J. Korean Math. Soc. **45** (2008), No. 4, pp. 993–1005

PRECISE RATES IN THE LAW OF THE LOGARITHM FOR THE MOMENT CONVERGENCE OF I.I.D. RANDOM VARIABLES

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Abstract. Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables. Set $S_n = X_1 + X_2 + \cdots + X_n$, $M_n = \max_{k \leq n} |S_k|$, $n \geq 1$. Then we obtain that for any $-1 < b < 1/2$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{M_n - \varepsilon\sigma\sqrt{n \log n}\}_+ = \frac{2\sigma}{(b+1)(2b+3)} \mathsf{E}|N|^{2b+3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}}$$

if and only if $\mathsf{E}X = 0$ and $\mathsf{E}X^2 = \sigma^2 < \infty$.

1. Introduction and main results

Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables with mean zero and positive finite variance σ^2. Set $S_n = \sum_{k=1}^{n} X_k$, $M_n = \max_{k \leq n} |S_k|$, $n \geq 1$ and let $\log n = \log(n \vee e)$, $\log\log n = \log\log(n \vee e^e)$.

Gut and Spătaru [5] obtained the following precise asymptotics on the law of the logarithm.

Theorem A. *Let $\{X, X_n; n \geq 1\}$ be a sequence of* i.i.d. *random variables and suppose that $EX = 0$, $EX^2 = \sigma^2 < \infty$. Then, for $0 \leq \delta \leq 1$,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\delta+2} \sum_{n=1}^{\infty} \frac{(\log n)^\delta}{n} P(|S_n| \geq \varepsilon\sqrt{n \log n}) = \frac{\mu^{(2\delta+2)}}{\delta+1} \sigma^{2\delta+2},$$

where $\mu^{2\delta+2}$ is the $(2\delta+2)$th absolute moment of the standard normal distribution.

Received November 13, 2006.

2000 *Mathematics Subject Classification.* Primary 60F15; Secondary 60G50.

Key words and phrases. law of the logarithm, moment convergence, rate of convergence, strong approximation, i.i.d. random variables.

Project supported by NSFC (10571159), NSFC (10671176), KRF-2006-353-C00004 and the Post-doctoral Fellowship Program of KOSEF.

The purpose of this paper is to consider the precise asymptotics of moment convergence of the law of logarithm, which is an analogue of Theorem A. We state our result as follows.

Theorem 1.1. *For any $-1 < b < 1/2$, the following are equivalent:*

$$EX = 0 \quad \text{and} \quad EX^2 = \sigma^2 \quad (0 < \sigma < \infty), \tag{1.1}$$

$$\begin{aligned}&\lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} E\{M_n - \varepsilon\sigma\sqrt{n \log n}\}_+ \\ &= \frac{2\sigma}{(b+1)(3+2b)} E|N|^{3+2b} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{3+2b}}\end{aligned} \tag{1.2}$$

and

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} E\{|S_n| - \varepsilon\sigma\sqrt{n \log n}\}_+ = \frac{\sigma}{(b+1)(3+2b)} E|N|^{3+2b}, \tag{1.3}$$

where N is a standard normal variable.

Throughout this paper, we let C denote a generic constant, which can vary from one place to another. $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.

2. Proofs

First, we give four lemmas which will be used in the proof of Theorem 1.1. Lemma 2.1 is well known, see Billingsley [1], page 97.

Lemma 2.1. *Let $\{W(t); t \geq 0\}$ be a standard Wiener process, and N be a standard normal variable. Then*

$$\begin{aligned}P\{\sup_{0 \leq s \leq 1} |W(s)| \geq x\} &= 1 - \sum_{k=-\infty}^{\infty} (-1)^k P\{(2k-1)x \leq N \leq (2k+1)x\} \\ &= 4\sum_{k=0}^{\infty} (-1)^k P\{N \geq (2k+1)x\} \\ &= 2\sum_{k=0}^{\infty} (-1)^k P\{|N| \geq (2k+1)x\}.\end{aligned} \tag{2.1}$$

Proof. It is well-known, see Billingsley [1], page 97. □

Lemma 2.2. *Let $\{W(t); t \geq 0\}$ be a standard Wiener process. For any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that*

$$P(\sup_{0 \leq s \leq 1-h} \sup_{0 < t \leq h} |W(s+t) - W(s)| \geq \nu\sqrt{h}) \leq \frac{C}{h} e^{-\frac{\nu^2}{2+\varepsilon}}$$

for every $\nu > 0$ and $h < 1$.

Proof. It is Lemma 1.1.1 of Csörgő and Révész [2]. □

Lemma 2.3. *For any sequence of independent random variables $\{\xi_n; n \geq 1\}$ with mean zero and finite variance, there exists a sequence of independent normal variables $\{\eta_n; n \geq 1\}$ with $E\eta_n = 0$ and $E\eta_n^2 = E\xi_n^2$ such that, for all $q > 2$ and $y > 0$,*

$$P(\max_{k \leq n} |\sum_{i=1}^{k} \xi_i - \sum_{i=1}^{k} \eta_i| \geq y) \leq (Aq)^q y^{-q} \sum_{i=1}^{n} E|\xi_i|^q, \tag{2.2}$$

where A is a universal constant.

Proof. The proof can be found in Sakhaneko [6, 7]. □

The purpose of this section is to employ strong approximation and truncation methods of Feller [4] and Einmahl [3] to show that the moment of M_n can be approximated by that of $\sqrt{n} \sup_{0 \leq s \leq 1} |W(s)|$ and the moment of S_n can be approximated by that of $\sqrt{n}N$. In the sequel, without losing of generality, we assume that $\sigma = 1$. Moreover, For each n and $1 \leq j \leq n$, we denote

$$X'_{nj} = X_j I\{|X_j| \leq \sqrt{n} \log n\}, \qquad X^{(1)}_{nj} = X'_{nj} - \mathsf{E}X'_{nj},$$

$$S^{(1)}_{nj} = \sum_{i=1}^{j} X^{(1)}_{ni}, \quad M^{(1)}_n = \max_{1 \leq k \leq n} |S^{(1)}_{nk}|, \qquad B_n = \sum_{j=1}^{n} \mathsf{Var} X^{(1)}_{nj}.$$

Theorem 1.1 will be proved by the following propositions.

Proposition 2.1. *For any $b > -1$,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} E\{|N| - \varepsilon\sqrt{\log n}\}_+ = \frac{1}{(b+1)(2b+3)} E|N|^{2b+3} \tag{2.3}$$

and

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} E\{\sup_{0 \leq s \leq 1} |W(s)| - \varepsilon\sqrt{\log n}\}_+ \\ &= \frac{2}{(b+1)(2b+3)} E|N|^{2b+3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}}. \end{aligned} \tag{2.4}$$

Proof. In light of (2.1) and the fact $\mathsf{P}(|N| \geq x) = 2\mathsf{P}(N \geq x)$ we have for any $m \geq 1$,

$$\begin{aligned} 2\sum_{k=0}^{2m+1} (-1)^k \mathsf{P}\{|N| \geq (2k+1)x\} &\leq \mathsf{P}\{\sup_{0 \leq s \leq 1} |W(s)| \geq x\} \\ &\leq 2\sum_{k=0}^{2m} (-1)^k \mathsf{P}\{|N| \geq (2k+1)x\}. \end{aligned}$$

Hence, for any $t > 0$ and $m \geq 1$,

$$\begin{aligned}
\mathsf{E}\{\sup_{0\leq s\leq 1}|W(s)|-t\}_+ &= \int_0^\infty \mathsf{P}(\sup_{0\leq s\leq 1}|W(s)|\geq t+x)dx \\
&\leq 2\sum_{k=0}^{2m}(-1)^k\int_0^\infty \mathsf{P}\{|N|\geq (2k+1)(t+x)\}dx \\
&= 2\sum_{k=0}^{2m}\frac{(-1)^k}{2k+1}\int_0^\infty \mathsf{P}\{|N|\geq (2k+1)t+x\}dx \\
&= 2\sum_{k=0}^{2m}\frac{(-1)^k}{2k+1}\mathsf{E}\{|N|-(2k+1)t\}_+.
\end{aligned}$$

Similarly, we have

$$\mathsf{E}\{\sup_{0\leq s\leq 1}|W(s)|-t\}_+ \geq 2\sum_{k=0}^{2m+1}\frac{(-1)^k}{2k+1}\mathsf{E}\{|N|-(2k+1)t\}_+.$$

So we only need to prove (2.3). For any $\beta \geq 1$ and $b > -1$,

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{2b+2}\sum_{n=1}^\infty \frac{(\log n)^b}{n}\mathsf{E}\{|N|-\beta\varepsilon\sqrt{\log n}\}_+ \\
&= \lim_{\varepsilon\searrow 0}\varepsilon^{2b+2}\int_e^\infty \frac{(\log y)^b}{y}\int_{\beta\varepsilon\sqrt{\log y}}^\infty \mathsf{P}(|N|\geq x)dxdy \\
&= \lim_{\varepsilon\searrow 0}\varepsilon^{2b+2}\int_{\beta\varepsilon}^\infty 2(\beta\varepsilon)^{-(2b+2)}t^{2b+1}\int_t^\infty \mathsf{P}(|N|\geq x)dxdt \\
&= 2\beta^{-(2b+2)}\lim_{\varepsilon\searrow 0}\int_{\beta\varepsilon}^\infty \mathsf{P}(|N|\geq x)\int_{\beta\varepsilon}^x t^{2b+1}dtdx \\
&= 2\beta^{-(2b+2)}\lim_{\varepsilon\searrow 0}\int_{\beta\varepsilon}^\infty \mathsf{P}(|N|\geq x)\cdot\frac{1}{2b+2}(x^{2b+2}-(\beta\varepsilon)^{2b+2})dx \\
&= \frac{2\beta^{-(2b+2)}}{2b+2}\lim_{\varepsilon\searrow 0}\int_{\beta\varepsilon}^\infty x^{2b+2}\mathsf{P}(|N|\geq x)dx \\
&= \frac{\beta^{-(2b+2)}}{(b+1)(2b+3)}\mathsf{E}|N|^{2b+3}.
\end{aligned} \tag{2.5}$$

By letting $\beta = 1$, the proposition is proved. □

Proposition 2.2. *For any $b \leq 1$, we have*

$$\sum_{n=1}^\infty \frac{(\log n)^b}{\sqrt{n}}E|X|I\{|X|>\sqrt{n}\log n\}<\infty. \tag{2.6}$$

Proof. It is easy to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\log n)^b}{\sqrt{n}} \mathsf{E}|X| I\{|X| > \sqrt{n}\log n\} \\
=\ & \sum_{n=1}^{\infty} \frac{(\log n)^b}{\sqrt{n}} \sum_{j=n}^{\infty} \mathsf{E}|X| I\{\sqrt{j}\log j < |X| \le \sqrt{(j+1)}\log(j+1)\} \\
=\ & \sum_{j=1}^{\infty} \mathsf{E}|X| I\{\sqrt{j}\log j < |X| \le \sqrt{(j+1)}\log(j+1)\} \sum_{n=1}^{j} \frac{(\log n)^b}{\sqrt{n}} \\
\le\ & C\sum_{j=1}^{\infty} \mathsf{E}|X| I\{\sqrt{j}\log j < |X| \le \sqrt{(j+1)}\log(j+1)\} \cdot \sqrt{j}(\log j)^b \\
\le\ & C\mathsf{E}\big(X^2(\log|X|)^{b-1}\big) \le C\mathsf{E}X^2 < \infty.
\end{aligned}$$

□

Proposition 2.3. *There exists $\alpha > 0$ small enough such that for any $x > 0$*

$$\begin{aligned}
&\sqrt{B_n} E\{\sup_{0\le s\le 1} |W(s)| - x - (\log n)^{p'}\}_+ - p_n \\
\le\ & E\{\max_{k\le n} |\sum_{j=1}^{k} X_{nj}^{(1)}| - x\sqrt{B_n}\}_+ \\
\le\ & \sqrt{B_n} E\{\sup_{0\le s\le 1} |W(s)| - x + (\log n)^{p'}\}_+ + p_n. \qquad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{B_n} E\{|N| - x - (\log n)^{p'}\}_+ - p_n \\
\le\ & E\{|\sum_{j=1}^{n} X_{nj}^{(1)}| - x\sqrt{B_n}\}_+ \\
\le\ & \sqrt{B_n} E\{|N| - x + (\log n)^{p'}\}_+ + p_n, \qquad (2.8)
\end{aligned}$$

where $p' = 1/2 - \alpha$ and $p_n \ge 0$ satisfying

$$\sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} p_n < \infty \qquad (2.9)$$

with $b < 1/2$.

Proof. Obviously, $n > B_n \sim n$. In view of Lemma 2.3, there exist a universal constant A and a sequence of independent standard Wiener processes $\{W_n(\cdot)\}$ such that for all fixed $q > 2$, one has

$$p_{n1} := \mathsf{E}\{\max_{k\le n} |\sum_{j=1}^{k} X_{nj}^{(1)} - W_n(\frac{k}{n}B_n)| - \frac{1}{2}\sqrt{B_n}(\log n)^{p'}\}_+ \qquad (2.10)$$

$$
\begin{aligned}
&= \int_{\frac{1}{2}\sqrt{B_n}(\log n)^{p'}}^{\infty} \mathsf{P}(\max_{k\le n} |\sum_{j=1}^{k} X_{nj}^{(1)} - W_n(\frac{k}{n}B_n)| \ge x)dx \\
&\le \int_{\frac{1}{2}\sqrt{B_n}(\log n)^{p'}}^{\infty} (Aq)^q x^{-q} \sum_{k=1}^{n} \mathsf{E}|X_{nk}^{(1)}|^q dx \\
&\le Cn(\frac{1}{2}\sqrt{B_n}(\log n)^{p'})^{-q+1} \mathsf{E}|X|^q I\{|X| \le \sqrt{n}\log n\} \\
&\le Cn^{3/2-q/2}(\log n)^{p'(1-q)} \mathsf{E}|X|^q I\{|X| \le \sqrt{n}\log n\},
\end{aligned}
$$

hence,

(2.11)

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} p_{n1} \\
&\le C\sum_{n=1}^{\infty} \frac{(\log n)^{b+p'(1-q)}}{n^{q/2}} \mathsf{E}|X|^q I\{|X| \le \sqrt{n}\log n\} \\
&= C\sum_{n=1}^{\infty} \frac{(\log n)^{b+p'(1-q)}}{n^{q/2}} \sum_{j=1}^{n} \mathsf{E}|X|^q I\{\sqrt{(j-1)}\log(j-1) < |X| \le \sqrt{j}\log j\} \\
&= C\sum_{j=1}^{\infty} \mathsf{E}|X|^q I\{\sqrt{(j-1)}\log(j-1) < |X| \le \sqrt{j}\log j\} \sum_{n=j}^{\infty} \frac{(\log n)^{b+p'(1-q)}}{n^{q/2}} \\
&\le C\sum_{j=1}^{\infty} \mathsf{E}|X|^q I\{\sqrt{(j-1)}\log(j-1) < |X| \le \sqrt{j}\log j\} \cdot j^{1-q/2}(\log j)^{b+p'(1-q)} \\
&\le C\mathsf{E}\left(X^2(\log|X|)^{b+p'(1-q)-(2-q)}\right) \\
&\le C\mathsf{E}X^2 < \infty
\end{aligned}
$$

by choosing α small enough and letting $q = 2+\alpha$ such that

$$b + p'(1-q) - (2-q) = 3\alpha/2 + \alpha^2 + b - 1/2 \le 0.$$

On the other hand, write

$$
\begin{aligned}
p_{n2} &:= \mathsf{E}\{\sup_{0\le s\le 1} |W_n(sB_n) - W_n(\frac{[ns]}{n}B_n)| - \frac{1}{2}\sqrt{B_n}(\log n)^{p'}\}_+ \\
&= \sqrt{B_n}\mathsf{E}\{\sup_{0\le s\le 1} |W_n(s) - W_n(\frac{[ns]}{n})| - \frac{1}{2}(\log n)^{p'}\}_+ \\
&= \sqrt{B_n}\int_0^1 \mathsf{P}(\sup_{0\le s\le 1} |W_n(s) - W_n(\frac{[ns]}{n})| \ge x + \frac{1}{2}(\log n)^{p'})dx \\
&\quad + \sqrt{B_n}\int_1^{\infty} \mathsf{P}(\sup_{0\le s\le 1} |W_n(s) - W_n(\frac{[ns]}{n})| \ge x + \frac{1}{2}(\log n)^{p'})dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sqrt{n}\mathsf{P}\big(\sup_{0\leq s\leq 1}|W_n(s)-W_n(\tfrac{[ns]}{n})|\geq \tfrac{1}{2}(\log n)^{p'}\big)\\
&\quad+\sqrt{n}\int_1^\infty \mathsf{P}\big(\sup_{0\leq s\leq 1}|W_n(s)-W_n(\tfrac{[ns]}{n})|\geq x\big)dx\\
&=: D_{n1}+D_{n2}. \qquad (2.12)
\end{aligned}
$$

By Lemma 2.2, we have

$$
\begin{aligned}
&\sum_{n=1}^\infty \frac{(\log n)^b}{n^{3/2}}D_{n1}\\
&= \sum_{n=1}^\infty (\log n)^b n^{-1}\mathsf{P}\big(\sup_{0\leq s\leq 1}|W_n(s)-W_n(\tfrac{[ns]}{n})|\geq \frac{1}{\sqrt{n}}\frac{\sqrt{n}(\log n)^{p'}}{2}\big)\\
&\leq C\sum_{n=1}^\infty (\log n)^b n^{-1}\cdot n\exp\Big\{-\frac{(\sqrt{n}(\log n)^{p'})^2/4}{3}\Big\}\\
&= C\sum_{n=1}^\infty (\log n)^b\exp\Big\{-\frac{n(\log n)^{2p'}}{12}\Big\}<\infty \qquad (2.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n=1}^\infty \frac{(\log n)^b}{n^{3/2}}D_{n2}\\
&= \sum_{n=1}^\infty \frac{(\log n)^b}{n}\int_1^\infty \mathsf{P}\big(\sup_{0\leq s\leq 1}|W_n(s)-W_n(\tfrac{[ns]}{n})|\geq \sqrt{\frac{1}{n}}\sqrt{n}x\big)dx\\
&\leq C\sum_{n=1}^\infty \frac{(\log n)^b}{n}\int_1^\infty n\exp\Big\{-\frac{nx^2}{3}\Big\}dx\\
&\leq C\int_e^\infty (\log y)^b\int_1^\infty \exp\Big\{-\frac{yx^2}{3}\Big\}dxdy\\
&= C\int_e^\infty (\log y)^b y^{-1/2}\int_{\sqrt{y/3}}^\infty \exp\{-t^2\}dtdy \ \text{(by letting } t=\sqrt{\tfrac{y}{3}}x)\\
&= C\int_{\sqrt{e/3}}^\infty \exp\{-t^2\}\int_e^{3t^2} y^{-1/2}(\log y)^b dydt\\
&\leq C\int_{\sqrt{e/3}}^\infty \exp\{-t^2\}t(\log t)^b dt<\infty. \qquad (2.14)
\end{aligned}
$$

Combining the last two inequalities leads to

$$
\sum_{n=1}^\infty \frac{(\log n)^b}{n^{3/2}}p_{n2}<\infty. \tag{2.15}
$$

Write

$$\begin{aligned}
&\mathsf{E}\{\max_{k\le n}|\sum_{j=1}^{k}X_{nj}^{(1)}|-x\sqrt{B_n}\}_+\\
=\ &\mathsf{E}\{\sup_{0\le s\le 1}|\sum_{j=1}^{[ns]}X_{nj}^{(1)}-W_n(sB_n)+W_n(sB_n)|-x\sqrt{B_n}\}_+.
\end{aligned}$$

Since $\{\frac{W_n(tB_n)}{\sqrt{B_n}};t\ge 0\}\stackrel{\mathcal{D}}{=}\{W(t);t\ge 0\}$ for each n, so for any real t one has

$$\begin{aligned}
\mathsf{E}\{\sup_{0\le s\le 1}|W_n(sB_n)|-t\sqrt{B_n}\}_+ &= \sqrt{B_n}\mathsf{E}\{\sup_{0\le s\le 1}|\frac{W_n(sB_n)}{\sqrt{B_n}}|-t\}_+\\
&= \sqrt{B_n}\mathsf{E}\{\sup_{0\le s\le 1}|W(s)|-t\}_+.
\end{aligned}$$

Denote

$$p_n = \mathsf{E}\left\{\sup_{0\le s\le 1}|\sum_{k=1}^{[ns]}X_{nk}^{(1)}-W_n(sB_n)|-\sqrt{B_n}(\log n)^{p'}\right\}_+ .$$

Then

$$\begin{aligned}
&\sqrt{B_n}\mathsf{E}\{\sup_{0\le s\le 1}|W(s)|-x-(\log n)^{p'}\}_+-p_n\\
\le\ &\mathsf{E}\{\max_{k\le n}|\sum_{j=1}^{k}X_{nj}^{(1)}|-x\sqrt{B_n}\}_+\\
\le\ &\sqrt{B_n}\mathsf{E}\{\sup_{0\le s\le 1}|W(s)|-x+(\log n)^{p'}\}_++p_n,
\end{aligned}$$

and from (2.11) and (2.15), we have

$$\sum_{n=1}^{\infty}\frac{(\log n)^b}{n^{3/2}}p_n\le\sum_{n=1}^{\infty}\frac{(\log n)^b}{n^{3/2}}(p_{n1}+p_{n2})<\infty.$$

(2.8) can be proved in the same way. The proposition is proved. □

Now, we turn to prove the theorem.

Proof of Theorem 1.1. By $\mathsf{E}X=0$, we have

$$\begin{aligned}
&M_n-\varepsilon\sqrt{B_n\log n}\\
=\ &\max_{k\le n}|\sum_{i=1}^{k}X_{ni}^{'}+\sum_{i=1}^{k}(X_i-X_{ni}^{'})|-\varepsilon\sqrt{B_n\log n}\\
=\ &\max_{k\le n}|\sum_{i=1}^{k}X_{ni}^{(1)}+\sum_{i=1}^{k}X_iI\{|X_i|>\sqrt{n}\log n\}-\sum_{i=1}^{k}\mathsf{E}X_iI\{|X_i|>\sqrt{n}\log n\}|\\
&-\varepsilon\sqrt{B_n\log n}.
\end{aligned}$$

Denote

$$q_n = \sum_{j=1}^{n} \mathsf{E}|X_j| I\{|X_j| > \sqrt{n}\log n\} = n\mathsf{E}|X| I\{|X| > \sqrt{n}\log n\}.$$

Then we have

$$\begin{aligned} & \mathsf{E}\{\max_{k\le n} |\sum_{i=1}^{k} X_{ni}^{(1)}| - \varepsilon\sqrt{B_n \log n}\}_+ - 2q_n \\ \le\ & \mathsf{E}\{M_n - \varepsilon\sqrt{B_n\log n}\}_+ \\ \le\ & \mathsf{E}\{\max_{k\le n} |\sum_{i=1}^{k} X_{ni}^{(1)}| - \varepsilon\sqrt{B_n \log n}\}_+ + 2q_n. \end{aligned}$$

In view of Proposition 2.3, we have for any $p' = 1/2 - \alpha$,

$$\begin{aligned} & \sqrt{B_n}\mathsf{E}\{\sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{\log n} - (\log n)^{p'}\}_+ - p_n - 2q_n \\ \le\ & \mathsf{E}\{M_n - \varepsilon\sqrt{B_n\log n}\}_+ \\ \le\ & \sqrt{B_n}\mathsf{E}\{\sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{\log n} + (\log n)^{p'}\}_+ + p_n + 2q_n, \end{aligned}$$

where $p_n \ge 0$ satisfying

$$\sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} p_n < \infty.$$

Moreover, by Proposition 2.2, we have

$$\sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} q_n = \sum_{n=1}^{\infty} \frac{(\log n)^b}{\sqrt{n}} \mathsf{E}|X| I\{|X| > \sqrt{n}\log n\} < \infty.$$

On the other hand, let $\varepsilon' = \varepsilon \pm 1/(\log n)^{\alpha} \sim \varepsilon$ and notice that $B_n/n \to 1$ as $n \to \infty$, then by Propositions 2.3, 2.2 and 2.1, we have

$$\begin{aligned} (2.16)\quad & \lim_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{M_n - \varepsilon\sqrt{B_n \log n}\}_+ \\ =\ & \lim_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \sqrt{B_n}\mathsf{E}\left\{\sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{\log n} \pm (\log n)^{p'}\right\}_+ \\ =\ & \lim_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \sqrt{B_n}\mathsf{E}\left\{\sup_{0\le s\le 1} |W(s)| - \varepsilon'\sqrt{\log n}\right\}_+ \\ =\ & \lim_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} \mathsf{E}\left\{\sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{\log n}\right\}_+ \end{aligned}$$

$$= \frac{2}{(b+1)(2b+3)} \mathsf{E}|N|^{2b+3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}}.$$

Moreover, for n large enough and all $\varepsilon > 0$ and $\delta > 0$ we have

$$\begin{aligned} & \mathsf{E}\{M_n - \varepsilon(1+\delta)\sqrt{B_n \log n}\}_+ \\ \leq\ & \mathsf{E}\{M_n - \varepsilon\sqrt{n \log n}\}_+ \\ \leq\ & \mathsf{E}\{M_n - \varepsilon\sqrt{B_n \log n}\}_+. \end{aligned}$$

So, by the proof of Propositions 2.1 (taking $\beta = 1+\delta$ in (2.5)),

$$\begin{aligned} & (1+\delta)^{-2(b+1)} \frac{2}{(b+1)(2b+3)} \mathsf{E}|N|^{2b+3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}} \\ \leq\ & \liminf_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} n^{-1/2} \mathsf{E}\{M_n - \varepsilon\sqrt{n \log n}\}_+ \\ \leq\ & \limsup_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} n^{-1/2} \mathsf{E}\{M_n - \varepsilon\sqrt{n \log n}\}_+ \\ \leq\ & \frac{2}{(b+1)(2b+3)} \mathsf{E}|N|^{2b+3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}}. \end{aligned}$$

Letting $\delta \to 0$, we obtain that (1.1) implies (1.2). Similarly, (1.1) implies (1.3).

Now, we only need to show (1.3) implies (1.1). First, we show that $0 < \mathsf{E}X^2 < \infty$. $\mathsf{E}X^2 > 0$ is obvious, otherwise, (1.3) can not hold. Let $\{X', X_n'; n \geq 1\}$ be an independent copy of $\{X, X_n; n \geq 1\}$, and

$$S_n' = \sum_{k=1}^{n} X_k', \quad \tilde{X}_n = X_n - X_n', \quad \tilde{S}_n = S_n - S_n'.$$

From (1.3), for all $\varepsilon > 0$, we have

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{|\tilde{S}_n| - 4\sigma\varepsilon\sqrt{n \log n}\}_+ \\ \leq\ & 2 \lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{|S_n| - \sigma\varepsilon\sqrt{n \log n}\}_+ \\ <\ & \infty. \end{aligned}$$

For any $M > 0$, $n \geq 1$, we denote

$$Y_n^{(1)} = \tilde{X}_n I\{|\tilde{X}_n| \leq M\},$$

$$Y_n^{(2)} = \tilde{X}_n I\{|\tilde{X}_n| \leq M\} - \tilde{X}_n I\{|\tilde{X}_n| > M\}.$$

Then the sequence $\{Y_n^{(2)}; n \geq 1\}$ has the same distribution as $\{\tilde{X}_n; n \geq 1\}$, and $\tilde{X}_n + Y_n^{(2)} = 2Y_n^{(1)}$. It follows that

$$\limsup_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{|2\sum_{k=1}^{n} Y_k^{(1)}| - 2\sigma\varepsilon\sqrt{n\log n}\}_+$$

$$(2.17) \quad \leq \quad 2\limsup_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{|\tilde{S}_n| - 2\sigma\varepsilon\sqrt{n\log n}\}_+ < C < \infty.$$

However, since $\{Y_n^{(1)}; n \geq 1\}$ is a sequence of i.i.d. bounded random variables with $\mathsf{E}Y_1^{(1)} = 0$, by the proof of the first part of the theorem we have
(2.18)

$$0 < \lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{|\sum_{k=1}^{n} Y_k^{(1)}| - \varepsilon\sigma\frac{\sqrt{\mathsf{E}(Y_1^{(1)})^2}}{\sigma}\sqrt{n\log n}\}_+ < C.$$

By (2.17) and (2.18) we obtain

$$\Big(\frac{\sqrt{\mathsf{E}(Y_1^{(1)})^2}}{2\sigma}\Big)^{2b+2} \lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}}$$

$$\cdot\, \mathsf{E}\{|\sum_{k=1}^{n} Y_k^{(1)}| - 2\sigma\varepsilon\frac{\sqrt{\mathsf{E}(Y_1^{(1)})^2}}{2\sigma}\sqrt{n\log n}\}_+$$

$$= \limsup_{\varepsilon_1 \searrow 0} \varepsilon_1^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{|\sum_{k=1}^{n} Y_k^{(1)}| - 2\sigma\varepsilon_1\sqrt{n\log n}\}_+$$

$$\leq \limsup_{\varepsilon_1 \searrow 0} \varepsilon_1^{2b+2} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{|2\sum_{k=1}^{n} Y_k^{(1)}| - 2\sigma\varepsilon_1\sqrt{n\log n}\}_+ < C < \infty,$$

which leads to

$$\frac{\sqrt{\mathsf{E}\tilde{X}_1^{\,2} I\{|\tilde{X}_1| \leq M\}}}{\sigma} = \frac{\sqrt{\mathsf{E}(Y_1^{(1)})^2}}{\sigma} \leq C.$$

Letting $M \to \infty$ yields $\mathsf{E}X^2 < \infty$.

"$\mathsf{E}X = 0$" is obvious, otherwise, for any $\varepsilon > 0$, by the law of large numbers we have

$$\mathsf{P}(|S_n| \geq 2\varepsilon\sigma\sqrt{n\log n}) \to 1.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\{|S_n| - \varepsilon\sigma\sqrt{n\log n}\}_+$$

$$= \quad \sigma\sum_{n=1}^{\infty} \frac{(\log n)^b}{n} \int_0^{\infty} \mathsf{P}(\frac{|S_n|}{\sqrt{n}\sigma} \geq \varepsilon\sqrt{\log n} + x)dx$$

$$\begin{aligned}
&\geq \sigma \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} \int_0^{\varepsilon\sqrt{\log n}} \mathsf{P}(\frac{|S_n|}{\sqrt{n}\sigma} \geq \varepsilon\sqrt{\log n} + x)dx \\
&\geq \sum_{n=1}^{\infty} \frac{(\log n)^{b+1/2}}{n} \mathsf{P}(|S_n| \geq 2\varepsilon\sigma\sqrt{n \log n}) = \infty,
\end{aligned}$$

which is a contradiction to (1.3).

Suppose $\mathsf{E}X = 0$, $\mathsf{E}X^2 < \infty$ and (1.3) holds for some constant σ. By the direct part of Theorem 1.1, (1.3) should hold with $\mathsf{E}X^2$ taking the place of σ^2, it is obviously a contradiction if $\mathsf{E}X^2 \neq \sigma^2$. The proof is completed. □

References

[1] P. Billingsley, *Convergence of Probability Measures: Second edition*, Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.

[2] M. Csörgő and P. Révész, *Strong Approximations in Probability and Statistics*, Probability and Mathematical Statistics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[3] U. Einmahl, *The Darling-Erdös theorem for sums of i.i.d. random variables*, Probab. Theory Related Fields **82** (1989), no. 2, 241–257.

[4] W. Feller, *The law of the iterated logarithm for identically distributed random variables*, Ann. of Math. (2) **47** (1946), 631–638.

[5] A. Gut and A. Spătaru, *Precise asymptotics in the law of the iterated logarithm*, Ann. Probab. **28** (2000), no. 4, 1870–1883.

[6] A. I. Sakhanenko, *On estimates of the rate of convergence in the invariance principle*, In Advances in Probability Theory: Limit Theorems and Related Problems (A. A. Borovkov, ed.) 124–135, Springer, New York, 1984.

[7] ———, *Convergence rate in the invariance principle for nonidentically distributed variables with exponential moment*, In Advances in Probability Theory: Limit Theorems for Sums of Random Variables (A. A. Borovkov, ed.) 2–73. Springer, New York, 1985. Springer, New York.

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