LINEAR PRESERVERS OF SPANNING COLUMN RANK OF MATRIX PRODUCTS OVER SEMIRINGS

SEOK-ZUN SONG, GI-SANG CHEON, AND YOUNG-BAE JUN

ABSTRACT. The spanning column rank of an $m \times n$ matrix A over a semiring is the minimal number of columns that span all columns of A. We characterize linear operators that preserve the sets of matrix ordered pairs which satisfy multiplicative properties with respect to spanning column rank of matrices over semirings.

1. Introduction

Let $\mathbb{F}$ be a field and $\mathbb{M}_n(\mathbb{F})$ be the vector space of all $n \times n$ matrices. In the last few decades a lot of work has been done on the problems of determining the linear maps on $\mathbb{M}_n(\mathbb{F})$ that leave certain matrix relations, subsets or properties invariant. For a survey of these types of problems see [6]. Although the linear preservers concerned are mostly linear operators on matrix spaces over fields or rings, the same problem has been extended to matrices over various semirings.

Marsaglia and Styan [5] studied the inequalities for rank of matrices. Recently, Beasley and Guterman [1] investigated rank inequalities of matrices over semirings. And they characterized the equality cases for some inequalities in [2]. These characterization problems are open even over fields (see [5]). The structure of matrix varieties which arise as extremal cases in these inequalities is far from being understood over fields, as well as over semirings. A usual way to generate elements of such a variety is to find a tuple of matrices which belongs to it and to act on this tuple by various linear operators that preserve this variety. The complete classification of linear operators that preserve equality cases in matrix inequalities over fields was obtained in [3]. Song and Hwang ([8]) characterized the linear operators that preserve spanning column ranks of matrices over nonnegative reals. Recently Song ([7]) obtained characterizations of the linear operators of spanning column rank of matrix sums over semirings.

In this paper, we characterize linear operators that preserve the sets of matrix ordered pairs which satisfy multiplicative properties with respect to spanning column rank of matrices over semirings.

2. Preliminaries

A *semiring* $\mathbb{S}$ is essentially a ring in which only the zero is required to have an additive inverse ([9]). Thus all rings are semirings. A semiring is called *antinegative* if the zero element is the only element with an additive inverse. The set of nonnegative integers is an example of antinegative semiring but it is not a ring.

A semiring $\mathbb{S}$ is called *Boolean* if $\mathbb{S}$ is equivalent to a set of subsets of a given set M, the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set M.

It is straightforward to see that a Boolean semiring is commutative and antinegative. If $\mathbb{S}$ consists of only the empty subset and M then it is called a binary Boolean semiring (or $\{0,1\}$-semiring) and is denoted by $\mathbb{B}$.

A semiring is called a *chain* if the set $\mathbb{S}$ is totally ordered with universal lower and upper bounds and the operations are defined by $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$.

It is straightforward to see that any chain semiring $\mathbb{S}$ is a Boolean semiring on the set of appropriate subsets of $\mathbb{S}$.

Let $\mathbb{M}_{m,n}(\mathbb{S})$ denote the set of $m \times n$ matrices with entries from the semiring $\mathbb{S}$. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{S})$ instead of $\mathbb{M}_{n,n}(\mathbb{S})$.

A vector space is usually only defined over fields or division rings, and modules are generalizations of vector spaces defined over rings. We generalize the concept of vector spaces to *semiring vector spaces* defined over arbitrary semirings.

Given a semiring $\mathbb{S}$, we define a *semiring vector space*, $V(\mathbb{S})$, to be a nonempty set with two operations, addition and scalar multiplication such that $V(\mathbb{S})$ is closed under addition and scalar multiplication, addition is associative and commutative, and such that for all $\mathbf{u}$ and $\mathbf{v}$ in $V(\mathbb{S})$ and $r, s \in \mathbb{S}$:

1. There exists a $\mathbf{0}$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$,
2. $1\mathbf{v} = \mathbf{v} = \mathbf{v}1$,
3. $rs\mathbf{v} = r(s\mathbf{v})$,
4. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$, and
5. $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.

A set of vectors, $\mathbb{W}$, from a semiring vector space, $V(\mathbb{S})$ is called *linearly independent* if there is no vector in $\mathbb{W}$ that can be expressed as a nontrivial linear combination of the others. The set is *linearly dependent* if it is not independent.

Note that, unlike vectors over fields, there are several ways to define independence, we will use the definition above.

A collection, B, of linearly independent vectors is said to be a *basis* of the semiring vector space $V(\mathbb{S})$ if its linear span is $V(\mathbb{S})$. The *dimension* of $V(\mathbb{S})$ is a minimal number of vectors in any basis of $V(\mathbb{S})$.

The following rank functions are usual in the semiring context.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *factor rank* k ($\mathrm{rank}(A) = k$) if there exist matrices $B \in \mathbb{M}_{m,k}(\mathbb{S})$ and $C \in \mathbb{M}_{k,n}(\mathbb{S})$ such that $A = BC$ and k is the smallest positive integer for which such factorization exists. By definition, the only matrix with factor rank 0 is the zero matrix, O.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *column rank* k ($c(A) = k$) if the dimension of the linear span of the columns of A is equal to k.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *spanning column rank* k ($\mathrm{sc}(A) = k$) if the minimal number of columns that span all columns of A is k.

It follows that

$$1 \leq \mathrm{rank}(A) \leq c(A) \leq \mathrm{sc}(A) \leq n$$

for all nonzero matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ (see [1, 4, 8]). These inequalities may be strict: let

$$A = \begin{bmatrix} 3 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 3 - \sqrt{7} & \sqrt{7} - 2 \end{bmatrix} \in \mathbb{M}_{1,2}(\mathbb{S}),$$

where $\mathbb{S} = (\mathbb{Z}\,[\sqrt{7}])^+$ is the semiring of nonnegative elements of the ring $\mathbb{Z}\,[\sqrt{7}]$. Then $\mathrm{rank}(A) = 1 < 2 = c(A)$, and $c(B) = 1 < 2 = \mathrm{sc}(B)$ since $(3 - \sqrt{7}) + (\sqrt{7} - 2) = 1$ but any one column of B does not span the other column over $\mathbb{S} = (\mathbb{Z}\,[\sqrt{7}])^+$.

A *line* of a matrix A is a row or a column of A.

If $\mathbb{S}$ is a subsemiring of a field then there is a usual rank function $\rho(A)$ for any matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$. Easy examples show that over semirings these functions are not equal in general. However, the inequalities $\mathrm{sc}(A) \geq \rho(A)$ always hold. The behavior of the function ρ with respect to matrix multiplication and addition is given by well-known Frobenius, Schwartz and Sylvester inequalities. Arithmetic properties of spanning column ranks depend on the structure of semiring of entries.

For matrices $X = [x_{i,j}]$ and $Y = [y_{i,j}]$ in $\mathbb{M}_{m,n}(\mathbb{S})$, the matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the $(i,j)^{\text{th}}$ entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

We say that the matrix A *dominates* the matrix B if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$ in this case.

If A and B are matrices with $A \geq B$, then we let $A \backslash B$ denote the matrix C, where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0; \\ a_{i,j} & \text{otherwise.} \end{cases}$$

Let $\mathbb{Z}(\mathbb{S})$ denote the center of the semiring $\mathbb{S}$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context

and we write I, J, and O, respectively. The matrix $E_{i,j}$, called a *cell*, denotes the matrix with 1 in (i,j) position and zero elsewhere. A *weighted cell* is any nonzero scalar multiple of a cell, i.e., $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{S}$. Let R_i denote the matrix whose i^{th} row is all ones and all other rows are zero, and C_j denote the matrix whose j^{th} column is all ones and all other columns are zero. We let $|A|$ denote the number of nonzero entries in the matrix A. We denote by $A[i_1, \ldots, i_k | j_1, \ldots, j_l]$ the $k \times l$-submatrix of A which lies in the intersection of the $i_1, \ldots, i_k$ rows and $j_1, \ldots, j_l$ columns.

Let $\Delta_{m,n} = \{(i,j) | i = 1, \ldots, m; j = 1, \ldots, n\}$. If $m = n$, we use the notation Δ_n instead of $\Delta_{n,n}$.

Let $\mathbb{S}$ be a semiring, not necessary commutative. A map $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ is called *linear operator* if T preserves matrix addition and scalar multiplication on both sides.

We say that a linear operator T *preserves* a set $\mathbb{P}$ if $X \in \mathbb{P}$ implies that $T(X) \in \mathbb{P}$, or, if $\mathbb{P}$ is a set of ordered pairs, that $(X, Y) \in \mathbb{P}$ implies $(T(X), T(Y)) \in \mathbb{P}$.

An operator T on $\mathbb{M}_{m,n}(\mathbb{S})$ is called a (P, Q, B)-*operator* if there exist permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$, and a matrix $B \in \mathbb{M}_{m,n}(\mathbb{S})$ with $B \geq J$ such that

$$(2.1) \qquad\qquad T(X) = P(X \circ B)Q$$

for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$ or, $m = n$ and

$$(2.2) \qquad\qquad T(X) = P(X \circ B)^t Q$$

for all $X \in \mathbb{M}_n(\mathbb{S})$, where X^t denotes the transpose of X. Operators of the form (2.1) are called *non-transposing* (P, Q, B)-*operators*; operators of the form (2.2) are *transposing* (P, Q, B)-*operators*.

Unless otherwise specified, we will assume that $\mathbb{S}$ is an antinegative semiring without zero divisors in the following.

We recall some results proved in [2] for later use.

Theorem 2.1 ([2, Theorem 2.14]). *Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ be a linear operator. Then the following are equivalent:*

(1) *T is bijective.*
(2) *T is surjective.*
(3) *There exists a permutation σ on $\Delta_{m,n}$ and units $b_{i,j} \in \mathbb{Z}(\mathbb{S})$ such that $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ for all $(i,j) \in \Delta_{m,n}$.*

Lemma 2.2 ([2, Lemma 2.16]). *Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ be an operator which maps lines to lines and is defined by $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$, where σ is a permutation on $\Delta_{m,n}$, and $b_{i,j} \in \mathbb{Z}(\mathbb{S})$ are nonzero elements. Then T is a (P, Q, B)-operator.*

One can easily check that if $m = 1$ or $n = 1$ then all operators under consideration are (P, Q, B)-operators, if $m = n = 1$ then all operators under consideration are (P, P^t, B)-operators.

Henceforth we will always assume that $m, n \geq 2$.

We say that $\mathbb{M}_{m,n}(\mathbb{S})$ has *full spanning column rank* if for each $k \leq \min\{m, n\}$, $\mathbb{M}_{m-k,n-k}(\mathbb{S})$ contains a matrix of spanning column rank $n - k$.

If $m \geq n$, then we can easily show that $\mathbb{M}_{m,n}(\mathbb{S})$ has full spanning column rank. But, for $m < n$, $\mathbb{M}_{m,n}(\mathbb{S})$ may or may not have full spanning column rank according to a given semiring $\mathbb{S}$. For example, $\mathbb{M}_{2,3}(\mathbb{Z}^+)$ has full spanning column rank, while $\mathbb{M}_{2,3}(\mathbb{B})$ is not.

The spanning column ranks of matrix products over semirings are restricted by the following list of inequalities established in [1] :

If $O \neq X \in \mathbb{M}_{m,n}(\mathbb{S}), O \neq Y \in \mathbb{M}_{n,k}(\mathbb{S})$

$$(2.3) \qquad\qquad \mathrm{sc}(XY) \leq \mathrm{sc}(Y).$$

If $\mathrm{sc}(X) + \mathrm{sc}(Y^t) > n$, then

$$(2.4) \qquad\qquad \mathrm{sc}(XY) \geq 1.$$

Let $\mathbb{S}$ be a subsemiring of $\mathbb{R}^+$, the nonnegative reals.
If $\rho(X) + \rho(Y) \leq n$, then

$$(2.5) \qquad\qquad \mathrm{sc}(XY) \geq 0.$$

If $\rho(X) + \rho(Y) > n$, then

$$(2.6) \qquad\qquad \mathrm{sc}(XY) \geq \rho(X) + \rho(Y) - n.$$

As it was proved in [1] the above inequalities $(2.3) \sim (2.6)$ are sharp and the best possible.

The following example shows that standard analogs for the upper bound of the rank of a product of two matrices over a field do not work for spanning column ranks.

Example 2.3. Let

$$A = (3, 7, 7) \in \mathbb{M}_{1,3}(\mathbb{Z}^+), \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{M}_3(\mathbb{Z}^+),$$

where $\mathbb{Z}^+$ is the semiring of nonnegative integers. Then $\mathrm{sc}(A) = 2$, $\mathrm{sc}(B) = 3$, and $\mathrm{sc}(AB) = \mathrm{sc}(3, 10, 17) = 3$ over $\mathbb{Z}^+$.

Lemma 2.4. Let B be a matrix in $\mathbb{M}_{m,n}(\mathbb{S})$ with $\mathrm{sc}(B) = 1$. If all elements of B are units in $\mathbb{Z}(\mathbb{S})$, then $\mathrm{sc}(X) = \mathrm{sc}(P(X \circ B)Q)$ for all permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$.

Proof. Let X be any matrix in $\mathbb{M}_{m,n}(\mathbb{S})$. If $Q \in \mathbb{M}_n(\mathbb{S})$ is a permutation matrix, it is clear that $\mathrm{sc}(X) = \mathrm{sc}(XQ)$. Thus, for all permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$, we have

$$\mathrm{sc}(X) = \mathrm{sc}(P^t P X Q) \leq \mathrm{sc}(PXQ) \leq \mathrm{sc}(XQ) = \mathrm{sc}(X)$$

from (2.5), and hence $\mathrm{sc}(X) = \mathrm{sc}(PXQ)$. Thus, we claim that $\mathrm{sc}(X) = \mathrm{sc}(X \circ B)$.

Since $\mathrm{sc}(B) = 1$, there exists a column $\mathbf{b_i} = (b_{1,i}, \ldots, b_{m,i})^t \in \mathbb{S}^m$ such that $B = \mathbf{b_i}\mathbf{e} = [e_1\mathbf{b_i}, e_2\mathbf{b_i}, \ldots, e_n\mathbf{b_i}]$ with $\mathbf{e} = (e_1, \ldots, 1, \ldots, e_n) \in \mathbb{S}^n$. Thus, for any matrix $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] \in \mathbb{M}_{m,n}(\mathbb{S})$, we have $X \circ B = [(\mathbf{x}_1 \circ e_1\mathbf{b_i}), (\mathbf{x}_2 \circ e_2\mathbf{b_i}), \ldots, (\mathbf{x}_n \circ e_n\mathbf{b_i})]$. Let $\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k}$ be any columns of X. Then it suffices to show that $\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k}$ are spanning columns for all columns of X if and only if $(\mathbf{x}_{i_1} \circ e_{i_1}\mathbf{b_i}), \ldots, (\mathbf{x}_{i_k} \circ e_{i_k}\mathbf{b_i})$ are spanning columns for all columns of $X \circ B$. Let $\mathrm{sc}(X) = k$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be spanning columns for all the columns of X. Then $\mathbf{x}_r = \sum_{j=1}^{k} c_{r_j}\mathbf{x}_j$ for all $r = 1, \ldots, n$ with $c_{r_j} \in \mathbb{S}$. Then we have $(\mathbf{x}_r \circ e_r\mathbf{b_i}) = \sum_{j=1}^{k} c_{r_j}(\mathbf{x}_j \circ e_r\mathbf{b_i})$, equivalently, $\mathbf{x}_1 \circ e_1\mathbf{b_i}, \ldots, \mathbf{x}_k \circ e_k\mathbf{b_i}$ are spanning columns for all columns of $X \circ B$.

Conversely, assume that $\mathrm{sc}(X \circ B) = k$ and $(\mathbf{x}_1 \circ e_1\mathbf{b_i}), \ldots, (\mathbf{x}_k \circ e_k\mathbf{b_i})$ are spanning columns for all the columns of $X \circ B$ without loss of generality. Then for any column of $X \circ B$ we can write $(\mathbf{x}_r \circ e_r\mathbf{b_i}) = \sum_{j=1}^{k} f_j(\mathbf{x}_j \circ e_j\mathbf{b_i})$ for $f_j \in \mathbb{S}$. Let $\mathbf{b_i}' = (b_{1,i}^{-1}, \ldots, b_{m,i}^{-1})^t \in \mathbb{S}^m$. Then

$$(\mathbf{x}_r \circ e_r\mathbf{b_i}) \circ \mathbf{b_i}' = \left(\sum_{j=1}^{k} f_j(\mathbf{x}_j \circ e_j\mathbf{b_i}) \right) \circ \mathbf{b_i}',$$

equivalently $(e_r\mathbf{b_i} \circ \mathbf{b_i}') \circ \mathbf{x}_r = \sum_{j=1}^{k} f_j e_j(\mathbf{b_i} \circ \mathbf{b_i}') \circ \mathbf{x}_j$ since all entries of B are in $Z(\mathbb{S})$. Hence $e_r\mathbf{x}_r = \sum_{j=1}^{k} f_j e_j\mathbf{x}_j$ because $\mathbf{b_i} \circ \mathbf{b_i}' = (1, \ldots, 1)^t \in \mathbb{S}^m$. That is, $\mathbf{x}_r = \sum_{j=1}^{k} e_r^{-1} f_j e_j\mathbf{x}_j$, equivalently, $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are spanning columns for all columns of X. $\qquad\square$

Let $X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be a matrix in $\mathbb{M}_{2,1}(\mathbb{Z}^+)$. Then we have that $\mathrm{sc}(X) = 1$, but $\mathrm{sc}(X^t) = 2$. Thus, in general, it is not true that for a matrix $X \in \mathbb{M}_{m,n}(\mathbb{S})$, $\mathrm{sc}(X) = 1$ if and only if $\mathrm{sc}(X^t) = 1$.

Below, we use the following notations in order to denote sets of matrices that arise as extremal cases in the inequalities $(2.3) \sim (2.6)$ listed above:

$\Pi_L = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid \mathrm{sc}(XY) = \mathrm{sc}(Y)\};$
$\Pi_0 = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid \mathrm{sc}(XY) = 0\};$
$\Pi_1 = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid \mathrm{sc}(X) + \mathrm{sc}(Y^t) > n \text{ and } \mathrm{sc}(XY) = 1\};$
$\Pi_R = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid \mathrm{sc}(XY) = \rho(X) + \rho(Y) - n\}.$

In the following sections, we characterize linear operators that preserve the sets Π_L, Π_0, Π_1, and Π_R.

3. Linear preservers of Π_L

Lemma 3.1. *If $T : \mathbb{M}_n(\mathbb{S}) \to \mathbb{M}_n(\mathbb{S})$ is a surjective linear operator which preserves Π_L, then T preserves lines.*

Proof. By Theorem 2.1, there exist a permutation σ on Δ_n and units $b_{i,j} \in \mathbb{Z}(\mathbb{S})$ such that $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ for all $(i,j) \in \Delta_n$. Suppose that T^{-1} does not map columns to lines, say, without loss of generality, that $T^{-1}(E_{1,1} + E_{2,1}) \geq E_{1,1} + E_{2,2}$. Then $T(I)$ has nonzero entries in at most $n-1$ columns. Suppose $T(I)$ has all zero entries in column j. Then for $X = I$ and $Y = T^{-1}(E_{j,1})$, we have $XY = Y$ however, $T(X)T(Y) = O$. This contradicts the fact that T preserves Π_L. Suppose that T^{-1} does not map rows to lines. Say, without loss of generality, that $T^{-1}(E_{1,1} + E_{1,2}) \geq E_{1,1} + E_{2,2}$. That is $T(E_{1,1} + E_{2,2}) = b_{1,1}E_{1,1} + b_{2,2}E_{1,2}$. Then for $X = b_{1,1}^{-1}E_{1,1} + b_{2,2}^{-1}E_{2,2} + [O_2 \oplus I_{n-2}]$, $T(X)$ has spanning column rank at most $n-1$ since either the first two columns of $T(X)$ are equal or at least one of the columns from the 3rd through the nth is zero. Let $Y = T^{-1}(I)$, then we have that $(X,Y) \in \Pi_L$, since $\text{sc}(XZ) = \text{sc}(Z)$ for any Z, while $\text{sc}(T(X)I) = \text{sc}(T(X)) = n - 1 < \text{sc}(I) = \text{sc}(T(Y))$ so that $(T(X), T(Y)) \notin \Pi_L$, a contradiction.

Similarly, if $T^{-1}(E_{1,1} + E_{2,1}) \geq E_{1,1} + E_{2,2}$ then the second column of $T(X)$ is zero and the same pair $(X,Y) \in \Pi_L$ gives the contradiction.

Thus T^{-1} and hence T map lines to lines. $\square$

Lemma 3.2. *Let $T : \mathbb{M}_n(\mathbb{S}) \to \mathbb{M}_n(\mathbb{S})$ be a linear operator defined by $T(X) = X \circ B$, where $B = [b_{i,j}] \in \mathbb{M}_n(\mathbb{Z}(\mathbb{S}))$, $b_{i,j}$ are units for all $(i,j) \in \Delta_n$. If T preserves Π_L, then $\text{sc}(B) = 1$. If, in addition, $\mathbb{S}$ is commutative then there exist diagonal matrices D, E with the invertible elements on the main diagonal such that $T(X) = DXE$.*

Proof. Suppose to the contrary that $\text{sc}(B) \geq 2$. Since all $b_{i,j}$ are units it follows that there exist k, l such that $B[1, \ldots, n \mid k]$ does not span $B[1, \ldots, n \mid l]$ and conversely. Without loss of generality one may assume that $k = 1, l = 2$.

Let T preserve Π_L. Consider the pair $X = E_{1,1}$, $Y = C_1 + C_2$. One has that $XY = E_{1,1} + E_{1,2}$ and $\text{sc}(XY) = 1 = \text{sc}(Y)$, i.e., $(X,Y) \in \Pi_L$. However, the spanning column rank of $(X \circ B)(Y \circ B) = b_{1,1}^2 E_{1,1} + b_{1,1}b_{1,2}E_{1,2}$ is 1 since $b_{i,j}$ are units and commute with all elements from $\mathbb{S}$. Thus $\text{sc}((X \circ B)(Y \circ B)) = 1 \neq 2 = \text{sc}(Y \circ B)$ since the first two columns of $Y \circ B$ are the same as those of B and the other columns are zero. Hence, $(T(X), T(Y)) \notin \Pi_L$, which contradicts the assumption that T preserves Π_L. Therefore $\text{sc}(B) = 1$.

Thus it follows that there exist column $\mathbf{b_i} = (b_{1,i}, \ldots, b_{n,i})^t$ such that $B = \mathbf{b_i}\mathbf{e}$ with $\mathbf{e} = (e_1, \ldots, e_{i-1}, 1, e_{i+1}, \ldots, e_n) \in \mathbb{S}^n$. Let $\mathbb{S}$ be commutative,

$$D = \text{diag}(b_{1,i}, \ldots, b_{n,i}) \in \mathbb{M}_n(\mathbb{S}),$$
$$E = \text{diag}(e_1, \ldots, e_{i-1}, 1, e_{i+1}, \ldots, e_n) \in \mathbb{M}_n(\mathbb{S})$$

be diagonal matrices. Then it is straightforward to check that $X \circ B = DXE$ for all $X \in \mathbb{M}_n(\mathbb{S})$. $\square$

Theorem 3.3. *Let $T : \mathbb{M}_n(\mathbb{S}) \to \mathbb{M}_n(\mathbb{S})$ be a surjective linear operator with $n \geq 4$. If T preserves Π_L, then T is a nontransposing (P, P^t, B)-operator and $\text{sc}(B) = 1$ with $B = [b_{i,j}] \in \mathbb{M}_n(\mathbb{Z}(\mathbb{S}))$ and units $b_{i,j}$ for all $(i,j) \in \Delta_n$.*

Proof. Assume that surjective operator T preserves Π_L. By applying Lemma 3.1 and Theorem 2.1 to Lemmas 2.2 and 3.2 we have that if T preserves Π_L then T is a (P, Q, B)-operator and $\mathrm{sc}(B) = 1$.

To see that the operator $T(X) = P(X \circ B)^t Q$ does not preserve Π_L, it suffies to consider $T_0(X) = X^t D$, where $D = QP$, since a similarity and a Hadamard product with a matrix of spanning column rank 1 and invertible entries preserve Π_L. Let

$$X = (D^{-1})^t \left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bigoplus I_{n-3} \right) \text{ and } Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \bigoplus I_{n-4}.$$

Then $(X, Y) \in \Pi_L$ while $(X^t D, Y^t D) \notin \Pi_L$. Thus T_0 and hence T does not preserve Π_L.

It remains to prove that $Q = P^t$. Assume to the contrary that $QP \neq I$. Suppose $X \to (QP)X$ transforms the rth row into the tth for some $r \neq t$. We consider the matrix $X = \sum_{i \neq t} E_{i,i}$, $Y = E_{r,r}$. Thus $(X, Y) \in \Pi_L$. Then for certain invertible elements $b_{i,i} \in \mathbb{S}$ we have that

$$T(X)T(Y) = P(X \circ B)QP(Y \circ B)Q = P\left(\sum_{i \neq t} b_{i,i}E_{i,i}\right)(b_{r,r}E_{t,r})Q = 0,$$

i.e., $(T(X), T(Y)) \notin \Pi_L$, which is a contradiction. $\qquad\square$

Corollary 3.4. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$ with $n \geq 4$ and $\mathbb{S}$ be commutative with $1 + 1 \neq 1$. If T preserves Π_L, then there exist an invertible matrix U and an invertible element α such that $T(X) = \alpha U X U^{-1}$ for all $X \in \mathbb{M}_n(\mathbb{S})$.*

Proof. Suppose T preserves Π_L. By Theorem 3.3, T is a non-transposing (P, P^t, B)-operator, where $\mathrm{sc}(B) = 1$ and all elements of B are units in $\mathbb{Z}(\mathbb{S})$. That is,

$$(3.1) \qquad\qquad T(X) = P(X \circ B)P^t$$

for all $X \in \mathbb{M}_n(\mathbb{S})$. Since $\mathrm{sc}(B) = 1$, there exists $\mathbf{b_i} = (b_{1,i}, \ldots, b_{m,i})^t$ among the columns of B such that $B = \mathbf{b_i}\mathbf{e}$ with $\mathbf{e} = (e_1, \ldots, e_{i-1}, 1, e_{i+1}, \ldots, e_n)$. Since b_{ji} are units, e_j are invertible elements in $\mathbb{S}$ for all $j = 1, \ldots, n$. Let $D = \mathrm{diag}(b_{1,i}, \ldots, b_{m,i}) \in \mathbb{M}_m(\mathbb{S})$ and $E = \mathrm{diag}(e_1, \ldots, e_n) \in \mathbb{M}_n(\mathbb{S})$ be diagonal matrices. Since $\mathbb{S}$ is commutative, it is straightforward to check that $X \circ B = DXE$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. Thus (3.1) becomes $T(X) = PDXEP^t$. Let us show that ED is an invertible scalar matrix.

Define $L(X) = (EP^t)T(X)(EP^t)^{-1} = EDX$ for all $X \in \mathbb{M}_n(\mathbb{S})$. Since T preserves Π_L if and only if L does, it suffices to consider $L(X) = EDX$. Let $G = ED$. Then $G = \mathrm{diag}(g_1, \ldots, g_n)$ is an invertible diagonal matrix. Assume

that $g_1 \neq g_2$. Consider matrices

(3.2) $\qquad A = \begin{bmatrix} 0 & 4 & 1 & 1 \\ 4 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Let $X = A \oplus O_{n-4}$ and $Y = G^{-1}(B \oplus O_{n-4})$. Since all columns of A are linearly independent, it follows that

$$\mathrm{sc}(A) = \mathrm{sc}(X) = \mathrm{sc}(L(X)) = 4 \quad \text{and} \quad \mathrm{sc}(B) = \mathrm{sc}(Y) = \mathrm{sc}(L(Y)) = 2.$$

Furthermore,

$$XY = \begin{bmatrix} 4g_2^{-1} & 4g_2^{-1} & g_3^{-1} + g_4^{-1} & g_3^{-1} + g_4^{-1} \\ 4g_1^{-1} & 4g_1^{-1} & g_3^{-1} + g_4^{-1} & g_3^{-1} + g_4^{-1} \\ g_1^{-1} + g_2^{-1} & g_1^{-1} + g_2^{-1} & g_4^{-1} & g_4^{-1} \\ g_1^{-1} + g_2^{-1} & g_1^{-1} + g_2^{-1} & g_3^{-1} & g_3^{-1} \end{bmatrix} \oplus O_{n-4}$$

has the spanning column rank 2 since $g_1 \neq g_2$. That is $(X, Y) \in \Pi_L$. But

$$L(X)L(Y) = G\left(\begin{bmatrix} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \oplus O_{n-4} \right)$$

has spanning column rank 1 and hence $(L(X), L(Y)) \notin \Pi_L$. This contradiction shows that $g_1 = g_2$. Similarly, if we consider a matrix

$$A' = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 1 & 4 & 0 \end{bmatrix},$$

then the parallel argument shows that $g_3 = g_4$. Generally, if $n \geq 5$, then we can split zero block into two parts and take $X' = O_r \oplus A \oplus O_{n-r-4}$ or $X' = O_r \oplus A' \oplus O_{n-r-4}$ for appropriate r. Therefore we have that G is an invertible scalar matrix. That is, $G = ED = \alpha I$ for some invertible element α, equivalently $E = \alpha D^{-1}$. If we let $U = PD$, then

$$T(X) = P(DXE)P^t = \alpha(PD)X(PD)^{-1} = \alpha UXU^{-1}$$

for all $X \in \mathbb{M}_n(\mathbb{S})$. Thus the result follows. $\square$

Over chain semirings, the above results have the following improvement.

Theorem 3.5. *Let $\mathbb{S}$ be a chain semiring and T be a linear operator on $\mathbb{M}_n(\mathbb{S})$ with $n \geq 4$. Then T strongly preserves Π_L if and only if there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$.*

Proof. One can easily show that all operators of the form $T(X) = PXP^t$ strongly preserves Π_L.

Suppose T strongly preserves Π_L. We want to show that there exists $\beta \in \mathbb{S}$ such that βT is surjective on $\mathbb{M}_n(\beta\mathbb{S})$. In order to show this, it suffices to check that for each pair of indices (i, j) there exist $Y \in \mathbb{M}_n(\mathbb{S})$ and $a \in \mathbb{S}$ such that $T(Y) = aE_{i,j}$. If this is not the case or if there is a cell whose image is not dominated by a cell, then there exists a $(0, 1)$-matrix M and pair (r, s) such that $m_{r,s} = 0$ and $T(M) \geq T(J)$. Denote $G = T(M) = [g_{i,j}]$. Then, for $A = J \backslash E_{r,s}$, and $\alpha = \min\{g_{i,j} \mid g_{i,j} \neq 0\}$ we have that $T(\alpha A) = T(\alpha J)$. Since $\mathrm{sc}(\alpha A) = 2$ and $\mathrm{sc}((\alpha A)^2) = \mathrm{sc}(\alpha J) = \mathrm{sc}((\alpha J)^2) = 1$, $(\alpha A, \alpha A) \notin \Pi_L$, while $(\alpha J, \alpha J) \in \Pi_L$, a contradiction since $(T(\alpha J), T(\alpha J)) = (T(\alpha A), T(\alpha A))$. Thus there is no such a matrix M with a zero entry such that $T(M) \geq T(J)$. It follows that the image of a cell dominates only one cell and that for $\beta = \min\{h_{i,j}|T(J) = H = [h_{i,j}]\}$, βT is surjective on $\mathbb{M}_n(\beta\mathbb{S})$. By Theorem 3.3, $\beta T(X) = PXP^t$ for all $X \in \mathbb{M}_n(\beta\mathbb{S})$. Thus, there exists a matrix $B \in \mathbb{M}_n(\mathbb{S})$ with $B \geq J$ such that $T(X) = P(X \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$.

Suppose that $b_{i,j} < 1$ for some (i, j). Consider a matrix $X = J \backslash E_{i,j} + b_{i,j}E_{i,j}$. Then we have $T(X) = T(J)$. Since $\mathrm{sc}(X) = 2$ and $\mathrm{sc}(X^2) = \mathrm{sc}(J) = 1$, $(X, X) \notin \Pi_L$, while $(T(X), T(X)) = (T(J), T(J)) \in \Pi_L$, a contradiction. Thus $B = J$ and the theorem follows. $\qquad\square$

4. Linear preservers of Π_0

Recall that

$$\Pi_0 = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid \mathrm{sc}(XY) = 0\}.$$

Lemma 4.1. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$. If T preserves Π_0, then T maps columns to columns and rows to rows.*

Proof. By Theorem 2.1, there exist a permutation σ on Δ_n and units $b_{i,j} \in \mathbb{Z}(\mathbb{S})$ such that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_n$.

Suppose T does not map columns to columns. Say $T(C_j)$ is not a column. Then $T(J \setminus C_j)$ has no zero column. It follows that $(J \setminus C_j, E_{j,j}) \in \Pi_0$, while $(T(J \setminus C_j), T(E_{j,j})) \notin \Pi_0$, a contradiction.

If T does not map rows to rows, then $T(R_i)$ is not a row for some i. Thus, $T(J \backslash R_i)$ has no zero row. It follows that $(E_{i,i}, J \backslash R_i) \in \Pi_0$, while $(T(E_{i,i}), T(J \backslash R_i)) \notin \Pi_0$, a contradiction. $\qquad\square$

Theorem 4.2. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$. Then T preserves Π_0 if and only if T is a non-transposing (P, P^t, B)-operator, where all elements of B are units in $\mathbb{Z}(\mathbb{S})$.*

Proof. Suppose T preserves Π_0. Since, by Lemma 4.1, T preserves columns and rows, it preserves lines and hence, by applying Theorem 2.1 to Lemma 2.2, T is a (P, Q, B)-operator, where all elements of B are units in $\mathbb{Z}(\mathbb{S})$. Since T maps columns to columns, T must be a non-transposing (P, Q, B)-operator and hence $T(X) = P(X \circ B)Q$ for all $X \in \mathbb{M}_n(\mathbb{S})$. Suppose that $QP \neq I$. Then

$QPE_{r,s} = E_{l,s}$ for some distinct indices r and l. Now, $(E_{l,l}, E_{r,s}) \in \Pi_0$, while $(T(E_{l,l}), T(E_{r,s})) \notin \Pi_0$ because

$$T(E_{l,l})T(E_{r,s}) = b_{l,l}b_{r,s}(PE_{l,s}Q) \neq O.$$

This contradiction shows that $QP = I$ and hence $Q = P^t$.

The converse is easily established since $\mathbb{S}$ is antinegative.   □

5. Linear preservers of Π_1

Recall that

$$\Pi_1 = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid \mathrm{sc}(X) + \mathrm{sc}(Y^t) > n \text{ and } \mathrm{sc}(XY) = 1\}.$$

Lemma 5.1. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$. If T preserves Π_1, then T preserves lines.*

Proof. By Theorem 2.1, there exist a permutation σ on Δ_n and units $b_{i,j} \in \mathbb{Z}(\mathbb{S})$ such that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_n$.

Suppose T does not preserve lines. Without loss of generality, we may assume that $T(E_{i,j})$ and $T(E_{k,l})$ lie in a line, where $i \neq k$ and $j \neq l$. Let $T(E_{i,j}) = b_{i,j}E_{s,t}$. Then either $T(E_{k,l}) = b_{k,l}E_{s,t'}$ or $T(E_{k,l}) = b_{k,l}E_{s',t}$. In both cases, we have $\mathrm{sc}(T(E_{i,j} + E_{k,l})) = 1$ (in the first case since $b_{i,j}$ is invertible, in the second case, there is only one non-zero column in the matrix). Let $Y' \in \mathbb{M}_n(\mathbb{S})$ be a matrix such that $Y' + E_{i,j} + E_{k,l}$ is a permutation matrix. Consider matrices $X = E_{i,j} + E_{k,l}$ and $Y^t = Y' + E_{k,l}$. Then we have $XY = E_{k,k}$, and hence $(X, Y) \in \Pi_1$ since $\mathrm{sc}(X) + \mathrm{sc}(Y^t) = 2 + (n - 1) = n + 1$ and $\mathrm{sc}(XY) = 1$. But we have $\mathrm{sc}(T(X)) + \mathrm{sc}(T(Y)^t) \leq n$ since $\mathrm{sc}(T(X)) = 1$ and $\mathrm{sc}(T(Y)^t) \leq n - 1$. Thus, $(T(X), T(Y)) \notin \Pi_1$, a contradiction.   □

Theorem 5.2. *Let $n \geq 3$ and T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$. If T preserves Π_1 then T is a non-transposing (P, P^t, B)-operator, where $\mathrm{sc}(B) = 1$ and all elements of B are units in $\mathbb{Z}(\mathbb{S})$.*

Proof. Suppose T preserves Π_1. By applying Theorem 2.1 to Lemma 5.1, T is a (P, Q, B)-operator, where $\mathrm{sc}(B) = 1$ and all elements of B are units in $\mathbb{Z}(\mathbb{S})$. First, we show that the operator $T(X) = P(X \circ B)^t Q$ does not preserve Π_1. Let

$$D = QP, \; A = \begin{bmatrix} O & I_2 \\ O & O \end{bmatrix}, \; X = DA$$

and $Y = I_{n-1} \oplus [0]$. Then we can easily show that $(X, Y) \in \Pi_1$ and

$$\begin{aligned}
(X \circ B)^t D(Y \circ B)^t &= (A^t D^t \circ B^t)D(Y \circ B^t) = (A^t \circ B^t D)(Y \circ B^t) \\
&= c_1 E_{n-1,1} + c_2 E_{n,2}
\end{aligned}$$

for some units $c_1, c_2 \in \mathbb{Z}(\mathbb{S})$. It follows from Lemma 2.4 that

$$\mathrm{sc}(T(X)T(Y)) = \mathrm{sc}((X \circ B)^t D(Y \circ B)^t) = 2,$$

a contradiction. Thus, we have established that T is a non-transposing (P, Q, B)-operator; $T(X) = P(X \circ B)Q$, where all elements of B are units in $\mathbb{Z}(\mathbb{S})$.

Next, we claim that $QP = I$. Assume to the contrary that $QP \neq I$. Then there exist $p, s, r, t \in \{1, 2, \ldots, n\}$ with $p \neq s$, $r \neq t$, $s \neq t$ and $p \neq r$ such that $(QP)R_p = R_s$ and $(QP)R_r = R_t$. Consider matrices $X = \sum_{i \neq r} E_{i,i}$ and $Y = E_{p,p} + E_{r,r}$. Then $(X, Y) \in \Pi_1$ since $\mathrm{sc}(X) + \mathrm{sc}(Y^t) = (n-1) + 2 > n$ and $\mathrm{sc}(XY) = \mathrm{sc}(E_{p,p}) = 1$. But

$$
\begin{aligned}
T(X)T(Y) &= P\left(\sum_{i \neq r} E_{i,i} \circ B\right) QP((E_{p,p} + E_{r,r}) \circ B)Q \\
&= P\left(\sum_{i \neq r} b_{i,i} E_{i,i}\right)(b_{p,p} E_{s,p} + b_{r,r} E_{t,r})Q \\
&= P(b_{s,s} b_{p,p} E_{s,p} + b_{t,t} b_{r,r} E_{t,r})Q
\end{aligned}
$$

has spanning column rank 2 since $s \neq t$ and $p \neq r$, a contradiction. Hence $QP = I$ and hence $Q = P^t$. That is, $T(X) = P(X \circ B)P^t$.

It remains to check that $\mathrm{sc}(B) = 1$. Consider $(J, I) \in \Pi_1$. Then we have $\mathrm{sc}(T(J)) + \mathrm{sc}(T(I)^t) = \mathrm{sc}(J) + \mathrm{sc}(I^t) > n$. Since T preserves Π_1, it follows that $\mathrm{sc}(T(J)T(I)) = 1$. Let $V = I \circ B$. Then $V = \mathrm{diag}(b_{1,1}, \ldots, b_{n,n})$ is an invertible matrix, and hence it follows from Lemma 2.4 that

$$
1 = \mathrm{sc}(T(J)T(I)) = \mathrm{sc}((J \circ B)(I \circ B)) = \mathrm{sc}(BV) = \mathrm{sc}(B).
$$

Thus the Theorem follows.   □

Corollary 5.3. *Let* $\mathbb{S} = \mathbb{B}$, $\mathbb{Z}^+$ *or a chain semiring, and* T *be a surjective linear operator on* $\mathbb{M}_n(\mathbb{S})$ *with* $n \geq 3$. *Then* T *preserves* Π_1 *if and only if there is a permutation matrix* $P \in \mathbb{M}_n(\mathbb{S})$ *such that* $T(X) = PXP^t$ *for all* $X \in \mathbb{M}_n(\mathbb{S})$.

Proof. Suppose T preserves Π_1. By Theorem 5.2, T is a non-transposing (P, P^t, B)-operator, where all elements of B are units. Note that if $\mathbb{S} = \mathbb{B}$, $\mathbb{Z}^+$ or a chain semiring, "1" is the only unit element in $\mathbb{S}$, and hence $B = J$. Thus, there exists a permutation matrix $P \in \mathbb{M}_n(\mathbb{S})$ such that $T(X) = PXP^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$.

The converse is easily established.   □

6. Linear preservers of Π_R

Recall that

$$
\Pi_R = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid \mathrm{sc}(XY) = \rho(X) + \rho(Y) - n\}.
$$

Lemma 6.1. *Let* $\mathbb{S}$ *be any subsemiring of* $\mathbb{R}^+$, σ *be a permutation of* Δ_n, *and* T *be defined by* $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ *for all* $(i, j) \in \Delta_n$, *where all* $b_{i,j}$ *are units. If* T *preserves* Π_R, *then* T *preserves lines.*

Proof. If T does not preserve lines, then, as in the proof of Lemma 5.1, there exist indices i, j, k, l, $i \neq k$, $j \neq l$ such that $T(E_{i,j})$ and $T(E_{k,l})$ lie in a line. Let $X' \in \mathbb{M}_n(\mathbb{S})$ be a matrix such that $X = X' + E_{i,j} + E_{k,l}$ is a permutation

matrix. Then $(X, O) \in \Pi_R$. However, $\mathrm{sc}(T(X)) \leq n - 1$ since either $T(X)$ has a zero column or $T(X)$ has two proportional columns since $b_{i,j}$ is a unit. Thus, $\rho(T(X)) \leq n - 1$ and hence $(T(X), O) \notin \Pi_R$, a contradiction.   □

Theorem 6.2. *Let $\mathbb{S}$ be a subsemiring of $\mathbb{R}^+$, and T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$. If T preserves Π_R then T is a non-transposing (P, P^t, B)-operator, where $\mathrm{sc}(B) = 1$ and all elements of B are units.*

Proof. Suppose T preserves Π_R. By applying Theorem 2.1 to Lemma 6.1 and 2.2, T is a (P, Q, B)-operator, where all elements of B are units. First, we show all transposing (P, Q, B)-operators $T(X) = P(X \circ B)^t Q$ do not preserve Π_R. Let $QP = D$, $X = DE_{1,1}$, $Y = \sum_{i=1}^{n-1} E_{i+1,i}$. Then $(X, Y) \in \Pi_R$. But

$$T(X)T(Y) = P(DE_{1,1} \circ B)^t D(Y \circ B)^t Q$$

$$= P(E_{1,1} \circ B^t)(Y^t \circ B^t)Q = P(b_{1,1}b_{2,1}E_{1,2})Q \neq O.$$

Hence $\mathrm{sc}(T(X)T(Y)) > 0 = \rho(T(X)) + \rho(T(Y)) - n$; that is, $(T(X), T(Y)) \notin \Pi_R$, a contradiction. Thus, T is a non-transposing (P, Q, B)-operator; $T(X) = P(X \circ B)Q$.

Let us check that $Q = P^t$. If $QP \neq I$, then there exist two distinct indices s and t in $\{1, \ldots, n\}$ such that $(QP)R_s = R_t$. Let $X = \sum_{i \neq s} E_{i,i}$, $Y = E_{s,s}$. Then it follows from Lemma 2.4 that $\mathrm{sc}(T(X)) = \mathrm{sc}(X) = n - 1$ and $\mathrm{sc}(T(Y)) = \mathrm{sc}(Y) = 1$. Furthermore, we have $XY = O$ and $T(X)T(Y) = P(b_{t,t}b_{s,s}E_{t,s})Q \neq O$. Thus, $(X, Y) \in \Pi_R$, while $(T(X), T(Y)) \notin \Pi_R$, a contradiction. Hence $QP = I$ or $Q = P^t$.

It remains to show that $\mathrm{sc}(B) = 1$. Assume to the contrary that $\mathrm{sc}(B) \geq 2$. We lose no generality in assuming that $\mathrm{sc}(B[1, 2|1, 2]) = 2$. Consider the matrix

$$X = \left[\begin{array}{cc} b_{1,1}^{-1} & b_{1,2}^{-1} \\ b_{2,1}^{-1} & b_{2,2}^{-1} \end{array} \right] \oplus I_{n-2}$$

in $\mathbb{M}_{m,n}(\mathbb{S})$. Then $\mathrm{sc}(X) = \mathrm{sc}(X^2) = n$. Note that from the invertibility of $b_{i,j}$ it follows that $\rho(X) = n$. Indeed, if $b_{i,1}^{-1} = \lambda b_{i,2}^{-1}$, $i = 1, 2$, for some $\lambda \in \mathbb{R}^+$, then $\lambda = b_{i,1}^{-1}b_{i,2} \in \mathbb{S}$ which contradicts $\mathrm{sc}(B[1, 2|1, 2]) = 2$. Thus $(X, X) \in \Pi_R$ because $\mathrm{sc}(X^2) = n = 2\rho(X) - n$. But $\mathrm{sc}(T(X)) = \mathrm{sc}(T(X)^2) = \rho(T(X)) = n - 1$, while $2\rho(T(X)) - n = n - 2$, a contradiction.   □

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SEOK-ZUN SONG
DEPARTMENT OF MATHEMATICS
CHEJU NATIONAL UNIVERSITY
JEJU 690-756, KOREA
E-mail address: szsong@cheju.ac.kr

GI-SANG CHEON
DEPARTMENT OF MATHEMATICS
SUNGKYUNKWAN UNIVERSITY
SUWON 440-746, KOREA
E-mail address: gscheon@skku.edu

YOUNG-BAE JUN
DEPARTMENT OF MATHEMATICS EDUCATION
GYEONGSANG NATIONAL UNIVERSITY
JINJU 660-701, KOREA
E-mail address: ybjun@nongae.gsnu.ac.kr