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IDEAL CELL-DECOMPOSITIONS FOR A HYPERBOLIC SURFACE AND EULER CHARACTERISTIC

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ABSTRACT. In this article, we constructively prove that on a surface S with genus $g \geq 2$, there exit maximal geodesic laminations with $7g - 7, \ldots, 9g - 9$ leaves. Thus, S can have *ideal cell-decompositions* (i.e., S can be (ideally) triangulated by maximal geodesic laminations) with $7g - 7, \ldots, 9g - 9$ (ideal) 1-cells.

Once there is a triangulation for a compact surface, the Euler characteristic for the surface can be calculated as the alternating sum $F - E + V$, where F, E, and V denote the number of faces, edges, and vertices, respectively. We also prove that the same formula holds for the ideal cell-decompositions.

Introduction

Let S be a compact Riemann surface with genus at least 2. By the Uniformization Theorem, the only interesting Riemannian structure is hyperbolic metrics; that is, Riemannian metrics with constant Gaussian curvature -1. The upper-half plane with the hyperbolic metric can be considered as the universal covering of such surfaces. The geodesics in this hyperbolic plane model are either vertical lines perpendicular to the x-axis or the semicircles in the upper half-plane with centers on the x-axis [1].

Geodesic laminations are generalizations of deformation classes of simple closed curves on S. More precisely, a geodesic lamination λ on the surface S is by definition a closed subset of S which can be decomposed into family of disjoint simple geodesics, possibly infinite, called its *leaves* [2, 4, 5, 6].

Geodesic laminations are fundamental tools in low-dimensional topology and geometry. An important property of geodesic laminations is that they are in fact topological object, and independent of the hyperbolic metrics which we put on the surface. In other words, for any two hyperbolic metrics m_1, m_2 in the same isotopy class, the set of m_1-geodesic laminations on S can naturally be identified with that of m_2-geodesic laminations on S [2].

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A geodesic lamination is *maximal* if it is maximal with respect to inclusion; this is equivalent to the property that the complement $S - \lambda$ is union of finitely many triangles with vertices at infinity [2, 5]. For example, Figure 1d and Figure 1f are maximal geodesic laminations with 7 and 9 leaves for the surface S of genus 2.

The motivation for this study is if genus 2 surface S can have maximal geodesic lamination with 8 leaves. Actually, for genus 2, one can obtain maximal geodesic laminations with only 7, 8, and 9 leaves. Moreover, we realize the simple relation between the Euler characteristic of the surface S and the *ideal cell-decompositions* (or *ideal-triangulation*) of S obtained by these maximal geodesic laminations. By an ideal cell-decomposition of S, we mean a triangulation of the surface obtained from a maximal geodesic lamination, where the (*ideal*) vertices are ∞'s and the edges are the closed leaves and the infinite leaves of the lamination spiralling toward these ∞'s , and the faces of the finitely many ideal triangles forming $S - \lambda$. For example, in Figure 1f, there are 3 zero-cells, 9 one-cells, i.e., leaves, three of which are closed, and 4 two-cells, i.e., ideal triangles.

We, furthermore, prove that the same type relations hold for genus $g \geq 3$.

This paper explains the above discussion. The main result of this paper is:

Theorem 0.0.1. *Let S be a compact hyperbolic surface of genus g without boundary. Then,*

(1) *There are maximal geodesic laminations on S with $7g-7, \ldots, 9g-9$ leaves. Thus, S has ideal-triangulations with $7g-7, \ldots, 9g-9$ edges.*
(2) *Euler characteristic formula holds for the ideal-triangulations, too.*

□

We will constructively present the proof of the main result Theorem 0.0.1.

The paper consists of two parts. In §1, we will provide the preliminary definitions and results. The proof of Theorem 0.0.1 will be in §2.

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1. Ideal triangulation of a compact surface

In this section, S will denote a compact surface of genus at least 2. By the Uniformization Theorem, the upper-half plane (or Poincare disk) with the hyperbolic metric is the universal covering of such surfaces. Recall that geodesics (or hyperbolic lines) in the hyperbolic plane are either the vertical rays perpendicular to the x-axis or the semi-circles in the upper-half plane that are perpendicular to x-axis [1].

A *geodesic lamination* is a closed subset of S which can be decomposed as a union of disjoint complete geodesics which have no self-intersection points. We refer the reader [2], [4], [5], [6] for more information about the geodesic laminations on surfaces. Such a notion is actually a topological object, independent of the metric, in the sense that there is a natural identification

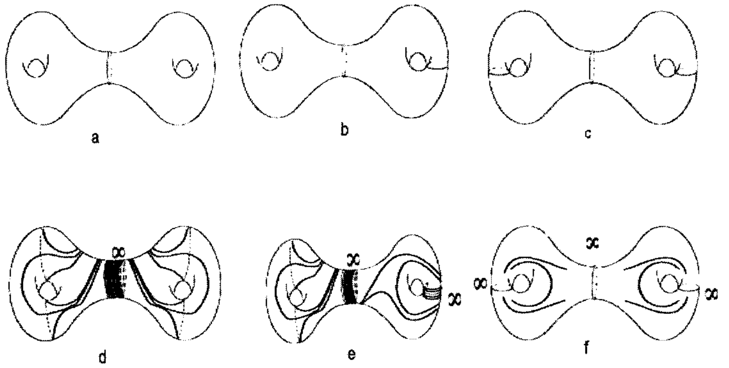

FIGURE 1. Geodesic Laminations on genus 2 surface with 1, 2, and 3 leaves. Maximal geodesic laminations with 7, 8, and 9 leaves.

between m-geodesic laminations and m'-geodesic laminations for any two negatively curved metrics m and m' [2]. For example, a geodesic lamination can consist of three disjoint simple closed geodesics (see Figure 1c) on a Riemann surface S. However, a typical geodesic lamination may have uncountably many leaves, for instance a Cantor set of leaves [7].

A geodesic lamination is *maximal* if it is maximal for inclusion among all geodesic laminations, which is equivalent to the property that the complement $S - \lambda$ consists of finitely many infinite triangles. A fundamental example of a maximal geodesic lamination is obtained as follows. Start with a family λ_1 of disjoint simple closed geodesics decomposing S into pairs of pants. Each pair of pants can be divided into two infinite triangles by two infinite geodesics spiralling around some boundary components. The union of λ_1 and of these spiralling geodesics forms a maximal geodesic lamination λ (see Figure 1f).

An *ideal triangulation* of S is a triangulation of the surface obtained from a maximal geodesic lamination, where the (*ideal*) vertices are ∞'s and the edges are the closed leaves and the infinite leaves of the lamination spiralling toward these ∞'s and the faces are the finitely many ideal triangles forming $S - \lambda$.

For example, in Figure 1f, we have an ideal triangulation of the surface obtained from so-called a *pant-decomposition* of the surface. The edges of the triangulation are the ∞'s where the leaves of the lamination spiral towards. The (*ideal*-)vertices of the triangulation are the leaves of the lamination and the faces are the (finitely many) ideal triangles forming $S - \lambda$. The number of vertices is $V = 3$. There are 9 leaves, 3 of which are closed. Thus, the number of edges is $E = \mathbf{9}$. If we cut along the closed leaves, we will obtain 2 so-called pair-of-pants. There are 2 ideal triangles in each pair-of-pants, one in the front and one on the back (see Figure 2). Hence the number of faces in Figure 1f will be $F = 2 \times 2 = 4 \times 1$. By the Gauss-Bonnet Theorem, the total surface area of a compact surface of genus g without boundary is $-2\pi\chi(S)$, where $\chi(S) = 2-2g$ is the Euler characteristic of S. Since the area of an ideal triangle is π [1], the number of faces in an ideal triangulation is actually fixed $F = 4(g-1)$ [5]. Thus $V - E + F$ is -2 or the Euler characteristic $\chi(S) = 2 - 2g$ of S, where $g = 2$ is the genus.

To clarify the subject, we shall give another example. Consider, for instance, Figure 1d, there is 1 vertex, where the leaves are spiralling towards each sides. There is 1 closed leaf and $3+3=6$ infinite leaves spiralling towards the closed leave from both sides. The number of edges is $3 + 1 + 3 = \mathbf{7}$. If we cut the surface along the closed leaf, we have 2 punctured torus. On each of these punctured torus Figure 4, the infinite leaves of the maximal lamination we started with will give us 2 ideal triangles, one in the front and one in the back of the punctured torus Figure 4, Figure 2. Thus, the number of faces in Figure 1d is again $2\times 2 = 4\times 1$. As above, $V-E+F = 1-7+4 = -2$ or $\chi(S) = 2-2g$.

As third example, consider the Figure 1e, this is a maximal geodesic lamination with **8** leaves. There are 2 closed leaves and $3 + 3$ infinite leaves spiralling towards these closed leaves. Since $V = 2$, $E = \mathbf{8}$, and $F = 4$, thus $V - E + F = -2$ or $\chi(S) = 2 - 2g$.

In all these three examples, the number of two-cells is $F = 4(g - 1)$. The number of zero-cells $V = 1, 2$, and 3, respectively. Since for a genus g surface, $3(g - 1)$ is the maximum number of separating simple closed curves [1], we can not have more that 3 zero-cell in an ideal triangulation on the genus 2 surface. Therefore, the number of one-cells in an ideal triangulation on the genus 2 surface can be at least $7 = 7(g - 1)$ and at most $9 = 9(g - 1)$, i.e., $7(g - 1), \ldots, 9(g - 1)$. Thus, there are 3 ideal cell-decompositions for genus 2 surfaces.

2. Proof of the main result

Let S be a compact surface with genus at least 2 and without boundary. It is well-known that such surfaces (see Figure 3) can be constructed from so-called pair-of-pants, i.e., disk with two small open disks removed (Figure 2a).

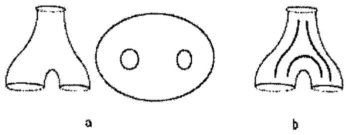

FIGURE 2. Pair-of-Pants and the only maximal geodesic lamination on Pair-of-Pants.

To prove to Theorem 0.0.1, we will use the following.

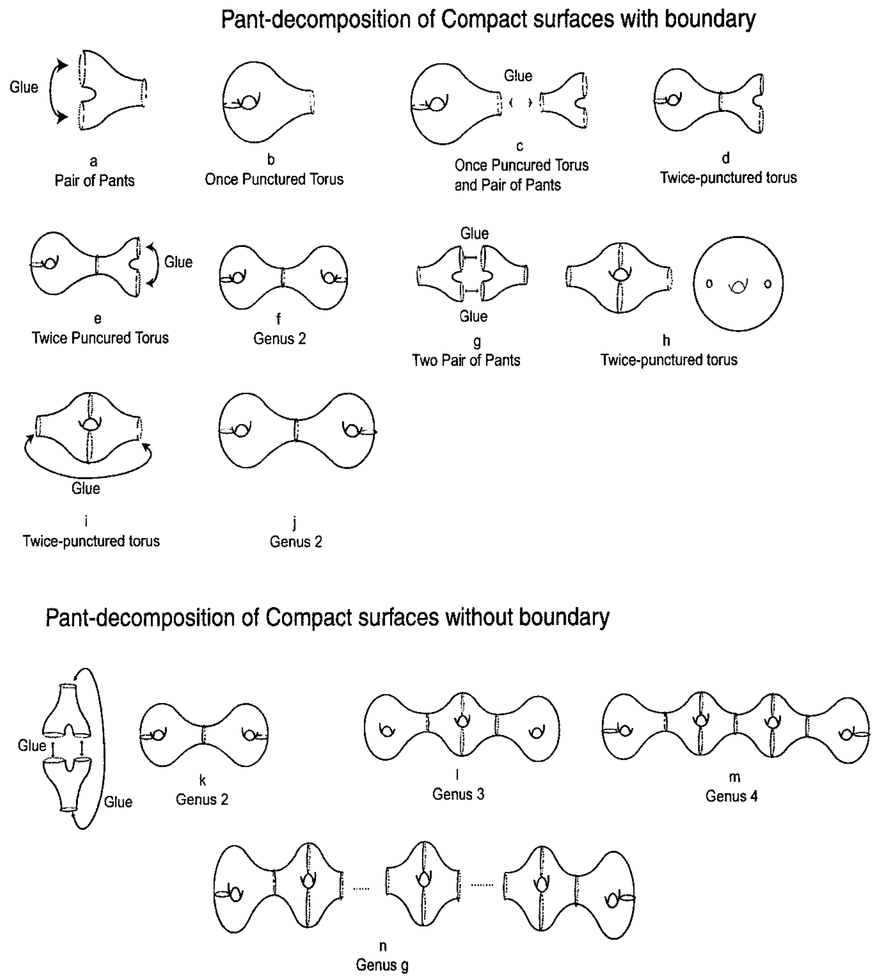

FIGURE 3. Pants Decomposition of Surfaces

Lemma 2.0.2. *Pair-of pants has only one ideal triangulation and the Euler characteristic formula holds.*

Proof. For a pair-of-pants $\mathcal{P}$, there is only one ideal triangulation Figure 2b. There are V=3 vertices and E=3+3 edges, 3 of which are closed curves (the boundary curves) and 3 infinite leaves spiralling towards the boundary curves, and F=2 ideal triangles, one in the front and one in the back of the pair of pants. Thus, $V - E + F$ is -1, i.e., the Euler characteristic $\chi(\mathcal{P})$ of the $\mathcal{P}$. □

We will also use the following lemma in our computations.

Lemma 2.0.3. *Once-punctured torus has only two idea-triangulations and Euler characteristic formula is true for punctured-torus.*

Proof. If we glue two of the boundary curves of $\mathcal{P}$, we obtain a once-punctured torus $\overset{\circ}{T}$. For $\overset{\circ}{T}$, there are only 2 ideal triangulations. See Figure 4. The first one has $V = 1$ (ideal)vertex, $E = 3 + 1 = 4$ edges, one of which is closed, and $F = 2$ faces. Then, $V - E + F = -1 = \chi(\overset{\circ}{T})$. In the second ideal-triangulation, $E = 3 + 2$ edges, 2 of which are closed and the 3 infinite edges spiralling towards the closed leaves. $V = 2$ and there are $F = 2$ ideal triangles in this ideal-triangulation. Note that $V - E + F = -1 = \chi(\overset{\circ}{T})$. □

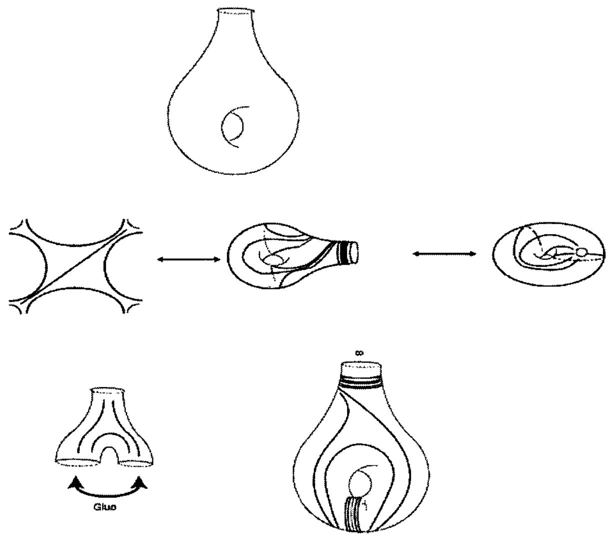

FIGURE 4. Punctured Torus, Maximal Laminations on Punctured Torus

Now, we will go back to the motivational examples and conclude that Theorem 0.0.1 is true for genus 2 surface.

Theorem 2.0.4. *Theorem 0.0.1 is true for genus 2 compact surface without boundary.*

Proof. Figure 1 shows how one can reconstruct genus 2 surface. In Figure 1 d,e,f, we respectively have $V = 1, E = 3 + 1 + 3 = \mathbf{7} = 7(g-1), F = 2 + 2$, and $V - E + F = -2 = \chi(S)$; $V = 2, E = 3 + 1 + 3 + 1 = \mathbf{8}, F = 2 + 2$, and $V - E + F = -2 = \chi(S)$; $V = 3, E = 1 + 3 + 1 + 3 + 1 = \mathbf{9} = 9(g-1), F = 2 + 2$, and $V - E + F = -2 = \chi(S)$. Here, the number of two-cells is fixed $F = 4(g-1)$.

The number of zero-cells is $V = 1, 2$, and 3, respectively. We can not have more than 3 zero-cell in an ideal triangulation on the genus 2 surface since there are at most $3(g-1)$ simple closed separating curves for genus 2 surface [1]. Thus, the number of one-cells in an ideal triangulation on the genus 2 surface can be at least $7 = 7(g-1)$ and at most $9 = 9(g-1)$.

This proves Theorem 0.0.1 is true for genus 2. □

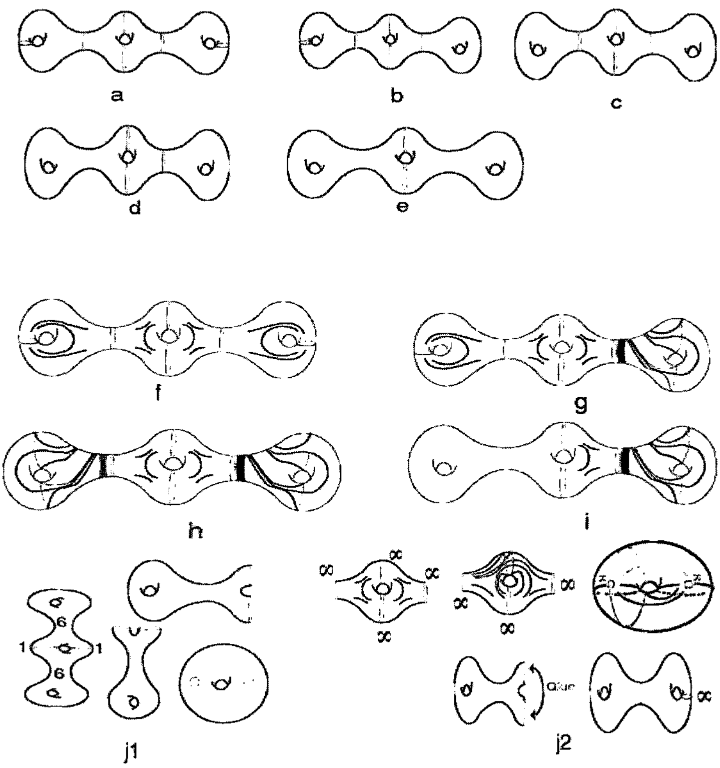

FIGURE 5. Pant-decomposition for genus 3

Lemma 2.0.5. *Twice-punctured torus has three ideal-triangulations and the Euler characteristic formula holds.*

Proof. There are three ideal triangulations for twice punctured torus (Figure 5f and Figure 3g,h). Consider two pair-of-pants $\mathcal{P}_1$ and $\mathcal{P}_2$ (see Figure 5f and Figure 3g,h). If we glue two boundary curves of $\mathcal{P}_1, \mathcal{P}_2$, we have twice-punctured torus $\overset{\circ\circ}{T}$. There are three ideal triangulations see Figure 5f. In the first one, we have $V = 4$, $E = 1+3+1+1+3+1 = 10$, and $F = 2+2$, so $V - E + F = -2 = \chi(\overset{\circ\circ}{T})$. In the second ideal-triangulation, there are $V = 3$

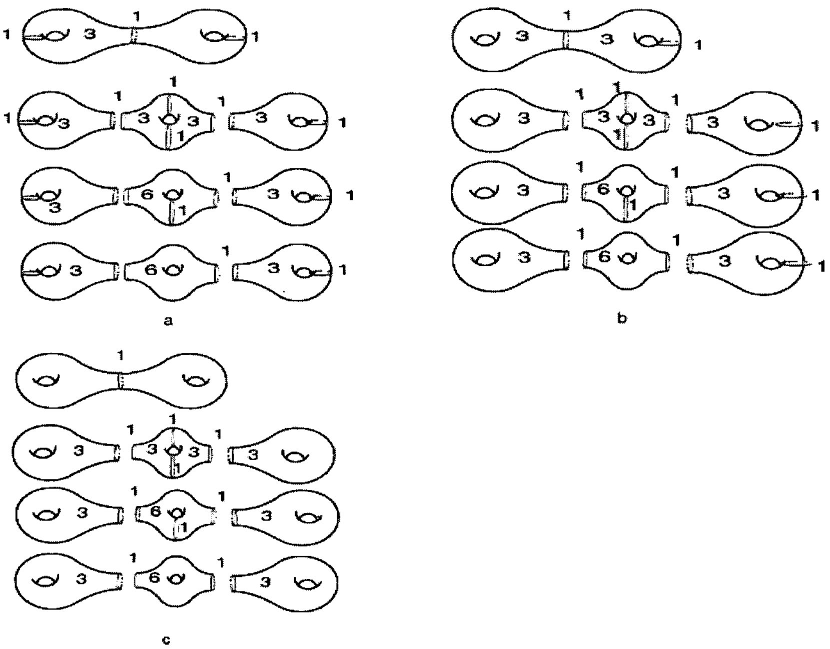

FIGURE 6. Ideal Triangulations for genus 3

(ideal) vertices, $E = 1+6+1+1 = 9$ edges and $F = 2+2$ faces. Thus, $V-E+F$ is -2 or $\chi(\overset{\circ\circ}{T})$. In the third ideal triangulation, $V = 2$, $E = 1+6+1 = 8$, $F = 2+2$, and $V - E + F = -2 = \chi(\overset{\circ\circ}{T})$. □

Theorem 2.0.6. *Theorem 0.0.1 is true for genus 3.*

Proof. One can obtain genus 3 surface as follows: 2 twice-punctured tori; 1 twice-punctured torus, 1 pair-of-pants,and 1 once-punctured torus; 2 once-punctured tori, 2 pair-of-pants; 3 pair-of-pants, 1 once-punctured torus; 4 pair-of-pants Figure 5a-e. In the ideal triangulation Figure 5j1, we have $V = 2, E = 6 + 2 + 6 = \mathbf{14} = \mathbf{7(g-1)}, F = 4 + 4 = 8$, so $V - E + F = \mathbf{-4} = \chi(S)$. In the second ideal-triangulation Figure 5i, $V = 3, E = 6 + 2 + 3 + 1 + 3 = \mathbf{15} = \mathbf{7(g-1)+1}, F = 4+2+2 = -2\chi(S)$, and so $V-E+F = -4 = \chi(S)$. The ideal-triangulation Figure 5h, $V = 4, E = 3+1+3+2+3+1+3 = \mathbf{16} = \mathbf{7(g-1)+2}$, and $F = 2 + 2 + 2 + 2 = 8 = -2\chi(S)$ hence $V - E + F = -4 = \chi(S)$. In the ideal-triangulation Figure 5g, we have $V = 5, E = 1 + 3 + 1 + 3 + 2 + 3 + 1 + 3 = \mathbf{17} = \mathbf{7(g-1)+3}$, and $F = 2 + 2 + 2 + 2 = 8 = -2\chi(S)$, again $V - E + F = -4 = \chi(S)$. The ideal-triangulation Figure 5f has $V = 6$,

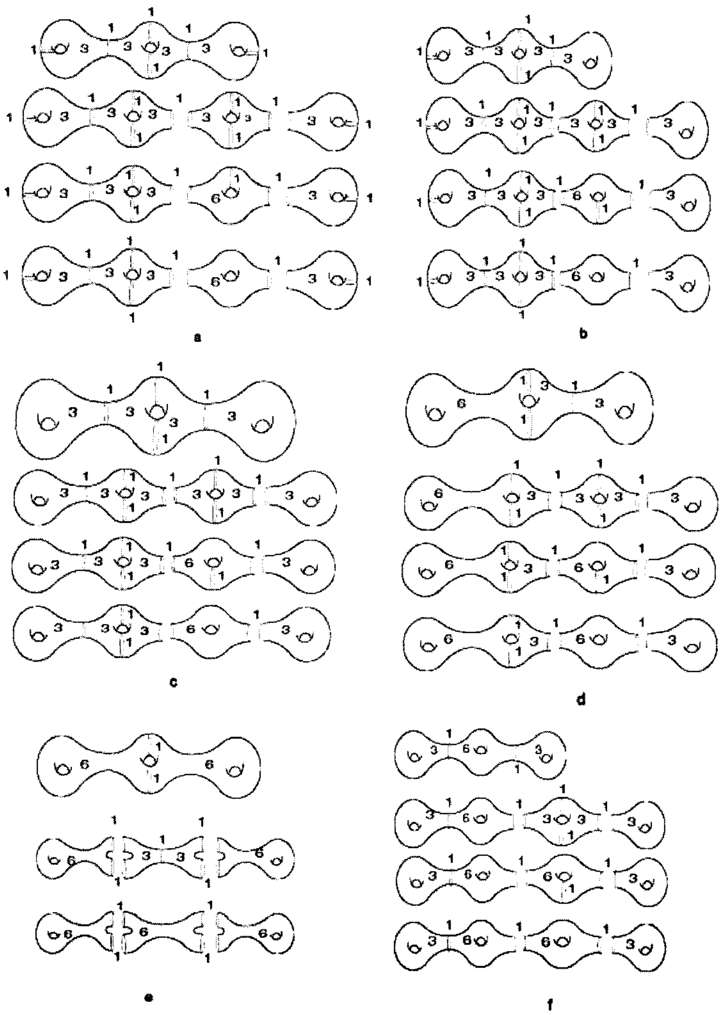

FIGURE 7. Ideal Triangulations for genus 4

$E = 1+3+1+3+2+3+1+3+1 = \mathbf{18} = \mathbf{7(g-1)+4} = 9(g-1)$ and $F = 2+2+2+2 = 8 = -2\chi(S)$ so $V - E + F = -4 = \chi(S)$. □

For the proof of Theorem 0.0.1 for genus 4, consider the Figure 7.

Theorem 2.0.7. *Theorem 0.0.1 is true for genus 4.*

Proof. The ideal triangulations in Figure 7a have respectively $V = 9, 8, 7$; $E = \mathbf{27}, 26, 25$; and $F = 12, 12, 12$. Thus, $V - E + F$ respectively is $9 - 27 + 12, 8 - 26 + 12, 7 - 25 + 12 = -6$ or $-\chi(S)$. In the ideal triangulations Figure 7b,

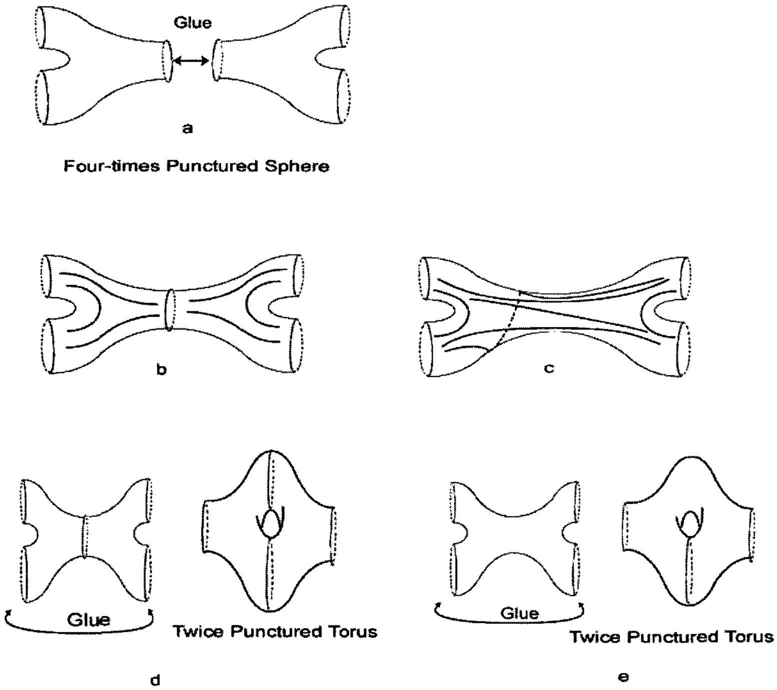

FIGURE 8. Pants To 4-Times Punctured Sphere

$V = 8, 7, 6$; $E = 26, 25, 24$; and $F = 12, 12, 12$. Hence, $V - E + F = \chi(S)$. Since the ideal triangulations in Figure 7c have respectively $V = 7, 6, 5$; $E = 25, 24, 23$; and $F = 12, 12, 12$, $V - E + F = \chi(S)$. In Figure 7d, the ideal triangulations have respectively $V = 6, 5, 4$; $E = 24, 23, 22$; $F = 12, 12, 12$ and thus $V - E + F = \chi(S)$. Ideal triangulations in Figure 7e respectively $V = 5, 4$; $E = 23, 22$; $F = 12, 12$ and hence $V - E + F = \chi(S)$. Finally, in Figure 7f, the ideal triangulations have respectively $V = 5, 4, 3$; $E = 23, 22, \mathbf{21}$; $F = 12, 12, 12$ and hence $V - E + F = \chi(S)$.

This finishes the proof of Theorem 0.0.1 for genus 4. □

We will finish by the proof of Theorem 0.0.1 genus ≥ 2.

Theorem 2.0.8. *Theorem 0.0.1 is true for genus* ≥ 2.

Proof. We can obtain the result for any genus $g \geq 2$ by starting with the maximal geodesic lamination corresponding to pant-decomposition of the surface Figure 3 and then delete the closed curves one at a time till we have $g - 1$ vertices. So, the number of vertices is at least $(g - 1)$ and at most $3(g - 1)$. $3(g - 1)$ is the number of closed curves in the pant-decomposition of a genus g surface [1]. The number of ideal triangles in a maximal geodesic lamination is

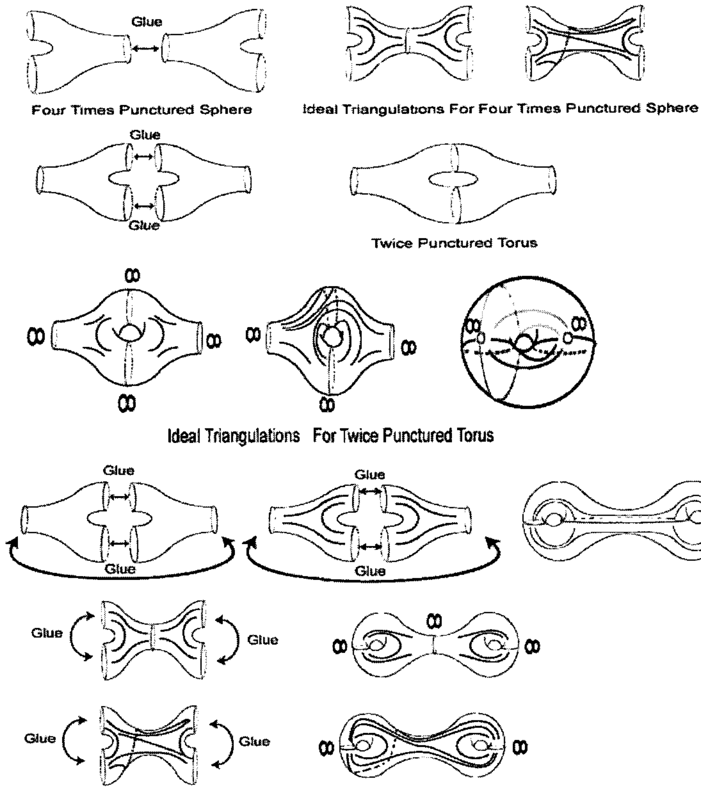

FIGURE 9. Maximal Laminations For Two Pairs of Pants

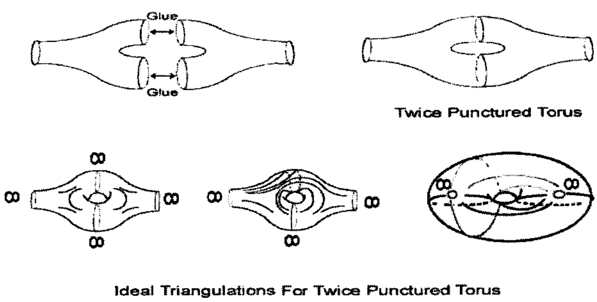

FIGURE 10. MaximalLaminations for Twice-Punctured Torus

fixed $4(g-1)$ [5]. Therefore, since $V - E + F = 2(1-g)$, $V - E = 6(1-g)$ and hence E can be $7(g-1), \dots, 9(g-1)$.

This finishes the proof of Theorem 0.0.1 for genus g. □

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