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HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. In this paper we consider the hyponormality of Toeplitz operators T_φ on the Bergman space $L_a^2(\mathbb{D})$ in the cases, where $\varphi := f + \overline{g}$ (f and g are polynomials). We present some necessary or sufficient conditions for the hyponormality of T_φ under certain assumptions about the coefficients of φ.

1. Introduction

The purpose of this paper is to study the hyponormality of Toeplitz operators acting on the Bergman space $L_a^2(\mathbb{D})$. Our interest is with Toeplitz operators with trigonometric polynomial symbols.

A bounded linear operator A on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^*A - AA^*$ is positive semidefinite. Let $\mathbb{D}$ denote the open unit disk in the complex plane, dA the area measure on the plane. The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f,\ g\rangle = \frac{1}{\pi}\int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

The Bergman space $L_a^2(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ consisting of functions analytic on $\mathbb{D}$. Let $L^\infty(\mathbb{D})$ be the space of bounded area measurable function on $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator M_φ on the Bergman space are defined by $M_\varphi(f) = \varphi \cdot f$, where f is in L_a^2. If P denotes the orthogonal projection of $L^2(\mathbb{D})$ onto the Bergman space L_a^2, the Toeplitz operator T_φ on the Bergman space is defined by

$$T_\varphi(f) = P(\varphi \cdot f),$$

where φ is measurable and f is in L_a^2. It is clear that those operators are bounded if φ is in $L^\infty(\mathbb{D})$. The Hankel operator $H_\varphi : L_a^2 \longrightarrow L_a^{2\perp}$ is defined by

$$H_\varphi(f) = (I - P)(\varphi \cdot f).$$

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Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T} = \partial\mathbb{D}$. Recall that given $\psi \in L^\infty(\mathbb{T})$, the Toeplitz operator on the Hardy space is the operator T_ψ on $H^2(\mathbb{T})$ defined by $T_\psi f = P_+(\psi \cdot f)$, where f is in $H^2(\mathbb{T})$ and P_+ denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Basic properties of the Bergman space and the Hardy space can be found in [1, 4, 5]. In [2], Cowen characterized the hyponormality of Toeplitz operator T_ψ on $H^2(\mathbb{T})$ by properties of the symbol $\psi \in L^\infty(\mathbb{T})$. Cowen's theorem states that if $\psi \in L^\infty(\mathbb{T})$, then the Toeplitz operator T_ψ is hyponormal if and only if the following 'Cowen' set $\mathcal{E}(\psi)$ is nonempty:

$$\mathcal{E}(\psi) = \{k \in H^\infty(\mathbb{T}) : ||k||_\infty \leq 1 \text{ and } \psi - k\overline{\psi} \in H^\infty(\mathbb{T})\}.$$

We record here some results on the hyponormality of Toeplitz operators on the Hardy space, which have been recently developed in [3, 6, 8, 9, 10].

Proposition 1.1. *Suppose that ψ is a trigonometric polynomial of the form $\psi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero.*

(i) *If T_ψ is a hyponormal operator, then $m \leq N$ and $|a_{-m}| \leq |a_N|$.*

(ii) *If T_ψ is a hyponormal operator, then $N - m \leq \operatorname{rank}[T_\psi^*, T_\psi] \leq N$.*

(iii) *The hyponormality of T_ψ is independent of the particular values of the Fourier coefficients $a_0, a_1, \ldots, a_{N-m}$ of ψ. Moreover the rank of the selfcommutator $[T_\psi^*, T_\psi]$ is also independent of those coefficients.*

(iv) *If $|a_{-m}| = |a_N| \neq 0$, then T_ψ is hyponormal if and only if the following equation holds:*

$$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m}\begin{pmatrix} \overline{a_{N-m+1}} \\ \overline{a_{N-m+2}} \\ \vdots \\ \overline{a_N} \end{pmatrix} \tag{1}$$

In this case, the rank of $[T_\psi^*, T_\psi]$ is $N - m$.

(v) *T_ψ is normal if and only if $m = N$, $|a_{-m}| = |a_N|$, and (1) holds with $m = N$.*

The solution (Cowen's theorem) of the hyponormality of T_ψ on the Hardy space is based on a dilation theorem of Sarason. It also exploited the fact that functions in $H^{2\perp}$ are conjugates of functions in zH^2. For the Bergman space, $L_a^{2\perp}$ is much larger than the conjugates of functions in zL_a^2, and no dilation theorem (similar to Sarason's theorem) is available. So we cannot get a similar version of Cowen's theorem for T_φ on the Bergman space. Therefore, at present, it seems to be quite difficult to determine the hyponormality of T_φ.

We will now consider the hyponormality of Toeplitz operators on the Bergman space with a symbol in the class of functions $\overline{g} + f$, where f and g are polynomials. Since the hyponormality of operators is translation invariant we may assume that $f(0) = g(0) = 0$. We shall list the well-known properties of Toeplitz operators T_φ on the Bergman space.

If f, g are in $L^\infty(\mathbb{D})$, then we can easily check that

$$\begin{aligned} &a)\ T_{f+g} = T_f + T_g \\ &b)\ T_f{}^* = T_{\overline{f}} \\ &c)\ T_{\overline{f}} T_g = T_{\overline{f}g} \text{ if } f \text{ or } g \text{ is analytic.} \end{aligned}$$

These properties enable us to establish several consequences of hyponormality.

Proposition 1.2 ([11]). *Let f, g be bounded and analytic. Then the followings are equivalent.*

(i) $T_{\overline{g}+f}$ *is hyponormal.*
(ii) $H^*_{\overline{g}} H_{\overline{g}} \le H^*_{\overline{f}} H_{\overline{f}}$.
(iii) $H_{\overline{g}} = C H_{\overline{f}}$ *, where C is of norm less than or equal to one.*

Very recently, in [7], it was shown that if $\varphi(z) = a_{-m}\overline{z}^m + a_{-N}\overline{z}^N + a_m z^m + a_N z^N$ $(0 < m < N)$ and $a_m \overline{a_N} = \overline{a_{-m}} a_{-N}$, then

(2)

$$T_\varphi \text{ is hyponormal}$$
$$\iff \begin{cases} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \ge \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) & \text{if } |a_{-N}| \le |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \le m^2(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \le |a_{-N}|. \end{cases}$$

In this paper we continue to examine the hyponormality of T_φ in the cases, where φ is a trigonometric polynomial.

2. Some necessary conditions for hyponormality of T_φ

In this section we present some necessary conditions for hyponormality of T_φ. First of all, observe that for any s, t nonnegative integers,

$$P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \ge t \\ 0 & \text{if } s < t. \end{cases}$$

Let $\varphi = \overline{g} + f$, where

$$f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.$$

For $m, n = 1, 2, \ldots, N$, define

$$A_{m,n} := \det \begin{pmatrix} a_m & a_{-m} \\ \overline{a_{-n}} & \overline{a_n} \end{pmatrix}$$

and we abbreviate $A_{n,n}$ to A_n.

The following lemma was shown in [7].

Lemma 2.1 ([7]). *Let $\varphi = \overline{g} + f$, where*

$$f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.$$

Suppose T_φ is hyponormal. Then

(i) *For each $i = 0, 1, 2, \dots, N-1$,*

$$\sum_{n=1}^{i} \frac{n^2 A_n}{(i+n+1)(i+1)^2} + \sum_{n=i+1}^{N} \frac{A_n}{i+n+1} \geq 0.$$

(ii) *For each $i \geq N$,*

$$\sum_{n=1}^{N} \frac{n^2 A_n}{(i+n+1)(i+1)^2} \geq 0.$$

Our main result treats the extremal cases in view of Lemma 2.1:

Theorem 2.2. *Let $\varphi = \overline{g} + f$, where $f(z) = \sum_{n=1}^{N} a_n z^n$ and $g(z) = \sum_{n=1}^{N} a_{-n} z^n$. Suppose that T_φ is hyponormal, and that for some $0 \leq i_0 \leq N-1$,*

$$\sum_{n=0}^{i_0} \frac{n^2 A_n}{(i_0+n+1)(i_0+1)^2} + \sum_{n=i_0+1}^{N} \frac{A_n}{i_0+n+1} = 0. \tag{3}$$

Then the following conditions hold

(i) *$AB = C$, where*

$A = [a_{ij}]_{i_0 \times (N-1)}$ *with* $a_{ij} = \begin{cases} 0 & \text{if } i > j \text{ or } j > N - i_0 + i - 1 \\ A_{j, i_0+j-i+1} & \text{if } i \leq j, \end{cases}$

$B = [b_j]_{(N-1)\times 1}$ *with* $b_j = \frac{1}{i_0+j+1}$,

$C = [c_j]_{i_0 \times 1}$ *with* $c_1 = 0$ *and* $c_j = -\sum_{n=1}^{j-1} \frac{n(i_0-j+1+n)}{j(i_0+1)(i_0+n+1)} A_{n, i_0-j+1+n}$.

(ii) *$AB = D$, where*

$A = [a_{ij}]_{(N-i_0-1)\times(N-i_0-1)}$ *with* $a_{ij} = \begin{cases} 0 & \text{if } i > j \\ A_{i_0+j-i+1, i_0+j+1} & \text{if } i \leq j, \end{cases}$

$B = [b_j]_{(N-i_0-1)\times 1}$ *with* $b_j = \frac{1}{2(i_0+1)+j}$,

$D = [d_j]_{(N-i_0-1)\times 1}$ *with* $d_j := -\sum_{n=1}^{i_0} \frac{n(j+n)}{(i_0+1)(i_0+j+1)(i_0+j+n+1)} A_{n,n+j}$.

(iii)

$$\sum_{n=1}^{i_0-j} \frac{n(N+j-i_0+n)}{(i_0+1)(N+j+1)(N+j+n+1)} A_{n, N+j-i_0+n} = 0$$

for each $0 \leq j \leq i_0 - 1$.

Proof. Let T_φ be a hyponormal operator and suppose (3) holds for some $0 \le i_0 \le N-1$. Then it follows from Proposition 1.2 that for each non-negative integer $m \ne i_0$ and $c_{i_0}, c_m \in \mathbb{C}$, we have

$$\Big\langle \big(H_{\bar f}^* H_{\bar f} - H_{\bar g}^* H_{\bar g}\big)(c_{i_0} z^{i_0} + c_m z^m),\ c_{i_0} z^{i_0} + c_m z^m \Big\rangle \ge 0,$$

or equivalently

$$\begin{aligned} &|c_{i_0}|^2 \Big\langle \big(H_{\bar f}^* H_{\bar f} - H_{\bar g}^* H_{\bar g}\big) z^{i_0}, z^{i_0} \Big\rangle + |c_m|^2 \Big\langle \big(H_{\bar f}^* H_{\bar f} - H_{\bar g}^* H_{\bar g}\big) z^m, z^m \Big\rangle \\ &+ 2\mathrm{Re}\Big(c_{i_0} \overline{c_m} \Big\langle \big(H_{\bar f}^* H_{\bar f} - H_{\bar g}^* H_{\bar g}\big) z^{i_0}, z^m \Big\rangle \Big) \ge 0. \end{aligned} \tag{4}$$

Observe that for $0 \le i_0 \le N-1$,

$$\begin{aligned} &\Big\langle \big(H_{\bar f}^* H_{\bar f} - H_{\bar g}^* H_{\bar g}\big) z^{i_0}, z^{i_0} \Big\rangle \\ &= \sum_{n=1}^{N} \frac{1}{i_0+n+1}\big(|a_n|^2 - |a_{-n}|^2\big) - \sum_{n=1}^{i_0} \frac{i_0-n+1}{(i_0+1)^2}\big(|a_n|^2 - |a_{-n}|^2\big) \\ &= \sum_{n=1}^{i_0} \frac{n^2 A_n}{(i_0+n+1)(i_0+1)^2} + \sum_{n=i_0+1}^{N} \frac{A_n}{i_0+n+1}. \end{aligned}$$

Hence by the assumption,

$$\Big\langle \big(H_{\bar f}^* H_{\bar f} - H_{\bar g}^* H_{\bar g}\big) z^{i_0}, z^{i_0} \Big\rangle = 0. \tag{5}$$

Since c_{i_0} and c_m are arbitrary, it follows from (4) and (5) that

$$\Big\langle \big(H_{\bar f}^* H_{\bar f} - H_{\bar g}^* H_{\bar g}\big) z^{i_0}, z^m \Big\rangle = 0. \tag{6}$$

If $i_0 < m$ $(i_0 + 1 \le m \le N + i_0 - 1)$, then we have

$$\langle M_{\bar f} z^{i_0}, M_{\bar f} z^m \rangle = \sum_{n=1}^{N+i_0-m} \frac{1}{m+n+1} a_{m+n-i_0} \overline{a_n}. \tag{7}$$

If instead $i_0 < m < N$, then

$$\langle T_{\bar f} z^{i_0}, T_{\bar f} z^m \rangle = \sum_{n=1}^{i_0} \frac{i_0+1-n}{(i_0+1)(m+1)} a_{m+n-i_0} \overline{a_n}. \tag{8}$$

Also if $N \le m \le N + i_0 - 1$, then

$$\langle T_{\bar f} z^{i_0}, T_{\bar f} z^m \rangle = \sum_{n=1}^{N-m+i_0} \frac{i_0+1-n}{(i_0+1)(m+1)} a_{m+n-i_0} \overline{a_n}. \tag{9}$$

Therefore (7), (8) and (9) give that for $i_0 < m$ $(i_0+1 \le m \le N+i_0-1)$,
(10)

$$\langle H^*_{\overline{f}} H_{\overline{f}} z^{i_0}, z^m \rangle = \begin{cases} \displaystyle\sum_{n=1}^{i_0} \frac{n(m-i_0+n)}{(i_0+1)(m+1)(m+n+1)} a_{m+n-i_0}\overline{a_n} \\ \displaystyle + \sum_{n=i_0+1}^{N+i_0-m} \frac{1}{m+n+1} a_{m+n-i_0}\overline{a_n} & \text{if } i_0 < m < N \\ \displaystyle\sum_{n=1}^{N+i_0-m} \frac{n(m-i_0+n)}{(i_0+1)(m+1)(m+n+1)} a_{m+n-i_0}\overline{a_n} & \text{if } N \le m \le N+i_0-1. \end{cases}$$

Similarly, we have
(11)

$$\langle H^*_{\overline{g}} H_{\overline{g}} z^{i_0}, z^m \rangle = \begin{cases} \displaystyle\sum_{n=1}^{i_0} \frac{n(m-i_0+n)}{(i_0+1)(m+1)(m+n+1)} a_{-(m+n-i_0)}\overline{a_{-n}} \\ \displaystyle + \sum_{n=i_0+1}^{N+i_0-m} \frac{1}{m+n+1} a_{-(m+n-i_0)}\overline{a_{-n}} & \text{if } i_0 < m < N \\ \displaystyle\sum_{n=1}^{N+i_0-m} \frac{n(m-i_0+n)}{(i_0+1)(m+1)(m+n+1)} a_{-(m+n-i_0)}\overline{a_{-n}} & \text{if } N \le m \le N+i_0-1. \end{cases}$$

Thus by (10) and (11) we have that for $i_0 < m$ $(i_0+1 \le m \le N+i_0-1)$
(12)

$$\left\langle \left(H^*_{\overline{f}} H_{\overline{f}} - H^*_{\overline{g}} H_{\overline{g}}\right) z^{i_0}, z^m \right\rangle = \begin{cases} \displaystyle\sum_{n=1}^{i_0} \frac{n(m-i_0+n)}{(i_0+1)(m+1)(m+n+1)} \overline{A_{n,m-i_0+n}} \\ \displaystyle + \sum_{n=i_0+1}^{N+i_0-m} \frac{1}{m+n+1} \overline{A_{n,m-i_0+n}} & \text{if } i_0 < m < N \\ \displaystyle\sum_{n=1}^{N+i_0-m} \frac{n(m-i_0+n)}{(i_0+1)(m+1)(m+n+1)} \overline{A_{n,m-i_0+n}} & \text{if } N \le m \le N+i_0-1. \end{cases}$$

If $0 \le m < i_0$, then we get

$$\left\langle M_{\overline{f}} z^{i_0}, M_{\overline{f}} z^m \right\rangle = \sum_{n=1}^{N+m-i_0} \frac{1}{i_0+n+1} a_n \overline{a_{i_0-m+n}}$$

and

$$\left\langle T_{\overline{f}} z^{i_0}, T_{\overline{f}} z^m \right\rangle = \sum_{n=1}^{m} \frac{m+1-n}{(i_0+1)(m+1)} a_n \overline{a_{i_0-m+n}}.$$

Thus we have, for $0 \le m < i_0$,

$$
\begin{aligned}
(13)\qquad \langle H_{\bar f}^* H_{\bar f} z^{i_0}, z^m\rangle = &\sum_{n=1}^{m} \frac{n(i_0-m+n)}{(i_0+1)(m+1)(i_0+n+1)} a_n \overline{a_{i_0-m+n}} \\
&+ \sum_{n=m+1}^{N+m-i_0} \frac{1}{i_0+n+1} a_n \overline{a_{i_0-m+n}}.
\end{aligned}
$$

Similarly, we have that for $0 \le m < i_0$,

$$
\begin{aligned}
(14)\qquad \langle H_{\bar g}^* H_{\bar g} z^{i_0}, z^m\rangle = &\sum_{n=1}^{m} \frac{n(i_0-m+n)}{(i_0+1)(m+1)(i_0+n+1)} a_{-n} \overline{a_{-(i_0-m+n)}} \\
&+ \sum_{n=m+1}^{N+m-i_0} \frac{1}{i_0+n+1} a_{-n} \overline{a_{-(i_0-m+n)}}.
\end{aligned}
$$

Thus by (13) and (14) we also have, for $0 \le m < i_0$,
(15)

$$
\begin{aligned}
\Big\langle \big(H_{\bar f}^* H_{\bar f} - H_{\bar f}^* H_{\bar f}\big) z^{i_0}, z^m \Big\rangle = &\sum_{n=1}^{m} \frac{n(i_0-m+n)}{(i_0+1)(m+1)(i_0+n+1)} A_{n,i_0-m+n} \\
&+ \sum_{n=m+1}^{N+m-i_0} \frac{1}{i_0+n+1} A_{n,i_0-m+n}.
\end{aligned}
$$

It follows from (5), (12), and (15) that for $0 \le i_0 \le N-1$,

$$
\begin{aligned}
&\sum_{n=m+1}^{N+m-i_0} \frac{1}{i_0+n+1} A_{n,i_0-m+n} \\
&= -\sum_{n=1}^{m} \frac{n(i_0-m+n)}{(i_0+1)(m+1)(i_0+n+1)} A_{n,i_0-m+n} \qquad \text{if } 0 \le m < i_0, \\
&\sum_{n=i_0+1}^{N+i_0-m} \frac{1}{m+n+1} A_{n,m-i_0+n} \\
&= -\sum_{n=1}^{i_0} \frac{n(m-i_0+n)}{(i_0+1)(m+1)(m+n+1)} A_{n,m-i_0+n} \qquad \text{if } i_0 < m < N,
\end{aligned}
$$

and

$$
\sum_{n=1}^{N+i_0-m} \frac{n(m-i_0+n)}{(i_0+1)(m+1)(m+n+1)} A_{n,m-i_0+n} = 0 \ \text{ if } N \le m \le N+i_0-1.
$$

This proves (i), (ii), and (iii). □

Theorem 2.3. *Let $\varphi=\overline{g}+f$, where $f(z)=\sum_{n=1}^{N} a_n z^n$ and $g(z)=\sum_{n=1}^{N} a_{-n} z^n$. Suppose that T_φ is hyponormal, and that for some $i_0 \geq N$,*

$$\sum_{n=1}^{N} \frac{n^2 A_n}{(i_0+n+1)(i_0+1)^2}=0. \tag{16}$$

Then we have

$$\sum_{n=1}^{N-j} \frac{n(n+j)}{(i_0+j+1)(i_0+j+n+1)} A_{n,n+j}=0 \qquad \text{for } 1 \leq j \leq N-1; \tag{17}$$

$$\sum_{n=1}^{N-j} \frac{n(n+j)}{(i_0-j+1)(i_0+n+1)} A_{n,n+j}=0 \qquad \text{for } 1 \leq j \leq N-1. \tag{18}$$

Proof. Let T_φ be a hyponormal operator and suppose (16) holds for some $i_0 \geq N$. Then by assumption we have that for $i_0 \geq N$,

$$\left\langle \left(H_{\overline{f}}^* H_{\overline{f}}-H_{\overline{g}}^* H_{\overline{g}}\right) z^{i_0}, z^{i_0}\right\rangle=\sum_{n=1}^{N} \frac{n^2 A_n}{(i_0+n+1)(i_0+1)^2}=0.$$

Thus it follows from (4) that for each non-negative integer $m \neq i_0$, we have

$$\left\langle \left(H_{\overline{f}}^* H_{\overline{f}}-H_{\overline{g}}^* H_{\overline{g}}\right) z^{i_0}, z^{m}\right\rangle=0. \tag{19}$$

If $i_0<m \leq N+i_0-1$, then

$$\langle M_{\overline{f}} z^{i_0}, M_{\overline{f}} z^m\rangle=\sum_{n=1}^{N-m+i_0} \frac{1}{m+n+1} \overline{a_n} a_{m-i_0+n}$$

and

$$\langle T_{\overline{f}} z^{i_0}, T_{\overline{f}} z^m\rangle=\sum_{n=1}^{N-m+i_0} \frac{i_0+1-n}{(i_0+1)(m+1)} \overline{a_n} a_{m-i_0+n}.$$

Thus for $i_0<m \leq N+i_0-1$ $(i_0 \geq N)$, we get

$$\left\langle \left(H_{\overline{f}}^* H_{\overline{f}}-H_{\overline{g}}^* H_{\overline{g}}\right) z^{i_0}, z^{m}\right\rangle=\sum_{n=1}^{N+i_0-m} \frac{n(n+m-i_0)}{(i_0+1)(m+1)(m+n+1)} \overline{A_{n,m-i_0+n}}. \tag{20}$$

Similarly, for $i_0-N+1 \leq m<i_0$ $(i_0 \geq N)$ we have

$$\left\langle \left(H_{\overline{f}}^* H_{\overline{f}}-H_{\overline{g}}^* H_{\overline{g}}\right) z^{i_0}, z^{m}\right\rangle=\sum_{n=1}^{N+m-i_0} \frac{n(n+i_0-m)}{(i_0+1)(m+1)(i_0+n+1)} A_{n,i_0-m+n}. \tag{21}$$

By (19), (20), and (21), we see that for $i_0 \geq N$,

$$\sum_{n=1}^{N+i_0-m} \frac{n(n+m-i_0)}{(i_0+1)(m+1)(m+n+1)} A_{n,m-i_0+n}=0 \quad \text{if } i_0<m \leq N+i_0-1; \tag{22}$$

(23)
$$\sum_{n=1}^{N+m-i_0} \frac{n(n+i_0-m)}{(i_0+1)(m+1)(i_0+n+1)} A_{n,i_0-m+n} = 0 \quad \text{if } i_0 - N + 1 \le m < i_0.$$

Putting $j = m - i_0$ and $j = i_0 - m$, respectively, in (22) and (23) gives the result. □

From Theorems 2.2 and 2.3 we get the following corollaries.

Corollary 2.4. *Let $\varphi = \overline{g} + f$, where $f(z) = \sum_{n=1}^{N} a_n z^n$ and $g(z) = \sum_{n=1}^{N} a_{-n} z^n$. If T_φ is hyponormal and (3) holds for some $0 \le i_0 \le N-1$, then*

$$\sum_{n=1}^{N-i_0} \frac{1}{n+i_0+1} A_{n,n+i_0} = 0.$$

Corollary 2.5 ([7]). *Let $\varphi = \overline{g} + f$, where $f(z) = \sum_{n=1}^{N} a_n z^n$ and $g(z) = \sum_{n=1}^{N} a_{-n} z^n$. If T_φ is hyponormal and $||f|| = ||g||$, then we have*

$$\begin{pmatrix} A_{1,1} & A_{2,2} & \cdots & \cdots & \cdots & A_{N,N} \\ 0 & A_{1,2} & A_{2,3} & \cdots & \cdots & A_{N-1,N} \\ 0 & 0 & A_{1,3} & \cdots & \cdots & A_{N-2,N} \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & A_{1,N-1} & A_{2,N} \\ 0 & 0 & \cdots & \cdots & 0 & A_{1,N} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \vdots \\ \frac{1}{N} \\ \frac{1}{N+1} \end{pmatrix} = 0.$$

Proof. We have the result by putting $i_0 = 0$ in Theorem 2.2 (ii). □

Corollary 2.6. *Let $\varphi = \overline{g} + f$, where $f(z) = \sum_{n=1}^{N} a_n z^n$ and $g(z) = \sum_{n=1}^{N} a_{-n} z^n$ ($N \ge 3$). If T_φ is hyponormal and (16) holds for some $i_0 \ge N$, then we have*

$$A_{1,N} = A_{1,N-1} = A_{2,N} = 0.$$

Proof. Putting $j = N-1$ in (17) gives $A_{1,N} = 0$ and putting $j = N-2$ in (17) and (18) gives that

$$\begin{pmatrix} \frac{N-1}{(i_0+N-1)(N+i_0)} & \frac{2N}{(i_0+N-1)(N+i_0+1)} \\ \frac{N-1}{(i_0-N+3)(i_0+2)} & \frac{2N}{(i_0-N+3)(i_0+3)} \end{pmatrix} \begin{pmatrix} A_{1,N-1} \\ A_{2,N} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Observe that

$$\det \begin{pmatrix} \frac{N-1}{(i_0+N-1)(N+i_0)} & \frac{2N}{(i_0+N-1)(N+i_0+1)} \\ \frac{N-1}{(i_0-N+3)(i_0+2)} & \frac{2N}{(i_0-N+3)(i_0+3)} \end{pmatrix} = 0 \quad \text{if and only if } N = 2.$$

Thus we have that $A_{1,N-1} = A_{2,N} = 0$. □

Corollary 2.7. *Let $\varphi = \overline{g} + f$, where $f(z) = \sum_{n=1}^{3} a_n z^n$ and $g(z) = \sum_{n=1}^{3} a_{-n} z^n$. If (3) or (16) holds for some $i_0 \ge 1$, then T_φ is hyponormal if and only if $\varphi(z)$ satisfies one of the following two conditions:*

(i) $f(z) = \alpha g(z)$ *for some* $|\alpha| = 1$ (*in this case* T_φ *is normal*);
(ii) $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m} z^m + a_{-N} z^N$, $A_{m,N} = 0$ $(1 \le m \le N \le 3)$ *and* (2) *holds.*

Proof. Suppose T_φ is hyponormal. We will show that $A_{1,2} = A_{2,3} = A_{1,3} = 0$. If $i_0 \ge 3$, this follows from Corollary 2.6. If $i_0 = 1$, then putting $j = 0$ in Theorem 2.2 (iii) gives $A_{1,3} = 0$ and by Theorem 2.2 (i) and (ii) we have

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{12} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} A_{1,2} \\ A_{2,3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore $A_{1,2} = A_{2,3} = A_{1,3} = 0$. If $i_0 = 2$ then by Theorem 2.2 (i) we get $A_{1,3} = 0$ and $\frac{1}{12} A_{1,2} + \frac{1}{5} A_{2,3} = 0$. Putting $j = 0$ in Theorem 2.2 (iii) we see that $2A_{1,2} + 5A_{2,3} = 0$ and therefore $A_{1,2} = A_{2,3} = 0$. Thus by (2), (i) or (ii) holds. The converse follows from Proposition 1.2 and (2). This completes the proof. □

Example 2.8. Consider the polynomial

$$\varphi(z) = 4\overline{z}^3 + 2\overline{z}^2 + \overline{z} + z + 2z^2 + \beta z^3 \quad (|\beta| = 4).$$

Then $\varphi(z)$ satisfies the equality (3). Thus by Corollary 2.7, T_φ is hyponormal if and only if $\beta = 4$.

Example 2.9. Consider the polynomial

$$\varphi(z) = 8\overline{z}^3 + \overline{z}^2 + \beta\overline{z} + \gamma z + 7z^2 + 2z^3 \quad (|\beta| = |\gamma|).$$

Then $\varphi(z)$ satisfies the equality (3). Thus Corollary 2.7 shows that T_φ is not hyponormal.

3. Some sufficient conditions for hyponormality of T_φ

If $f(z) = \sum_{n=2}^N a_n z^n$ $(N \ge 2)$ and $h(z) = az + f(z)$, then the Toeplitz operator $T_{\overline{f}+h}$ on the Hardy space is hyponormal if and only if $a = 0$ (Proposition 1.1(iv)). On the contrary, the following theorem shows that the Toeplitz operator $T_{\overline{f}+h}$ on the Bergman space is hyponormal if $|a|$ is sufficiently large.

Theorem 3.1. *If* $f(z) = \sum_{n=2}^N a_n z^n$ $(N \ge 2)$, $h(z) = az + f(z)$, *and* $A := \max\{|a_i| : 2 \le i \le N\}$, *then* $T_{\overline{f}+h}$ *is hyponormal when* $|a| \ge 2N^2 A$.

Proof. Let $K_i := \{k_i(z) \in L^2_a : k_i(z) = \sum_{n=0}^\infty c_{Nn+i} z^{Nn+i}\}$ for $i = 0, 1, 2, \dots, N-1$. Then by Proposition 1.2, we have that $T_{\overline{f}+h}$ is hyponormal if and only if $\langle (H^*_{\overline{h}} H_{\overline{h}} - H^*_{\overline{f}} H_{\overline{f}}) \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z)\rangle \ge 0$ for all $k_i \in K_i$ $(i =$

$0, 1, 2, \ldots, N-1$), or equivalently

$$\begin{aligned}(24)\quad &\sum_{i=0}^{N-1}\left(2\mathrm{Re}\left\langle H_{\overline{f}}k_i(z), \overline{a}H_{\overline{z}}k_i(z)\right\rangle + |a|^2\left\langle H_{\overline{z}}k_i(z), H_{\overline{z}}k_i(z)\right\rangle\right)\\ &+\sum_{i\neq j,\ i,j\geq 0}^{N-1}\left(2\mathrm{Re}\left\langle H_{\overline{f}}k_i(z), \overline{a}H_{\overline{z}}k_j(z)\right\rangle + |a|^2\left\langle H_{\overline{z}}k_i(z), H_{\overline{z}}k_j(z)\right\rangle\right) \geq 0.\end{aligned}$$

But we have

$$(25)\qquad \left\langle H_{\overline{f}}k_i(z), \overline{a}H_{\overline{z}}k_i(z)\right\rangle = 0$$

and for $i \neq j$ $(i, j = 0, 1, 2, \ldots, N-1)$,

$$(26)\qquad \left\langle H_{\overline{z}}k_i(z), H_{\overline{z}}k_j(z)\right\rangle = 0.$$

Putting (25) and (26) in (24) we have that $T_{\overline{f}+h}$ is hyponormal if and only if

$$(27)\quad \sum_{i=0}^{N-1}|a|^2\left\langle H_{\overline{z}}k_i(z), H_{\overline{z}}k_i(z)\right\rangle + \sum_{i\neq j,\ i,j\geq 0}^{N-1} 2\mathrm{Re}\left(a\left\langle H_{\overline{f}}k_i(z), H_{\overline{z}}k_j(z)\right\rangle\right) \geq 0.$$

Observe that

$$(28)\qquad \sum_{i=0}^{N-1}\left\langle H_{\overline{z}}k_i(z), H_{\overline{z}}k_i(z)\right\rangle = \sum_{n=0}^{\infty}\frac{1}{(n+2)(n+1)^2}|c_n|^2$$

and

$$(29)\qquad \sum_{i\neq j\ i,j\geq 0}^{N-1}\left\langle H_{\overline{f}}k_i(z), H_{\overline{z}}k_j(z)\right\rangle = \sum_{m=2}^{N}\overline{a_m}\sum_{i\neq j,\ i,j\geq 0}^{N-1}\left\langle H_{\overline{z}^m}k_i(z), H_{\overline{z}}k_j(z)\right\rangle.$$

For $m = 2, 3, \ldots, N$, we have

$$(30)\qquad \sum_{i\neq j,\ i,j\geq 0}^{N-1}\left\langle M_{\overline{z}^m}k_i(z), M_{\overline{z}}k_j(z)\right\rangle = \sum_{j=0}^{N-1}\sum_{n=0}^{\infty}\frac{1}{Nn+j+m+1}c_{Nn+j+m-1}\overline{c_{Nn+j}}$$

and

$$\begin{aligned}(31)\qquad &\sum_{i\neq j,\ i,j\geq 0}^{N-1}\left\langle T_{\overline{z}^m}k_i(z), T_{\overline{z}}k_j(z)\right\rangle\\ &= \sum_{j=0}^{N-1}\sum_{n=0}^{\infty}\frac{Nn+j}{(Nn+j+m)(Nn+j+1)}c_{Nn+j+m-1}\overline{c_{Nn+j}}.\end{aligned}$$

Combining (30) and (31) gives that

$$(32)\qquad \sum_{i\neq j,\ i,j\geq 0}^{N-1}\left\langle H_{\overline{z}^m}k_i(z), H_{\overline{z}}k_j(z)\right\rangle = \sum_{n=0}^{\infty}\frac{m}{(n+m+1)(n+m)(n+1)}\overline{c_n}c_{n+m-1}.$$

Putting (32) in (29) and putting (28) and (29) in (27) we see that $T_{\overline{f}+h}$ is hyponormal if and only if

$$(33)\qquad \begin{aligned} &|a|^2\sum_{n=0}^{\infty}\frac{1}{(n+2)(n+1)^2}|c_n|^2\\ &+2\mathrm{Re}\left(a\sum_{m=2}^{N}\overline{a_m}\sum_{n=0}^{\infty}\frac{m}{(n+m+1)(n+m)(n+1)}\overline{c_n}c_{n+m-1}\right)\geq 0. \end{aligned}$$

Note that the inequality (33) holds if the following inequality holds for each $m=2,3,\ldots,N$,

$$(34)\qquad \begin{aligned} &\sum_{n=0}^{\infty}\frac{|c_n|^2}{(n+2)(n+1)^2}\\ &\geq\frac{2(N-1)|a_m|}{|a|}\sum_{n=0}^{\infty}\frac{m}{(n+m+1)(n+m)(n+1)}|c_n||c_{n+m-1}|. \end{aligned}$$

So it follows from (34) that $T_{\overline{f}+h}$ is hyponormal if for all $n\geq 0,\ m=2,3,\ldots,N$,

$$(35)\qquad \begin{aligned} &\frac{\alpha_m}{(n+m+1)(n+m)(n+1)}|c_n||c_{n+m-1}|\\ &\leq\frac{1}{(n+2)(n+1)^2}|c_n|^2+\frac{1}{(n+m+1)(n+m)^2}|c_{n+m-1}|^2, \end{aligned}$$

where $\alpha_m=\frac{4(N-1)|a_m|m}{|a|}$. Observe that the inequality (35) holds if

$$\alpha_m{}^2\leq\frac{4(n+m+1)}{n+2}.$$

Let $A:=\max\{|a_i|:i=2,3,\ldots,N\}$. Then $T_{\overline{f}+h}$ is hyponormal when $|a|\geq 2N^2A$. This completes the proof. □

Corollary 3.2. *Let $f(z)=\sum_{n=2}^{N}a_nz^n$ $(N\geq 2)$, $g\in H^\infty$ and $T_{\overline{g}+f}$ be a hyponormal operator. If $h(z)=az+f(z)$ and $|a|\geq 2(N-1)A$, where $A:=\max\{|a_i|:2\leq i\leq N\}$, then $T_{\overline{g}+h}$ is hyponormal.*

Proof. This follows from Proposition 1.2 and Theorem 3.1. □

Let $f(z)=\sum_{n=1}^{N-1}$ $(N\geq 2)$ and $h(z)=f(z)+az^N$. Then the Toeplitz operator $T_{\overline{f}+h}$ on the Hardy space is hyponormal if $|a|$ is sufficiently large ([6]). The following theorem shows that the Toeplitz operator $T_{\overline{f}+h}$ on the Bergman space has the same property.

Theorem 3.3. *Let $f(z)=\sum_{n=1}^{N-1}a_nz^n$ $(N\geq 2)$, $h(z)=f(z)+az^N$ and $A:=\max\{|a_i|:1\leq i\leq N-1\}$. If $|a|\geq 2\sqrt{2}(N-1)A$, then $T_{\overline{f}+h}$ is hyponormal.*

Proof. Let $K_i := \{k_i(z) \in L_a^2 : k_i(z) = \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}\}$ for $i = 0, 1, 2, \ldots, N-1$. Then Proposition 1.2 gives that $T_{\overline{f}+h}$ is hyponormal if and only if $\langle (H_{\overline{h}}^* H_{\overline{h}} - H_{\overline{f}}^* H_{\overline{f}}) \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z) \rangle \geq 0$ for all $k_i \in K_i$ $(i = 0, 1, 2, \ldots, N-1)$, or equivalently

$$\begin{aligned}(36)\qquad &\sum_{i=0}^{N-1} |a|^2 \big\langle H_{\overline{z}^N} k_i(z), H_{\overline{z}^N} k_i(z) \big\rangle \\ &+ \sum_{i\neq j,\ i,j\geq 0}^{N-1} 2\mathrm{Re}\Big(a \sum_{m=1}^{N-1} \overline{a_m} \langle H_{\overline{z}^m} k_i(z), H_{\overline{z}^N} k_j(z) \rangle \Big) \geq 0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(37)\qquad &\sum_{i=0}^{N-1} \big\langle H_{\overline{z}^N} k_i(z), H_{\overline{z}^N} k_i(z) \big\rangle \\ &= \sum_{n=0}^{N-1} \frac{1}{n+N+1} |c_n|^2 + \sum_{n=N}^{\infty} \frac{N^2}{(n+N+1)(n+1)^2} |c_n|^2,\end{aligned}$$

and for each $m = 1, 2, \ldots, N-1$,

(38)

$$\sum_{i\neq j,\ i,j\geq 0}^{N-1} \big\langle M_{\overline{z}^m} k_i(z), M_{\overline{z}^N} k_j(z) \big\rangle = \sum_{i=0}^{N-1} \sum_{n=0}^{\infty} \frac{1}{N(n+1)+i+1} c_{Nn+i} \overline{c_{N(n+1)-m+i}}$$

and

$$\begin{aligned}(39)\qquad &\sum_{i\neq j,\ i,j\geq 0}^{N-1} \big\langle T_{\overline{z}^m} k_i(z), T_{\overline{z}^N} k_j(z) \big\rangle \\ &= \sum_{i=0}^{m-1} \sum_{n=1}^{\infty} \frac{Nn-m+i+1}{(Nn+i+1)(N(n+1)-m+i+1)} c_{Nn+i} \overline{c_{N(n+1)-m+i}} \\ &\quad + \sum_{i=m}^{N-1} \sum_{n=0}^{\infty} \frac{Nn-m+i+1}{(Nn+i+1)(N(n+1)-m+i+1)} c_{Nn+i} \overline{c_{N(n+1)-m+i}}.\end{aligned}$$

Combining (38) and (39) we see that

$$\begin{aligned}(40)\qquad &\sum_{i\neq j,\ i,j\geq 0}^{N-1} \big\langle H_{\overline{z}^m} k_i(z), H_{\overline{z}^N} k_j(z) \big\rangle \\ &= \sum_{n=0}^{m-1} \frac{1}{n+N+1} c_n \overline{c_{n+N-m}} \\ &\quad + \sum_{n=m}^{\infty} \frac{mN}{(n+1)(n+N-m+1)(n+N+1)} c_n \overline{c_{n+N-m}}.\end{aligned}$$

Putting (37) and (40) in (36) we have that $T_{\overline{f}+h}$ is hyponormal if and only if

$$
(41) \qquad
\begin{aligned}
&|a|^2\left(\sum_{n=0}^{N-1}\frac{1}{n+N+1}|c_n|^2+\sum_{n=N}^{\infty}\frac{N^2}{(n+N+1)(n+1)^2}|c_n|^2\right)\\
&+2\mathrm{Re}\left\{a\sum_{m=1}^{N-1}\overline{a_m}\left(\sum_{n=0}^{m-1}\frac{1}{n+N+1}c_n\overline{c_{n+N-m}}\right.\right.\\
&\left.\left.+\sum_{n=m}^{\infty}\frac{mN}{(n+1)(n+N-m+1)(n+N+1)}c_n\overline{c_{n+N-m}}\right)\right\}\geq 0.
\end{aligned}
$$

The inequality (41) holds if for each $m = 1, 2, 3, \ldots, N-1$,

$$
(42) \qquad
\begin{aligned}
&\sum_{n=0}^{N-1}\frac{1}{n+N+1}|c_n|^2+\sum_{n=N}^{\infty}\frac{N^2}{(n+N+1)(n+1)^2}|c_n|^2\\
&\geq \alpha_m\left(\sum_{n=0}^{m-1}\frac{1}{n+N+1}|c_n||c_{n+N-m}|\right.\\
&\left.+\sum_{n=m}^{\infty}\frac{mN}{(n+1)(n+N-m+1)(n+N+1)}|c_n||c_{n+N-m}|\right),
\end{aligned}
$$

where $\alpha_m = \frac{2(N-1)|a_m|}{|a|}$. Note that (42) holds if for $m = 1, 2, \ldots, N-1$,

$$
(43) \qquad
\begin{cases}
|a|^2 \geq \frac{4(N-1)^2|a_m|^2(n+2N-m+1)}{N+n+1} & \text{if } n = 0, 1, 2, \ldots, m-1,\\
|a|^2 \geq \frac{4(N-1)^2|a_m|^2m^2(n+2N-m+1)}{(n+1)^2(n+N+1)} & \text{if } n = m, m+1, \ldots, N-1,\\
|a|^2 \geq \frac{4(N-1)^2|a_m|^2m^2(n+2N-m+1)}{(n+N+1)N^2} & \text{if } n \geq N.
\end{cases}
$$

Observe that (43) holds if $|a| \geq 2\sqrt{2}(N-1)|a_m|$ for all $m = 1, 2, \ldots, N-1$. This completes the proof. □

Corollary 3.4. *Let $f(z) = \sum_{n=1}^{N-1} a_n z^n$ $(n \geq 2)$, $g \in H^\infty$ and $T_{\overline{g}+f}$ be a hyponormal operator. If $|a| \geq 2\sqrt{2}(N-1)A$, where $A := \max\{|a_i| : 1 \leq i \leq N-1\}$ and $h(z) = f(z) + az^N$, then $T_{\overline{g}+h}$ is hyponormal.*

Proof. This follows from Proposition 1.2 and Theorem 3.3. □

Example 3.5. Consider the polynomial

$$\varphi(z) = 2\overline{z}^2 + 2\overline{z} + 4z + z^2.$$

Then (2) shows that T_φ is hyponormal. Put $\psi(z) = 2\overline{z}^2 + 2\overline{z} + 4z + z^2 + 32z^3$. Then Corollary 3.4 shows that T_ψ is hyponormal.

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