten as

$$H_k \eta = 2\exp\big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\big)(\eta - \eta^{(k)}).$$

This is the Glauber dynamics.

Proof of Theorem 3.4. Let $\eta_\Lambda = (\prod_{l\in\Lambda} S_l^z)\xi_0 \in \mathcal{H}_d$, $\Lambda \in \mathcal{F}$. Since for fixed $k \in \Lambda$, $\prod_{\substack{l\in\Lambda \\ l\neq k}} S_l^z$ commutes with H_k by (3.5) and the definition of H_k in (2.6), we consider only

$$\eta = S_k^z \xi_0 = \sigma_{t-i/4}(S_k^z)\xi_0.$$

It follows from $[N_k, S_k^z] = 0$ that

$$H_{k,3}\,\eta = 0. \tag{3.8}$$

Next, we consider $H_{k,1}\,\eta$. By (3.4) and (3.5), we have

$$\begin{aligned} & [\sigma_{t-i/4}(b_k^*) - j(\sigma_{t-i/4}(b_k))]\eta \\ = \; & \sigma_{t-i/4}([b_k^*, S_k^z])\xi_0 \\ = \; & \sigma_{t-i/4}(-2b_k^*)\xi_0 \\ = \; & -2\exp\big((it+\frac{1}{4})\beta\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\big)b_k^*\xi_0 \end{aligned}$$

and so

$$
\begin{aligned}
&[\sigma_{t+i/4}(b_k) - j(\sigma_{t+i/4}(b_k^*))][\sigma_{t-i/4}(b_k^*) - j(\sigma_{t-i/4}(b_k))]\eta \\
= \; & -2\exp\Big((it+\frac{1}{4})\beta \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\Big)[\sigma_{t+i/4}(b_k)b_k^* - b_k^* j(\sigma_{t+i/4}(b_k^*))]\xi_0 \\
= \; & -2\Big[\exp\Big(\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\Big) b_k b_k^* - \exp\Big(-\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\Big) b_k^* b_k\Big]\xi_0 \\
= \; & -2\exp\Big(\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\Big)(\mathbf{1}-N)_k\xi_0 + 2\exp\Big(-\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\Big)N_k\xi_0. \qquad (3.9)
\end{aligned}
$$

Here we have used $j(\sigma_{t+i/4}(b_k^*))\xi_0 = \sigma_{t-3i/4}(b_k)\xi_0$ and (3.5) in the second equality and (3.2) in the third equality.

Notice that by (3.1) and (3.2)

$$
\begin{aligned}
&\exp\Big(\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\Big)(\mathbf{1}-N)_k = \exp\Big(-\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\Big)(\mathbf{1}-N)_k, \qquad (3.10)\\
&\exp\Big(-\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\Big)N_k = \exp\Big(-\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\Big)N_k.
\end{aligned}
$$

Substituting (3.10) into (3.9), we get

$$
\begin{aligned}
&[\sigma_{t+i/4}(b_k) - j(\sigma_{t+i/4}(b_k^*))][\sigma_{t-i/4}(b_k^*) - j(\sigma_{t-i/4}(b_k))]\eta \\
&= -2\exp\Big(-\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\Big)(\mathbf{1}-N)_k\xi_0 + 2\exp\Big(-\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\Big)N_k\xi_0 \\
&= 2\exp\Big(-\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\Big)S_k^z\xi_0 \\
&= 2\exp\Big(-\frac{\beta}{2} \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\Big)\eta.
\end{aligned}
$$

Thus for $\eta_\Lambda = (\prod_{l\in\Lambda} S_l^z)\xi_0$, since $\int f(t)dt = 1$, we have

$$
H_{k,1}\,\eta_\Lambda = \begin{cases} 2\exp\Big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\Big)\eta_\Lambda & \text{if } k\in\Lambda \\ 0 & \text{if } k\notin\Lambda. \end{cases} \qquad (3.11)
$$

By the similar calculation we can also check

$$
H_{k,2}\,\eta_\Lambda = H_{k,1}\,\eta_\Lambda. \qquad (3.12)
$$

The relation (3.7) follows from (3.8), (3.11) and (3.12). The proof is completed. □

In the rest of this paper, $b_k^\sharp$ is either b_k^* or b_k. Recall that

$$(S_k^z)^2 = \mathbf{1},\ \ b_k^2 = (b_k^*)^2 = 0,\ \ b_k^* b_k = N_k = \frac{1}{2}(\mathbf{1} + S_k^z),\ \text{ etc.}$$

Let $A \in \mathcal{M}$ be a monomial in the algebra generated by $\{b_k,\ b_k^*,\ S_k^z : k \in \mathbb{Z}^\nu\}$. Then $A\xi_0$ can be written as a linear combination of the vectors of the form

$$\eta_{\Lambda_1,\Lambda_2} := (\prod_{l\in\Lambda_2} b_l^\sharp)(\prod_{m\in\Lambda_1} S_m^z)\xi_0,$$

where $\Lambda_1,\ \Lambda_2 \in \mathcal{F}$ are disjoint. Clearly the family of the above vectors is dense in $\mathcal{H}$. Recall that the diagonal subspace $\mathcal{H}_d$ is spanned by the vectors of the form $\eta_{\Lambda,\emptyset} := \eta_\Lambda = (\prod_{l\in\Lambda} S_l^z)\xi_0,\ \Lambda \in \mathcal{F}$.

Lemma 3.6. (a) *The Hilbert space $\mathcal{H}_{od}$ is the closure of the subspace spanned by the vectors of the form*

$$\eta_{\Lambda_1,\Lambda_2} = (\prod_{l\in\Lambda_2} b_l^\sharp)(\prod_{m\in\Lambda_1} S_m^z)\xi_0,$$

where $\Lambda_1,\ \Lambda_2 \in \mathcal{F}$ are disjoint and $\Lambda_2 \neq \emptyset$.
(b) $\mathcal{M}_d\mathcal{H}_{od} \subset \mathcal{H}_{od}$.

Recall $\mathcal{M}_d\mathcal{H}_d \subset \mathcal{H}_d$ (Remark 3.3 (4)).

Proof of Lemma 3.6. (a) Let $\eta_{\Lambda_1,\Lambda_2} = (\prod_{l\in\Lambda_2} b_l^\sharp)(\prod_{m\in\Lambda_1} S_m^z)\xi_0$ for $\Lambda_1,\ \Lambda_2 \in \mathcal{F}$, $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\Lambda_2 \neq \emptyset$. Notice that for any $\Lambda_3 \in \mathcal{F}$

$$\langle \eta_{\Lambda_3}, \eta_{\Lambda_1,\Lambda_2}\rangle = \langle \xi_0, (\prod_{l\in\Lambda_2} b_l^\sharp)(\prod_{m\in\Lambda_1\triangle\Lambda_3} S_m^z)\xi_0\rangle,$$

where we have used the KMS condition and $(S_k^z)^2 = \mathbf{1}$.

Since $\mathcal{H}_d$ is spanned by the vectors of the form $(\prod_{l\in\Lambda} S_l^z)\xi_0,\ \Lambda \in \mathcal{F}$, it suffices to show that

$$\langle \xi_0, (\prod_{l\in\Lambda_2} b_l^\sharp)(\prod_{m\in\Lambda_1\triangle\Lambda_3} S_m^z)\xi_0\rangle = 0 \text{ for } \Lambda_2 \neq \emptyset. \tag{3.13}$$

Using the KMS condition, we have

$$\langle \xi_0, b_l^\sharp S_l^z \eta_{\Lambda_4,\Lambda_5}\rangle = \langle \xi_0, S_l^z b_l^\sharp \eta_{\Lambda_4,\Lambda_5}\rangle$$

for $l \notin \Lambda_4 \cup \Lambda_5$, $\Lambda_4 \cap \Lambda_5 = \emptyset$, $\Lambda_4, \Lambda_5 \in \mathcal{F}$, and so

$$\langle \xi_0, [b_l^\sharp, S_l^z]\eta_{\Lambda_4,\Lambda_5}\rangle = 0,$$

which implies (3.13). We obtain from (3.4) that

$$\langle \xi_0, b_l^\sharp \eta_{\Lambda_4,\Lambda_5}\rangle = 0 \ \text{ if } \ l \notin \Lambda_4 \cup \Lambda_5.$$

The part (a) of lemma is proved.
(b) This follows from the facts $S_l^z b_l^* = b_l^*,\ S_l^z b_l = -b_l$. □

Theorem 3.7 (Action of the generator H on $\mathcal{H}_{od}$). *For each $k \in \mathbb{Z}^\nu$, let H_k be defined as in (2.6). Then it acts on $\mathcal{H}_{od}$ as follows: for any*

$$\eta_{\Lambda_1,\Lambda_2} = (\prod_{l\in\Lambda_2} b_l^\sharp)(\prod_{m\in\Lambda_1} S_m^z)\xi_0,$$

$\Lambda_1, \Lambda_2 \in \mathcal{F}$, $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\Lambda_2 \neq \emptyset$,

(3.14)

$$H_k\,\eta_{\Lambda_1,\Lambda_2} = \begin{cases} 0 & \text{if } k \notin \Lambda_1 \cup \Lambda_2, \\ \left[1 + 2\cosh(\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu\\|l-k|=1}} \varepsilon_l S_k^z S_l^z)\right]\eta_{\Lambda_1,\Lambda_2} & \text{if } k \in \Lambda_2, \\ 4\exp\left(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu\\|l-k|=1}} \varepsilon_l S_k^z S_l^z\right)\eta_{\Lambda_1,\Lambda_2} & \text{if } k \in \Lambda_1. \end{cases}$$

Here $\varepsilon_l = -1$ if $l \in \Lambda_2$ and $\varepsilon_l = 1$ if $l \notin \Lambda_2$.

Proof. Let

$$\eta_{\Lambda_1,\Lambda_2} = (\prod_{l\in\Lambda_2} b_l^\sharp)(\prod_{m\in\Lambda_1} S_m^z)\xi_0$$

for $\Lambda_1, \Lambda_2 \in \mathcal{F}$, $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\Lambda_2 \neq \emptyset$. Recall that $\prod_{\substack{l\in\Lambda_1\\l\neq k}} S_l^z$ and $\prod_{\substack{l\in\Lambda_2\\l\neq k}} b_l^\sharp$ commute with H_k. If $k \notin \Lambda_1 \cup \Lambda_2$, then

$$H_k\,\eta_{\Lambda_1,\Lambda_2} = 0. \tag{3.15}$$

To prove the case $k \in \Lambda_2$, let

$$\eta_{\Lambda_1,\Lambda_2} = b_k^*(\prod_{\substack{l\in\Lambda_2\\l\neq k}} b_l^\sharp)(\prod_{m\in\Lambda_1} S_m^z)\xi_0.$$

Since $[N_k, b_k^*] = b_k^*$ and $\int f(t)dt = 1$, it is easily checked that

$$H_{k,3}\,\eta_{\Lambda_1,\Lambda_2} = \eta_{\Lambda_1,\Lambda_2}. \tag{3.16}$$

Also we get from the identity $(b_k^*)^2 = 0$ that

$$H_{k,1}\,\eta_{\Lambda_1,\Lambda_2} = 0. \tag{3.17}$$

Now, we calculate $H_{k,2}\,\eta_{\Lambda_1,\Lambda_2}$. We will again adopt the similar calculations used to get the relation (3.11) in the proof of Theorem 3.4. By (3.5) and $[b_k, b_k^*] = -S_k^z$, we have

$$\begin{aligned} &[\sigma_{t-i/4}(b_k) - j(\sigma_{t-i/4}(b_k^*))]b_k^*\xi_0 \\ = \;& \exp\left(-(it+\frac{1}{4})\beta \sum_{\substack{l\in\mathbb{Z}^\nu\\|l-k|=1}} S_l^z\right)[b_k, b_k^*]\xi_0 \\ = \;& -\exp\left(-(it+\frac{1}{4})\beta \sum_{\substack{l\in\mathbb{Z}^\nu\\|l-k|=1}} S_l^z\right)S_k^z\xi_0 \end{aligned}$$

and hence

$$\begin{aligned}
&[\sigma_{t+i/4}(b_k^*) - j(\sigma_{t+i/4}(b_k))][\sigma_{t-i/4}(b_k) - j(\sigma_{t-i/4}(b_k^*))]b_k^*\xi_0 \\
= \;& -\exp\big(-(it+\frac{1}{4})\beta \sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\big)[\sigma_{t+i/4}(b_k^*)S_k^z - S_k^z j(\sigma_{t+i/4}(b_k))]\xi_0 \\
= \;& \big[-\exp\big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\big)b_k^* S_k^z + \exp\big(\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_l^z\big)S_k^z b_k^*\big]\xi_0 \\
= \;& \exp\big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\big)b_k^*\xi_0 + \exp\big(\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\big)b_k^*\xi_0 \\
= \;& [\exp\big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\big) + \exp\big(\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} S_k^z S_l^z\big)]b_k^*\xi_0,
\end{aligned}$$

where we used (3.3) in the third equality.

Applying $b_k^\sharp \exp(\pm\frac{\beta}{2}S_k^z) = \exp(\mp\frac{\beta}{2}S_k^z)b_k^\sharp$ to the above, and since the function $f(t)$ is normalized admissible, we have

$$\begin{aligned}
H_{k,2}\eta_{\Lambda_1,\Lambda_2} &= [\exp\big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} \varepsilon_l S_k^z S_l^z\big) + \exp\big(\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} \varepsilon_l S_k^z S_l^z\big)]\eta_{\Lambda_1,\Lambda_2} \\
&= 2\cosh\big(\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} \varepsilon_l S_k^z S_l^z\big)\eta_{\Lambda_1,\Lambda_2}, \qquad (3.18)
\end{aligned}$$

where $\varepsilon_l = -1$ if $l \in \Lambda_2$ and $\varepsilon_l = 1$ if $l \notin \Lambda_2$.

For $\eta_{\Lambda_1,\Lambda_2} = b_k(\prod_{\substack{l\in\Lambda_2 \\ l\neq k}} b_l^\sharp)(\prod_{m\in\Lambda_1} S_m^z)\xi_0$, the similar calculation as the above gives

$$\begin{aligned}
&H_{k,3}\,\eta_{\Lambda_1,\Lambda_2} = \eta_{\Lambda_1,\Lambda_2}, \\
&H_{k,1}\,\eta_{\Lambda_1,\Lambda_2} = 0, \qquad (3.19) \\
&H_{k,2}\,\eta_{\Lambda_1,\Lambda_2} = 2\cosh\big(\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} \varepsilon_l S_k^z S_l^z\big)\eta_{\Lambda_1,\Lambda_2}.
\end{aligned}$$

Hence we get

$$H_k\,\eta_{\Lambda_1,\Lambda_2} = [\mathbf{1} + 2\cosh\big(\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu \\ |l-k|=1}} \varepsilon_l S_k^z S_l^z\big)]\eta_{\Lambda_1,\Lambda_2}.$$

Next, to prove the case $k \in \Lambda_1$, let

$$\eta_{\Lambda_1,\Lambda_2} = S_k^z\big(\prod_{l\in\Lambda_2} b_l^\sharp\big)\big(\prod_{\substack{m\in\Lambda_1 \\ m\neq k}} S_m^z\big)\xi_0.$$

By the similar calculation used to (3.16), (3.17) and (3.18), we have

$$H_{k,3}\,\eta_{\Lambda_1,\Lambda_2} = 0, \tag{3.20}$$

$$\begin{aligned} H_{k,1}\,\eta_{\Lambda_1,\Lambda_2} &= H_{k,2}\,\eta_{\Lambda_1,\Lambda_2} \\ &= 2\Big(\prod_{l\in\Lambda_2} b_l^\sharp\Big)\Big(\prod_{\substack{m\in\Lambda_1\\ m\neq k}} S_m^z\Big)\exp\Big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu\\ |l-k|=1}}\varepsilon_l S_k^z S_l^z\Big)S_k^z\xi_0, \end{aligned} \tag{3.21}$$

where ε_l is defined as in (3.18). Applying (3.3) to (3.21), we get

$$\begin{aligned} H_{k,1}\,\eta_{\Lambda_1,\Lambda_2} &= H_{k,2}\,\eta_{\Lambda_1,\Lambda_2} \\ &= 2\exp\Big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu\\ |l-k|=1}}\varepsilon_l S_k^z S_l^z\Big)\eta_{\Lambda_1,\Lambda_2}. \end{aligned}$$

Thus

$$H_k\,\eta_{\Lambda_1,\Lambda_2} = 4\exp\Big(-\frac{\beta}{2}\sum_{\substack{l\in\mathbb{Z}^\nu\\ |l-k|=1}}\varepsilon_l S_k^z S_l^z\Big)\eta_{\Lambda_1,\Lambda_2}.$$

The proof is completed. □

Remark 3.8. (1) $T_t\mathcal{H}_{od}\subset\mathcal{H}_{od}$ for all $t\in\mathbb{R}$.

(2) For any $\eta\in\mathcal{H}_{od}$, $\langle\eta, H\eta\rangle\geq\|\eta\|^2$. In particular, for any $\eta\in\mathcal{H}_{od}$, $T_t\eta$ decays to zero exponentially fast as $t\to\infty$.

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