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WEIGHTED COMPOSITION OPERATORS FROM $F(p,q,s)$ INTO LOGARITHMIC BLOCH SPACE

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Abstract. We characterize the boundedness and compactness of the weighted composition operator uC_φ from the general function space $F(p, q, s)$ into the logarithmic Bloch space β_L on the unit disk. Some necessary and sufficient conditions are given for which uC_φ is a bounded or a compact operator from $F(p,q,s)$, $F_0(p,q,s)$ into β_L, β_L^0 respectively.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and $H(D)$ denote the set of all analytic functions on D. For $f \in H(D)$, let

$$\|f\|_{\beta_L} = \sup\{(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| : z \in D\}.$$

As in [6], the logarithmic Bloch space β_L consists of all $f \in H(D)$ satisfying $\|f\|_{\beta_L} < +\infty$ and the little logarithmic Bloch space β_L^0 consists of all $f \in H(D)$ satisfying $\lim_{|z|\to 1^-}(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| = 0$. The β_L is a Banach space under the norm

$$\|f\|_L = |f(0)| + \|f\|_{\beta_L}.$$

In [6], the author proved that β_L^0 is a closed subspace and coincides with the closure of polynomials under the norm. Yoneda [8] studied the composition operator in the β_L space and the β_L^0 space.

We write β_α $(\alpha > 0)$ for the space of analytic function f for which $\|f\|_{\beta_\alpha} = |f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha|f'(z)| < \infty$ and β_α^0 $(\alpha > 0)$ for $f \in \beta_\alpha$ such that $\lim_{|z|\to 1^-}(1-|z|^2)^\alpha|f'(z)| = 0$. The functions in β_α and β_α^0 will be referred to as α-Bloch functions, little α-Bloch functions respectively. It is easily proved that for $0 < \alpha < 1$, $\beta_\alpha \subsetneqq \beta_L \subsetneqq \beta_1$. For more information about the β_α (see [12]).

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Let dm denote the Lebesgue area measure in D. Let $0 < p, s < \infty$ and $-2 < q < \infty$, then a function $f \in H(D)$ is said to belong to the general function space $F(p,q,s)$ (see [11]) if

$$\|f\|_{F(p,q,s)} = |f(0)| + \{ \sup_{a \in D} \int_D |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s \, dm(z) \}^{\frac{1}{p}} < \infty$$

and the little general function space $F_0(p,q,s)$ if $f \in F(p,q,s)$ and

$$\lim_{|a| \to 1^-} \int_D |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s \, dm(z) = 0.$$

Here φ_a is a conformal automorphism defined by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in D$. We can get many function spaces if we take some specific parameters of p, q, s. For example (see [11]), $F(p,q,s) = \beta_{\frac{q+2}{p}}$ and $F_0(p,q,s) = \beta^0_{\frac{q+2}{p}}$ for $s > 1$; $F(p,q,s) \subset \beta_{\frac{q+2}{p}}$ and $F_0(p,q,s) \subset \beta^0_{\frac{q+2}{p}}$ for $0 < s \leq 1$; $F(2,0,s) = Q_s$ and $F_0(2,0,s) = Q_{s,0}$; $F(2,0,1) = BMOA$ and $F_0(2,0,1) = VMOA$. For more information about the $BMOA$ and $VMOA$ (see [1]). If $q + s \leq -1$, then $F(p,q,s)$ is the space of constant functions.

An analytic self-map $\varphi : D \to D$ induces the composition operator C_φ on $H(D)$, defined by $C_\varphi(f) = f(\varphi(z))$ for f analytic on D. It is a well known consequence of Littlewood's subordination principle that the composition operator C_φ is bounded on the classical Hardy and Bergman spaces (see, for example, [2, 13]).

Recall that a linear operator is said to be bounded if the image of a bounded set is a bounded set, while a linear operator is compact if it takes bounded sets to sets with compact closure. It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces. The book [2] contains plenty of information on this topic.

Let u be a fixed analytic function on the open unit disk. Define a linear operator uC_φ on the space of analytic functions on D, called a weighted composition operator, by $uC_\varphi f = u \cdot (f \circ \varphi)$, where f is an analytic function on D. We can regard this operator as a generalization of a multiplication operator and a composition operator.

In [4, 5], Ohno, Stroethoff and Zhao have characterized the boundedness and compactness of weighted composition operators between α-Bloch spaces, between H^∞ and the Bloch space β_1, and from the little $F_0(p,q,s)$ into the little Bloch space β_1^0. The boundedness and compactness of weighted composition operators from $F(p,q,s)$ into the Bloch space β_α were investigated in [9].

In this paper we study the weighted composition operators from the general function spaces $F(p,q,s)$ into the logarithmic Bloch space β_L and the little $F_0(p,q,s)$ into the little logarithmic Bloch space β_L^0. Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Boundedness and compactness of $uC_\varphi : F(p,q,s) \to \beta_L$

In this section we characterize the boundedness and compactness of the weighted composition operator $uC_\varphi : F(p,q,s) \to \beta_L$. For this purpose we need some lemmas. The following lemma can be found in [11].

Lemma 2.1. *Suppose that $0 < p, s < \infty$, $-2 < q < \infty$, $q+s > -1$ and $f \in F(p,q,s)$. Then there is a positive constant C such that $\|f\|_{\beta_\alpha} \le C\|f\|_{F(p,q,s)}$, moreover, if $f \in F_0(p,q,s)$, then $f \in \beta_\alpha^0$, where $\alpha = \frac{q+2}{p}$.*

Lemma 2.2 ([7]). *Suppose $t > 0$ and $f \in H(D)$. Then $\sup_{z\in D}(1-|z|)^t|f(z)| < +\infty$ if and only if $\sup_{z\in D}(1-|z|)^{t+1}|f'(z)| < +\infty$.*

Lemma 2.3. *Let $0 < p, s < \infty$, $-2 < q < \infty$ and $q+s > -1$. The operator $uC_\varphi : F(p,q,s) \to \beta_L$ is compact if and only if for any bounded sequence $\{f_n\}$ in $F(p,q,s)$ which converges to zero uniformly on compact subsets of D, we have $\|uC_\varphi f_n\|_L \to 0$ as $n \to \infty$.*

Proof. Using [6, Lemma 2.1] and Montel's Theorem, one may prove the lemma. The details are omitted here. □

Lemma 2.4 ([7]). *Let $\alpha > 0$ and $f \in \beta_\alpha$. Then*
(1) $|f(z)| \le C\|f\|_{\beta_\alpha}$, *where* $\alpha < 1$;
(2) $|f(z)| \le C\log(\frac{2}{1-|z|^2})\|f\|_{\beta_\alpha}$, *where* $\alpha = 1$;
(3) $|f(z)| \le \frac{C}{(\alpha-1)(1-|z|)^{\alpha-1}}\|f\|_{\beta_\alpha}$, *where* $\alpha > 1$.

Theorem 2.1. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q+s > -1$, $u \in H(D)$ and φ be an analytic self-map of D.*

(i) *If $2+q > p$, then uC_φ is a bounded operator from $F(p,q,s)$ to β_L if and only if*

$$\sup_{z\in D} \frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^\alpha}|u(z)\varphi'(z)| < +\infty \tag{1}$$

and

$$\sup_{z\in D} \frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| < +\infty, \tag{2}$$

where $\alpha = \frac{q+2}{p}$.

(ii) *If $2+q < p$, then uC_φ is a bounded operator from $F(p,q,s)$ to β_L if and only if $u \in \beta_L$ and (1) holds.*

(iii) *If $2+q = p$, $s > 1$, then uC_φ is a bounded operator from $F(p,q,s)$ to β_L if and only if (1) holds and*

$$\sup_{z\in D} |u'(z)|(1-|z|^2)\log\Big(\frac{2}{1-|z|}\Big)\log\Big(\frac{2}{1-|\varphi(z)|^2}\Big) < \infty. \tag{3}$$

(iv) *Let $2+q = p$, $0 < s \le 1$. Then uC_φ is a bounded operator from $F(p,q,s)$ to β_L provided that (1) and (3) hold.*

Conversely, if uC_φ is a bounded operator from $F(p,q,s)$ to β_L, then $u \in \beta_L$ and (1) *holds.*

Proof. Suppose that uC_φ is bounded from $F(p,q,s)$ to β_L. Then we can easily obtain $u \in \beta_L$ by taking $f(z)=1$. Let $\alpha = \frac{q+2}{p}$. Fixed $w \in D$, we consider the function

$$f_w(z) = \frac{(z-\varphi(w))(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^{\alpha+1}}. \tag{4}$$

Then $\|f_w\|_{F(p,q,s)} \le C$ by [9], where C is not depended on w. Since that $f_w(\varphi(w)) = 0$ and $f'_w(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^\alpha}$, it follows that

$$\begin{aligned}
&\frac{(1-|w|^2)\log\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)^\alpha}|u(w)\varphi'(w)| \\
&= (1-|w|^2)\log(\frac{2}{1-|w|})|u(w)f'_w(\varphi(w))\varphi'(w)| \\
&\le \|uC_\varphi f_w\|_L \le \|uC_\varphi\|\|f_w\|_{F(p,q,s)} \le C\|uC_\varphi\| < +\infty,
\end{aligned}$$

which showing that (1) is necessary for all case.

(i) Let $\alpha = \frac{q+2}{p} > 1$. Fix $w \in D$, we take again the test function

$$g_w(z) = \frac{(\alpha+1)(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^\alpha} - \frac{\alpha(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\alpha+1}}. \tag{5}$$

Then we can easily prove that $\|g_w\|_{F(p,q,s)} \le C$ by using the same methods in [9], where C is not depended on w. Since $g'_w(\varphi(w)) = 0$ and $g_w(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\alpha-1}}$, it follows that

$$\begin{aligned}
&(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)g_w(\varphi(w))| \\
&= (1-|w|^2)\log\frac{2}{1-|w|}|(uC_\varphi g_w)'(w)| \\
&\le \|uC_\varphi g_w\|_L \le \|uC_\varphi\|\|g_w\|_{F(p,q,s)} \le C\|uC_\varphi\| < +\infty.
\end{aligned}$$

So,

$$\sup_{w\in D}\frac{(1-|w|^2)\log\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)^{\alpha-1}}|u'(w)| < +\infty.$$

This proves that (2) is also necessary.

Conversely, suppose that u and φ satisfy the condition in (i). For arbitrary $f \in F(p,q,s)$, by Lemma 2.1 and 2.2 we have

$$\sup_{z\in D}(1-|z|^2)^\alpha|f'(z)| \le C\|f\|_{F(p,q,s)}$$

and $\sup_{z\in D}(1-|z|^2)^{\alpha-1}|f(z)| \le C\|f\|_{F(p,q,s)}$. It follows that

$$\begin{aligned}
&\|uC_\varphi f\|_{\beta_L}\\
\le\ & \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|u'(z)f(\varphi(z))|\\
&+\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|u(z)||f'(\varphi(z))\varphi'(z)|\\
=\ & \sup_{z\in D}(1-|\varphi(z)|^2)^{\alpha-1}|f(\varphi(z))|\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)|\\
&+\sup_{z\in D}(1-|\varphi(z)|^2)^{\alpha}|f'(\varphi(z))|\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)|\\
\le\ & C\|f\|_{F(p,q,s)}
\end{aligned}$$

and by Lemma 2.4,

$$|u(0)f(\varphi(0))| \le \frac{C|u(0)|}{(\alpha-1)(1-|\varphi(0)|)^{\alpha-1}}\|f\|_{\beta_\alpha} \le C\|f\|_{F(p,q,s)}.$$

Hence uC_φ is bounded from $F(p,q,s)$ to β_L.

Next we will prove (ii). Let $\alpha=\frac{q+2}{p}<1$. Suppose $u\in\beta_L$ and that (1) holds. Assume that $f\in F(p,q,s)$. Then we have $\sup_{z\in D}|f(z)|\le C\|f\|_{F(p,q,s)}$ by Lemma 2.1 and 2.4. It follows that

$$\begin{aligned}
&\|uC_\varphi f\|_{\beta_L}\\
\le\ & \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|u'(z)f(\varphi(z))|\\
&+\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|u(z)||f'(\varphi(z))\varphi'(z)|\\
\le\ & C\|u\|_{\beta_L}\|f\|_{F(p,q,s)}+C\|f\|_{F(p,q,s)}\sup_{z\in D}\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)|\\
\le\ & C\|f\|_{F(p,q,s)}
\end{aligned}$$

and $|u(0)f(\varphi(0))|\le C|u(0)|\|f\|_{F(p,q,s)}$. Hence uC_φ is bounded from $F(p,q,s)$ to β_L.

Conversely, we have proved that (1) holds and $u\in\beta_L$ above.

Finally let $\alpha=\frac{q+2}{p}=1$ and $s>1$, we set

$$h_w(z)=2\log\frac{2}{1-\overline{\varphi(w)}z}-\frac{1}{\log\frac{2}{1-|\varphi(w)|^2}}(\log\frac{2}{1-\overline{\varphi(w)}z})^2 \tag{6}$$

for $z,w\in D$. Then

$$h_w'(z)=\frac{2\overline{\varphi(w)}}{1-\overline{\varphi(w)}z}-2\big(\log\frac{2}{1-\overline{\varphi(w)}z}\big)\frac{\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)}\frac{1}{\log\frac{2}{1-|\varphi(w)|^2}}.$$

By a direct calculation we get $\sup_{w\in D}\|h_w\|_{\beta_1} \le C < +\infty$. Noting $s > 1$, thus we have $h_w \in F(p,q,s)$ and $\sup_{w\in D}\|h_w\|_{F(p,q,s)} \le C < +\infty$ by introduction. Since $h'_w(\varphi(w)) = 0$ and $h_w(\varphi(w)) = \log\frac{2}{1-|\varphi(w)|^2}$, it follows that

$$\begin{aligned}
&\sup_{w\in D}|u'(w)|(1-|w|^2)\log(\frac{2}{1-|w|})\log(\frac{2}{1-|\varphi(w)|^2})\\
=\ &\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)h_w(\varphi(w))|\\
\le\ &\|uC_\varphi h_w\|_L \le \|uC_\varphi\|\|h_w\|_{F(p,q,s)} \le C\|uC_\varphi\| < +\infty,
\end{aligned}$$

which showing that (3) is necessary.

Conversely, suppose (1) and (3) hold. Assume that $f \in F(p,q,s)$. Then $f \in \beta_1$ by Lemma 2.1. From Lemma 2.4, it follows that

$$\begin{aligned}
&\|uC_\varphi f\|_{\beta_L}\\
\le\ &\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|u'(z)f(\varphi(z))|\\
&+\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|u(z)||f'(\varphi(z))\varphi'(z)|\\
\le\ &C\|f\|_{F(p,q,s)}\sup_{z\in D}|u'(z)|(1-|z|^2)\log(\frac{2}{1-|z|})\log(\frac{2}{1-|\varphi(z)|^2})\\
&+\|f\|_{F(p,q,s)}\sup_{z\in D}\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)}|\varphi'(z)u(z)| \le C\|f\|_{F(p,q,s)}
\end{aligned}$$

and

$$|u(0)f(\varphi(0))| \le C|u(0)|\log(\frac{2}{1-|\varphi(0)|^2})\|f\|_{\beta_1} \le C\|f\|_{F(p,q,s)}.$$

Hence uC_φ is bounded from $F(p,q,s)$ to β_L.

(iv) The results have been proved in case (ii) and (iii). □

Corollary 2.1. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$ and φ be an analytic self-map of D. Then C_φ is a bounded operator from $F(p,q,s)$ into β_L if and only if*

$$\sup_{z\in D}\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}|\varphi'(z)| < +\infty.$$

In the formulation of the corollary, we will use the notation $M(X,Y)$ to denote the set of all multipliers of X into Y: $M(X,Y) = \{u : fu \in Y \text{ for all } f \in X\}$. By M_u we denote the operator of multiplication by u: $M_u f = uf, f \in X$.

Corollary 2.2. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$.*

(1) *If $2 + q \ge p$, then $M(F(p,q,s), \beta_L) = \{0\}$.*

(2) *If $2 + q < p$, then $M(F(p,q,s), \beta_L) = \beta_L$.*

Proof. If $2+q<p$, then M_u is bounded from $F(p,q,s)$ to β_L if and only if $u \in \beta_L$ and

$$\sup_{z\in D}(1-|z|^2)^{1-\frac{q+2}{p}} \log\frac{2}{1-|z|}|u(z)| < \infty \tag{7}$$

by Theorem 2.1. However, if $u \in \beta_L$, by [6, Lemma 2.1], we get

$$|u(z)| \le (2+\log(\log\frac{2}{1-|z|}))\|u\|_L.$$

Hence (7) holds, since $\frac{q+2}{p}<1$. □

Theorem 2.2. *Let $0<p,s<\infty$, $-2<q<\infty$, $q+s>-1$, $u\in H(D)$ and φ be an analytic self-map of D.*

(i) *If $2+q>p$, then uC_φ is a compact operator from $F(p,q,s)$ to β_L if and only if u, $u\varphi \in \beta_L$,*

$$\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|u(z)\varphi'(z)| = 0 \tag{8}$$

and

$$\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| = 0, \tag{9}$$

where $\alpha=\frac{q+2}{p}$.

(ii) *If $2+q<p$, then uC_φ is a compact operator from $F(p,q,s)$ to β_L if and only if u, $u\varphi\in\beta_L$ and* (8) *holds.*

(iii) *If $2+q=p$, $s>1$, then uC_φ is a compact operator from $F(p,q,s)$ to β_L if and only if u, $u\varphi \in \beta_L$,* (8) *holds and*

$$\lim_{|\varphi(z)|\to 1^-} |u'(z)|(1-|z|^2)\log(\frac{2}{1-|z|})\log(\frac{2}{1-|\varphi(z)|^2}) = 0. \tag{10}$$

Proof. Suppose that uC_φ is compact from $F(p,q,s)$ to β_L. It is easily obtained that u, $u\varphi\in\beta_L$ by taking $f(z)=1$ and $f(z)=z$ respectively. Let $\alpha=\frac{q+2}{p}$ and $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)|\to 1^-$ as $n\to\infty$. We take the test functions

$$f_n(z) = \frac{(z-\varphi(z_n))(1-|\varphi(z_n)|^2)}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}}.$$

Then $f_n\in F(p,q,s)$ and $\sup_n \|f_n\|_{F(p,q,s)} \le C<\infty$ by using the same methods in [9]. Then $\{f_n\}$ is a bounded sequence in $F(p,q,s)$ which converges to 0 uniformly on compact subsets of D. Note that $f_n(\varphi(z_n))=0$ and $f_n'(\varphi(z_n))=$

$\frac{1}{(1-|\varphi(z_n)|^2)^\alpha}$, it follows that

$$\begin{aligned}\|uC_\varphi f_n\|_L &\geq \|uC_\varphi f_n\|_{\beta_L}\\ &\geq (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|u'(z_n)f_n(\varphi(z_n))+u(z_n)f_n'(\varphi(z_n))\varphi'(z_n)|\\ &= \frac{(1-|z_n|^2)\log\frac{2}{1-|z_n|}}{(1-|\varphi(z_n)|^2)^\alpha}|u(z_n)\varphi'(z_n)|.\end{aligned}$$

Then, by Lemma 2.3, (8) holds for all case.

(i) **Case $\alpha = \frac{q+2}{p} > 1$.** We consider the functions g_n defined by

$$g_n(z) = \frac{(\alpha+1)(1-|\varphi(z_n)|^2)}{(1-\overline{\varphi(z_n)}z)^\alpha} - \frac{\alpha(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}}. \tag{11}$$

Then $g_n \in F(p,q,s)$ and $\sup_n \|g_n\|_{F(p,q,s)} \leq C < \infty$, so $\{g_n\}$ is a bounded sequence in $F(p,q,s)$ which converges to 0 uniformly on compact subsets of D. Since $g_n'(\varphi(z_n)) = 0$ and $g_n(\varphi(z_n)) = \frac{1}{(1-|\varphi(z_n)|^2)^{\alpha-1}}$, it follows that

$$\begin{aligned}\|uC_\varphi g_n\|_L &\geq \|uC_\varphi g_n\|_{\beta_L}\\ &\geq (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|u'(z_n)g_n(\varphi(z_n))+u(z_n)g_n'(\varphi(z_n))\varphi'(z_n)|\\ &= (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|u'(z_n)|\frac{1}{(1-|\varphi(z_n)|^2)^{\alpha-1}}.\end{aligned}$$

Then (9) holds by Lemma 2.3.

(ii) **Case $\alpha = \frac{q+2}{p} < 1$.** The necessity in condition (ii) has been proved above.

(iii) **Case $\alpha = \frac{q+2}{p} = 1$ and $s > 1$.** We take the other test functions

$$h_n(z) = \frac{3}{a_n}(\log\frac{2}{1-\overline{\varphi(z_n)}z})^2 - \frac{2}{a_n^2}(\log\frac{2}{1-\overline{\varphi(z_n)}z})^3,$$

where $a_n = \log\frac{2}{1-|\varphi(z_n)|^2}$. Clearly $h_n(z) \to 0$ uniformly on compact subsets of D. By a direct calculation we get $\sup_n \|h_n\|_{\beta_1} \leq C < +\infty$. Noting $s > 1$, thus we have $h_n \in F(p,q,s)$ and $\sup_n \|h_n\|_{F(p,q,s)} < \infty$ by introduction. Then $\{h_n\}$ is a bounded sequence in $F(p,q,s)$ which converges to 0 uniformly on compact subsets of D. Note that $h_n'(\varphi(z_n)) \equiv 0$ and $h_n(\varphi(z_n)) = a_n$, it follows that

$$\begin{aligned}\|uC_\varphi h_n\|_L &\geq \|uC_\varphi h_n\|_{\beta_L}\\ &\geq (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|u'(z_n)h_n(\varphi(z_n))+u(z_n)h_n'(\varphi(z_n))\varphi'(z_n)|\\ &= (1-|z_n|^2)\log(\frac{2}{1-|z_n|})\log\frac{2}{1-|\varphi(z_n)|^2}|u'(z_n)|.\end{aligned}$$

Then (10) holds by Lemma 2.3.

Now we prove the sufficient conditions.

Case $\alpha = \frac{2+q}{p} > 1$. Assume that (8) and (9) hold and $u, u\varphi \in \beta_L$. Then it is not difficult to obtain that (1), (2) and

$$\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|u(z)\varphi'(z)| < \infty \tag{12}$$

hold. Let $\{f_n\}$ be a bounded sequence in $F(p,q,s)$ which converges to 0 uniformly on compact subsets of D. Let $M = \sup_n \|f_n\|_{F(p,q,s)} < +\infty$. We only prove $\lim_{n\to\infty}\|uC_\varphi(f_n)\|_L = 0$ by Lemma 2.3. This amounts to showing that both

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u(w)f_n'(\varphi(w))\varphi'(w)| \longrightarrow 0$$

and

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))| \longrightarrow 0$$

as $n\to\infty$.

If $|\varphi(w)| \le r < 1$, by (12), then

$$(1-|w|^2)\log(\frac{2}{1-|w|})|u(w)f_n'(\varphi(w))\varphi'(w)| \le C\max_{|z|\le r}|f_n'(z)|.$$

If $|\varphi(w)| > r$, then by Lemma 2.1 we get

$$\begin{aligned}&(1-|w|^2)\log(\frac{2}{1-|w|})|u(w)f_n'(\varphi(w))\varphi'(w)|\\ \le\ & CM\frac{(1-|w|^2)\log(\frac{2}{1-|w|})}{(1-|\varphi(w)|^2)^\alpha}|\varphi'(w)u(w)|.\end{aligned}$$

Thus

$$\begin{aligned}&\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u(w)f_n'(\varphi(w))\varphi'(w)|\\ \le\ & C\max_{|w|\le r}|f_n'(w)| + CM\sup_{|\varphi(w)|>r}\frac{(1-|w|^2)\log(\frac{2}{1-|w|})}{(1-|\varphi(w)|^2)^\alpha}|\varphi'(w)u(w)|.\end{aligned}$$

First letting n tend to infinity and subsequently r increase to 1, one obtains that

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u(w)f_n'(\varphi(w))\varphi'(w)| \longrightarrow 0$$

as $n\to\infty$. The other statement is proved similarly.

If $|\varphi(w)| \le r < 1$, by $u \in \beta_L$, then

$$(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))| \le \|u\|_L\max_{|z|\le r}|f_n(z)|.$$

If $|\varphi(w)| > r$, by Lemma 2.1 and 2.2, then

$$(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))| \le CM\frac{(1-|w|^2)\log(\frac{2}{1-|w|})}{(1-\varphi(w)|^2)^{\alpha-1}}|u'(w)|.$$

Thus

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))|$$
$$\leq \|u\|_L \max_{|w|\leq r}|f_n(w)| + CM \sup_{|\varphi(w)|>r}\frac{(1-|w|^2)\log(\frac{2}{1-|w|})}{(1-|\varphi(w)|^2)^{\alpha-1}}|u'(w)|,$$

which also implies that

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))| \longrightarrow 0$$

as $n \to \infty$.

Case $\alpha = \frac{2+q}{p} < 1$. Similarly, let $\{f_n\}$ be a bounded sequence in $F(p,q,s)$ which converges to 0 uniformly on compact subsets of D. By Lemma 2.1 and [10, Lemma 3.2] we have $\sup_{z\in D}|f_n(z)| \longrightarrow 0$ as $n \to \infty$. Then

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))| \leq \|u\|_L \sup_{z\in D}|f_n(z)| \longrightarrow 0$$

as $n \to \infty$. As the same proof as in case $\alpha = \frac{2+q}{p} > 1$, we get

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u(w)f_n'(\varphi(w))\varphi'(w)| \longrightarrow 0.$$

Hence uC_φ is a compact operator from $F(p,q,s)$ to β_L.

Case $\alpha = \frac{2+q}{p} = 1$ and $s > 1$. Let $\{f_n\}$ be a bounded sequence in $F(p,q,s)$ which converges to 0 uniformly on compact subsets of D. Let $M = \sup_n \|f_n\|_{F(p,q,s)} < +\infty$. By Lemma 2.1 and 2.4, we get

$$\sup_n |f_n(z)| \leq CM \log\frac{2}{1-|z|^2}. \tag{13}$$

If $|\varphi(w)| \leq r < 1$, by $u \in \beta_L$, then

$$(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))| \leq \|u\|_L \max_{|z|\leq r}|f_n(z)|.$$

If $|\varphi(w)| > r$, by (13), we have

$$(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))|$$
$$\leq CM(1-|w|^2)\log(\frac{2}{1-|w|})\log(\frac{2}{1-|\varphi(w)|^2})|u'(w)|.$$

Thus

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))|$$
$$\leq \|u\|_L \max_{|w|\leq r}|f_n(w)| + CM(1-|w|^2)\log(\frac{2}{1-|w|})\log(\frac{2}{1-|\varphi(w)|^2})|u'(w)|,$$

which also implies that

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u'(w)f_n(\varphi(w))| \longrightarrow 0$$

as $n\to\infty$. We also similarly prove that

$$\sup_{w\in D}(1-|w|^2)\log(\frac{2}{1-|w|})|u(w)f_n'(\varphi(w))\varphi'(w)| \longrightarrow 0$$

as $n\to\infty$. Hence $\lim_{n\to\infty}\|uC_\varphi(f_n)\|_L = 0$. This completes the proof of Theorem 2.2. □

3. Boundedness and compactness of $uC_\varphi : F_0(p,q,s)\to\beta_L^0$

In this section we characterize the boundedness and compactness of the weighted composition operator $uC_\varphi : F_0(p,q,s)\to\beta_L^0$. For this purpose we need another lemmas.

Lemma 3.1. *Let $0<s<\infty$, $1\le p<\infty$, $-2<q<\infty$ and $q+s>-1$. For $0<t<1$, $z\in D$, let $f_t(z)=f(tz)$. If $f\in F(p,q,s)$, then $f_t\in F(p,q,s)$ and $\|f_t\|_{F(p,q,s)}\le\|f\|_{F(p,q,s)}$.*

Proof. For $f\in F(p,q,s)$ and $0<t<1$, by the Poisson formula, we have

$$f_t(z)=\int_0^{2\pi} f(ze^{i\theta})\frac{1-t^2}{|e^{i\theta}-t|^2}\frac{d\theta}{2\pi},\qquad z\in D.$$

Then by Hölder's inequality we get,

$$\begin{aligned}
&\int_D |f_t'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s\,dm(z)\\
&\le \int_D\int_0^{2\pi}|f'(e^{i\theta}z)|^p\frac{1-t^2}{|e^{i\theta}-t|^2}\frac{d\theta}{2\pi}(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s\,dm(z)\\
&=\int_0^{2\pi}\int_D|f'(e^{i\theta}z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s\,dm(z)\frac{1-t^2}{|e^{i\theta}-t|^2}\frac{d\theta}{2\pi}\\
&\le \|f\|_{p,q,s}^p\int_0^{2\pi}\frac{1-t^2}{|e^{i\theta}-t|^2}\frac{d\theta}{2\pi}=\|f\|_{p,q,s}^p,
\end{aligned}$$

where $\|f\|_{p,q,s}^p=\sup_{a\in D}\int_D|f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s\,dm(z)$. That is,

$$\|f_t\|_{F(p,q,s)}\le\|f\|_{F(p,q,s)}.$$

□

Lemma 3.2. *Let u be an analytic function on the unit disk D and φ an analytic self-map of D. Let $0<s<\infty$, $1\le p<\infty$, $-2<q<\infty$ and $q+s>-1$. If uC_φ is a bounded operator from $F_0(p,q,s)$ into β_L^0, then uC_φ is bounded from $F(p,q,s)$ to β_L.*

Proof. Suppose uC_φ is bounded from $F_0(p,q,s)$ into β_L^0 and for $f \in F(p,q,s)$, we have $f_t \in F_0(p,q,s)$ for all $0 < t < 1$. Then, according to Lemma 3.1,

$$\|uC_\varphi(f_t)\|_{F(p,q,s)} \leq \|uC_\varphi\|\|f_t\|_{F(p,q,s)} \leq \|uC_\varphi\|\|f\|_{F(p,q,s)} < +\infty.$$

Hence $\|uC_\varphi(f)\|_{F(p,q,s)} \leq \|uC_\varphi\|\|f\|_{F(p,q,s)} < +\infty$, which shows uC_φ is bounded from $F(p,q,s)$ to β_L. □

Theorem 3.1. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Let $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$. Then $uC_\varphi : F_0(p,q,s) \longrightarrow \beta_L^0$ is a bounded operator provided that $u \in \beta_L^0$, $uC_\varphi : F(p,q,s) \longrightarrow \beta_L$ is bounded and*

$$\lim_{|z|\to 1^-}(1-|z|^2)\log\frac{2}{1-|z|}|u(z)\varphi'(z)| = 0. \tag{14}$$

Conversely, if $uC_\varphi : F_0(p,q,s) \longrightarrow \beta_L^0$ is a bounded operator, then $u \in \beta_L^0$ and (14) holds; moreover, if $1 \leq p < \infty$, then $uC_\varphi : F(p,q,s) \longrightarrow \beta_L$ is bounded.

Proof. Assume that $u \in \beta_L^0$ and $\lim_{|z|\to 1^-}(1-|z|^2)\log\frac{2}{1-|z|}|u(z)\varphi'(z)| = 0$. Then, for each polynomial $p(z)$, we have that

$$\begin{aligned}&(1-|z|^2)\log\frac{2}{1-|z|}|(uC_\varphi p)'(z)|\\ &\leq (1-|z|^2)\log\frac{2}{1-|z|}|u'(z)||p(\varphi(z))| + (1-|z|^2)\log\frac{2}{1-|z|}|u(z)\varphi'(z)p'(\varphi(z))|,\end{aligned}$$

from which it follows that $uC_\varphi p \in \beta_L^0$. Since the set of all polynomials is dense in $F_0(p,q,s)$, we have that for every $f \in F_0(p,q,s)$ there is a sequence of polynomials $\{p_n\}$ such that $\|f - p_n\|_{F(p,q,s)} \to 0$ as $n \to \infty$. Note that $uC_\varphi : F(p,q,s) \longrightarrow \beta_L$ is bounded, then

$$\|uC_\varphi f - uC_\varphi p_n\|_L \leq \|uC_\varphi\|\|f - p_n\|_{F(p,q,s)} \longrightarrow 0$$

as $n \to \infty$. Since β_L^0 is closed in subset of β_L, we obtain

$$uC_\varphi(F_0(p,q,q)) \subset \beta_L^0.$$

So $uC_\varphi : F_0(p,q,s) \longrightarrow \beta_L^0$ is bounded.

Conversely, assume that $uC_\varphi : F_0(p,q,s) \longrightarrow \beta_L^0$ is bounded. Taking the functions $f(z) = 1$ and $f(z) = z$ respectively, we obtain that $u \in \beta_L^0$ and $\lim_{|z|\to 1^-}(1-|z|^2)\log\frac{2}{1-|z|}|u(z)\varphi'(z)| = 0$. If $p \geq 1$, then $uC_\varphi : F(p,q,s) \longrightarrow \beta_L$ is bounded by Lemma 3.2. □

Lemma 3.3. *Let $U \subset \beta_L^0$. Then U is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1^-}\sup_{f\in U}(1-|z|^2)\log(\frac{2}{1-|z|})|f'(z)| = 0.$$

The proof is similar to [3, Lemma 1], so we omitted it.

Theorem 3.2. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $u \in H(D)$ and φ be an analytic self-map of D.*

(i) *If $2+q > p$, then uC_φ is a compact operator from $F_0(p,q,s)$ to β_L^0 if and only if*

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^\alpha}|u(z)\varphi'(z)| = 0 \tag{15}$$

and

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| = 0,$$

where $\alpha = \frac{q+2}{p}$.

(ii) *If $2+q < p$, then uC_φ is a compact operator from $F_0(p,q,s)$ to β_L^0 if and only if $u \in \beta_L^0$ and* (15) *holds.*

(iii) *If $2+q = p$, $s > 1$, then uC_φ is a compact operator from $F_0(p,q,s)$ to β_L^0 if and only if* (15) *holds and*

$$\lim_{|z|\to 1^-} |u'(z)|(1-|z|^2)\log\Big(\frac{2}{1-|z|}\Big)\log\Big(\frac{2}{1-|\varphi(z)|^2}\Big) = 0.$$

Proof. First we consider the case that $\alpha = \frac{q+2}{p} > 1$. Suppose that u and φ satisfy the condition in (i). By Theorem 2.1 and Theorem 3.1, we know that uC_φ is bounded from $F_0(p,q,s)$ to β_L^0. Suppose that $f \in F_0(p,q,s)$ is such that $\|f\|_{F(p,q,s)} \le 1$. By Lemma 2.1, we know that $f \in \beta_\alpha^0$ and $\|f\|_{\beta_\alpha} \le C\|f\|_{F(p,q,s)}$. Then from Lemma 2.2 it follows that

$$\begin{aligned}&(1-|z|^2)\log\frac{2}{1-|z|}|(uC_\varphi f)'(z)|\\ &\le C\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| + C\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)|,\end{aligned}$$

thus

$$\begin{aligned}&\sup\{(1-|z|^2)\log\frac{2}{1-|z|}|(uC_\varphi f)'(z)| : f \in F_0(p,q,s), \|f\|_{F(p,q,s)} \le 1\}\\ &\le C\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| + C\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)|,\end{aligned}$$

and it follows that

$$\lim_{|z|\to 1^-} \sup\{(1-|z|^2)\log\frac{2}{1-|z|}|(uC_\varphi f)'(z)| : f \in F_0(p,q,s), \|f\|_{F(p,q,s)} \le 1\} = 0,$$

so that uC_φ is compact from β_α^0 to β_L^0 by Lemma 3.3.

Conversely, suppose that uC_φ is compact from $F_0(p,q,s)$ to β_L^0. By Lemma 3.3, we have

$$\lim_{|z|\to 1^-} \sup\{(1-|z|^2)\log\frac{2}{1-|z|}|(uC_\varphi f)'(z)| : f \in F_0(p,q,s), \|f\|_{F(p,q,s)} \le M\} = 0$$

for some $M > 0$. Note that the proof of Theorem 2.1 and the fact that the functions given in (4) are in $F_0(p,q,s)$ and have norms bounded uniformly in w, we get

$$\lim_{|w|\to 1^-} \frac{(1-|w|^2)\log\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)^\alpha}|u(w)\varphi'(w)| = 0.$$

Similarly, note that the functions given in (5) are in $F_0(p,q,s)$ and have norms bounded uniformly in w, we get

$$\lim_{|w|\to 1^-} \frac{(1-|w|^2)\log\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)^{\alpha-1}}|u'(w)| = 0.$$

This completes the proof of (i).

Next, we consider the case $\alpha = \frac{q+2}{p} < 1$. Assume that $u \in \beta_L^0$ and (14) holds. Suppose that $f \in F_0(p,q,s)$ is such that $\|f\|_{F(p,q,s)} \le 1$. By Lemma 2.1 and 2.4, we get

$$\begin{aligned} &(1-|z|^2)\log\tfrac{2}{1-|z|}|(uC_\varphi f)'(z)| \\ \le\ & C(1-|z|^2)\log\tfrac{2}{1-|z|}|u'(z)| + C\tfrac{(1-|z|^2)\log\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)u(z)|, \end{aligned}$$

thus

$$\lim_{|z|\to 1^-} \sup\{(1-|z|^2)\log\frac{2}{1-|z|}|(uC_\varphi f)'(z)| : f \in F_0(p,q,s), \|f\|_{F(p,q,s)} \le 1\} = 0.$$

So that uC_φ is compact from $F_0(p,q,s)$ to β_L^0 by Lemma 3.3. The fact that the conditions in (ii) are necessary for compactness of operator uC_φ as an operator from $F_0(p,q,s)$ to β_L^0 is proved in Theorem 3.1 and the case that $\alpha > 1$.

Finally, we will prove (iii). Similarly, suppose that u and φ satisfy the condition in (iii). By Theorem 2.2, we know that uC_φ is bounded from $F_0(p,q,s)$ to β_L^0. Suppose that $f \in F_0(p,q,s)$ is such that $\|f\|_{F_0(p,q,s)} \le 1$. By Lemma 2.1 and 2.4, we get

$$\begin{aligned} &(1-|z|^2)\log\frac{2}{1-|z|}|(uC_\varphi f)'(z)| \\ &\le C(1-|z|^2)\log\frac{2}{1-|z|}\log\frac{2}{1-|\varphi(z)|^2}|u'(z)| \\ &\quad + C\frac{(1-|z|^2)\log\frac{2}{1-|z|}}{1-|\varphi(z)|^2}|\varphi'(z)u(z)|, \end{aligned}$$

thus

$$\lim_{|z|\to 1^-} \sup\{(1-|z|^2)\log\frac{2}{1-|z|}|(uC_\varphi f)'(z)| : f \in F_0(p,q,s), \|f\|_{F_0(p,q,s)} \le 1\} = 0,$$

so that uC_φ is compact from $F_0(p,q,s)$ to β_L^0 by Lemma 3.3.

Conversely, noting that the functions given in (6) are in $F_0(p,q,s)$ and have norms bounded uniformly in w, we get

$$\lim_{|w|\to 1^-} |u'(w)|(1-|w|^2)\log(\frac{2}{1-|w|})\log(\frac{2}{1-|\varphi(w)|^2}) = 0.$$

The another necessity in condition (iii) is similarly proved in (i). This completes the proof of Theorem 3.2 □

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