

ON PARAHOLOMORPHICALLY PSEUDOSYMMETRIC PARA-KÄHLERIAN MANIFOLDS

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ABSTRACT. We find necessary and sufficient conditions for a para-Kählerian manifold to be paraholomorphically pseudosymmetric in terms of the paraholomorphic projective and Bochner curvatures. New examples of such spaces are proposed.

1. Introduction

The semisymmetry and pseudosymmetry curvature conditions form an interesting object of investigations in geometry of Riemannian and pseudo-Riemannian manifolds, see [2, 5, 10, 12, 13, 19, 20, 25], etc.

However, in the class of para-Kählerian manifolds, the usual pseudosymmetry conditions are not essential in dimensions greater than 4. One of the author proved that in this case, they reduce to the semisymmetry conditions; see [16]. Therefore, the paraholomorphic versions of pseudosymmetry conditions was proposed in [17]. We recall the necessary definitions in the next sections.

In the presented paper, we find necessary and sufficient conditions for a para-Kählerian manifold to be paraholomorphically (in short, PH) pseudosymmetric in terms of the paraholomorphic projective and Bochner curvatures. More precise, we prove that a para-Kählerian manifold is PH pseudosymmetric if and only if its paraholomorphic projective curvature is PH pseudosymmetric. Moreover, when the paraholomorphic Bochner curvature does not vanish on an open dense subset of the manifold, then the manifold is PH pseudosymmetric if and only if its paraholomorphic Bochner curvature is PH pseudosymmetric.

2. Para-Kählerian manifolds

A connected differentiable manifold of even dimension $n = 2m$ is said to be para-Kählerian ([4, 6, 11], etc.) if it is endowed with a $(1, 1)$ -tensor field J and a pseudo-Riemannian metric g satisfying the conditions

$$J^2 = Id, \quad g(JX, JY) = -g(X, Y), \quad \nabla J = 0,$$

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∇ being the Levi-Civita connection of g and Id the identity tensor field. Such a manifold occurs to be symplectic since the fundamental form Ω , $\Omega(X, Y) = g(X, JY)$, is a skew-symmetric, non-degenerate and closed.

Let M be a para-Kählerian manifold. For M , the curvature operators $\mathcal{R}(X, Y)$, the Riemann curvature tensor R , the Ricci curvature tensor S , the Ricci operator Q and the scalar curvature r are defined

$$\begin{aligned}\mathcal{R}(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}, \quad R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W), \\ S(X, Y) &= g(QX, Y) = \text{Trace}\{Z \mapsto \mathcal{R}(Z, X)Y\}, \quad r = \text{Trace } Q.\end{aligned}$$

For these tensor fields, the following identities hold

$$\begin{aligned}\mathcal{R}(JX, JY) &= -\mathcal{R}(X, Y), \quad \mathcal{R}(JX, Y) = -\mathcal{R}(X, JY), \\ S(JX, JY) &= -S(X, Y), \quad S(JX, Y) = -S(X, JY), \\ \text{Trace}\{Z \mapsto \mathcal{R}(JZ, X)Y\} &= S(X, JY).\end{aligned}$$

Moreover, for any pair $U, V \in \mathfrak{X}(M)$ and a symmetric $(0, 2)$ -tensor field A , we consider the operator $U \wedge_A V: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ defined by

$$(U \wedge_A V)Z = A(V, Z)U - A(U, Z)V.$$

In the case when $A = g$, we shall write \wedge instead of \wedge_g .

Throughout this paper, $U, V, W, X, X_1, \dots, Y, Z, \dots$ indicate arbitrary smooth vector fields on a differentiable manifold M if it is not otherwise stated. $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields and $\mathfrak{F}(M)$ is the class of smooth functions on M .

3. Paraholomorphic pseudosymmetry

Let M be a para-Kählerian manifold. For $U, V \in \mathfrak{X}(M)$ and $f \in \mathfrak{F}(M)$, define the curvature type operator $\mathcal{R}^f(U, V): \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by

$$(1) \quad \mathcal{R}^f(U, V) = \mathcal{R}(U, V) - f(U \wedge V - JU \wedge JV - 2g(U, JV)J).$$

Algebraic properties of $\mathcal{R}^f(U, V)$ are analogous to those holding for $\mathcal{R}(U, V)$ ($= \mathcal{R}^0(U, V)$), that is,

$$\begin{aligned}\mathcal{R}^f(V, U) &= -\mathcal{R}^f(U, V), \quad \mathcal{R}^f(JU, JV) = -\mathcal{R}^f(U, V), \\ g(\mathcal{R}^f(U, V)W, X) &= g(\mathcal{R}^f(W, X)U, V), \\ \mathcal{R}^f(U, V)W + \mathcal{R}^f(V, W)U + \mathcal{R}^f(W, U)V &= 0.\end{aligned}$$

It is obvious that M is of constant paraholomorphic sectional curvature (a para-Kählerian space form) if and only if $R^c = 0$ with $c = \text{const.}$ (see [7]).

We extend the operators $\mathcal{R}^f(U, V)$ to derivations of the tensor algebra on M by assuming that they commute with contractions and $\mathcal{R}^f(U, V)h = 0$ for any $h \in \mathfrak{F}(M)$. Thus, for a $(0, k)$ -tensor field K , $\mathcal{R}^f(U, V)K$ is a $(0, k)$ -tensor

field,

$$\begin{aligned} & (\mathcal{R}^f(U, V)K)(X_1, \dots, X_k) \\ &= - \sum_{a=1}^k K(X_1, \dots, X_{a-1}, \mathcal{R}^f(U, V)X_a, X_{a+1}, \dots, X_k), \end{aligned}$$

and for a $(1, k)$ -tensor field K , $\mathcal{R}^f(U, V)K$ is a $(1, k)$ -tensor field such that

$$\begin{aligned} & (\mathcal{R}^f(U, V)K)(X_1, \dots, X_k) = \mathcal{R}^f(X, Y)(K(X_1, \dots, X_k)) \\ & \quad - \sum_{a=1}^k K(X_1, \dots, X_{a-1}, \mathcal{R}^f(U, V)X_a, X_{a+1}, \dots, X_k). \end{aligned}$$

It is important that $\mathcal{R}^f(U, V)g = 0$, $\mathcal{R}^f(U, V)J = 0$.

For a $(0, k)$ -tensor field K and $f \in \mathfrak{F}(M)$, define the $(0, k+2)$ -tensor field $\mathcal{R}^f \cdot K$ by

$$(\mathcal{R}^f \cdot K)(U, V, X_1, \dots, X_k) = (\mathcal{R}^f(U, V)K)(X_1, \dots, X_k).$$

We will say that a $(0, k)$ -tensor field K is paraholomorphically (PH in short) pseudosymmetric if there exists $f \in \mathfrak{F}(M)$ such that

$$(2) \quad \mathcal{R}^f \cdot K = 0.$$

A para-Kählerian manifold will be said to be ([17]): (a) PH pseudosymmetric if its Riemann curvature tensor R is PH pseudosymmetric, that is, for a certain $f \in \mathfrak{F}(M)$,

$$(3) \quad \mathcal{R}^f \cdot R = 0;$$

(b) PH Ricci pseudosymmetric if its Ricci curvature tensor S is PH pseudosymmetric, that is, for a certain $f \in \mathfrak{F}(M)$,

$$(4) \quad \mathcal{R}^f \cdot S = 0.$$

The notion of PH pseudosymmetric para-Kählerian manifolds was introduced by one of the author in [17], where certain examples were constructed. Below, we extend the class of examples of PH pseudosymmetric para-Kählerian structures.

By the commuting of the operators $\mathcal{R}^f(U, V)$ with contractions, it follows that (3) always implies (4) with the same structure function f , which means that the PH pseudosymmetry implies the PH Ricci pseudosymmetry. The converse statement does not hold in general.

Note that for a para-Kählerian manifold, the semisymmetry $\mathcal{R} \cdot R = 0$ and the Ricci semisymmetry $\mathcal{R} \cdot S = 0$ are respectively the PH pseudosymmetry and the PH Ricci pseudosymmetry with $f = 0$.

Remark. Tensor fields having the shape (1) as well as those realizing the condition (2), were independently studied by another authors in the class of Kählerian and para-Kählerian (hyperbolically Kählerian) spaces; see [18], where different denotations are used. With regard to a geometric meaning

of these tensor fields, it is important that they are invariant with respect to certain special paraholomorphically projective transformations.

Finally, we mention the following useful formula [18, p. 1339], which is a generalization of the famous Walker's identity [26],

$$(5) \quad \sum_{(U,V),(X,Y),(Z,W)} (\mathcal{R}^f \cdot R)(U, V, X, Y, Z, W) = 0,$$

where S denotes the cyclic sum with respect to the indicated pairs of arguments.

4. Paraholomorphic projective pseudosymmetry

The paraholomorphic projective curvature operator \mathcal{P} of a para-Kählerian manifold are defined by the formula (see [14, 21, 24])

$$(6) \quad \mathcal{P}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n+2} (X \wedge_S Y - (JX) \wedge_S (JY) - 2S(X, JY)J),$$

and the $(0, 4)$ -tensor P by

$$P(X, Y, Z, W) = g(P(X, Y)Z, W).$$

As it is already known, the tensors \mathcal{P}, P have the following algebraic properties

$$(7) \quad \begin{aligned} \mathcal{P}(X, Y) &= -\mathcal{P}(Y, X), \quad \mathcal{P}(JX, JY) = -\mathcal{P}(X, Y), \\ \text{Trace} \{ X \mapsto \mathcal{P}(X, Y)Z \} &= 0, \\ \text{Trace}_g \{ (Y, Z) \mapsto P(X, Y, Z, W) \} &= \frac{n}{n+2} S_0(X, W), \end{aligned}$$

where S_0 is the traceless Ricci tensor, that is, $S_0 = S - (r/n)g$.

A para-Kählerian manifold M will be called PH projectively pseudosymmetric if its paraholomorphic projective curvature tensor P is PH pseudosymmetric, i.e., for a certain function $f \in \mathfrak{F}(M)$,

$$(8) \quad \mathcal{R}^f \cdot P = 0.$$

Theorem 1. *A para-Kählerian manifold is PH pseudosymmetric if and only if it is PH projectively pseudosymmetric (with the same structure function f).*

Proof. Let M be a para-Kählerian manifold. Using (6), we can find

$$(9) \quad \begin{aligned} (\mathcal{R}^f \cdot P)(U, V, X, Y, Z, W) &= (\mathcal{R}^f \cdot R)(U, V, X, Y, Z, W) \\ &\quad - \frac{1}{n+2} ((\mathcal{R}^f \cdot S)(U, V, Y, Z)g(X, W) - (\mathcal{R}^f \cdot S)(U, V, X, Z)g(Y, W) \\ &\quad - (\mathcal{R}^f \cdot S)(U, V, JY, Z)g(JX, W) + (\mathcal{R}^f \cdot S)(U, V, JX, Z)g(JY, W) \\ &\quad + 2(\mathcal{R}^f \cdot S)(U, V, X, JY)S(Z, JW) + 2S(X, JY)(\mathcal{R}^f \cdot S)(U, V, Z, JW)). \end{aligned}$$

Let M be PH pseudosymmetric. Then the relations (3) and (4) applied to (9) imply (8). Thus, M is PH projectively pseudosymmetric.

Conversely, assume that M is PH projectively pseudosymmetric. Using (7) and knowing that $\mathcal{R}^f(U, V)$ commutes with contractions, we claim that

$$\begin{aligned} & \text{Trace}_g\{(Y, Z) \mapsto (\mathcal{R}^f \cdot P)(U, V, X, Y, Z, W)\} \\ &= \frac{n}{n+2}(\mathcal{R}^f \cdot S_0)(U, V, X, W). \end{aligned}$$

Hence, by (8), it follows that $\mathcal{R}^f \cdot S_0 = 0$. Furthermore, $\mathcal{R}^f \cdot S = 0$ since $\mathcal{R}^f \cdot r = 0$ and $\mathcal{R}^f \cdot g = 0$. This together with (8) applied into (9) leads to (3). \square

5. Paraholomorphic Bochner pseudosymmetry

On a para-Kählerian manifold, the paraholomorphic Bochner curvature operator \mathcal{B} is defined by (cf. [1, 8, 18, 22])

$$\begin{aligned} \mathcal{B}(X, Y) &= \mathcal{R}(X, Y) \\ &- \frac{1}{n+4} (X \wedge (QY) + (QX) \wedge Y - (JX) \wedge (QJY) \\ (10) \quad &- (QJX) \wedge (JY) + 2g(JX, Y)QJ + 2g(QJX, Y)J) \\ &+ \frac{r}{(n+4)(n+2)} (X \wedge Y - (JX) \wedge (JY) + 2g(JX, Y)J), \end{aligned}$$

and the $(0, 4)$ -tensor field B by

$$B(X, Y, Z, W) = g(\mathcal{B}(X, Y)Z, W).$$

The tensors \mathcal{B}, B have the same algebraic properties as the Riemann curvature tensor,

$$\begin{aligned} \mathcal{B}(Y, X) &= -\mathcal{B}(X, Y), \quad \mathcal{B}(JX, JY) = -\mathcal{B}(X, Y), \\ B(X, Y, Z, W) &= B(Z, W, X, Y), \\ \mathcal{B}(X, Y)Z + \mathcal{B}(Y, Z)X + \mathcal{B}(Z, X)Y &= 0. \end{aligned}$$

Moreover, \mathcal{B} is traceless, precisely,

$$\text{Trace}\{Z \mapsto \mathcal{B}(Z, X)Y\} = 0, \quad \text{Trace}\{Z \mapsto \mathcal{B}(JZ, X)Y\} = 0.$$

Bochner flat ($B = 0$) para-Kählerian manifolds are studied in [1, 17, 23]. Isotropic para-Kählerian manifolds considered in [3, 9] are also of this type, i.e., Bochner flat. Bochner semisymmetric ($\mathcal{R} \cdot B = 0$) para-Kählerian manifolds are studied in [15], where it is shown that such manifolds are semisymmetric on the set on which the Bochner curvature tensor does not vanish.

In the presented section, we are interested in para-Kählerian manifolds, which are PH Bochner pseudosymmetric, i.e., for which there is a function $f \in \mathfrak{F}(M)$ such that

$$(11) \quad \mathcal{R}^f \cdot B = 0.$$

Let M be a para-Kählerian manifold. To simplify denotations, let us suppose $T = \mathcal{R}^f \cdot S$, so that we have

$$(12) \quad \begin{aligned} T(X, Y, Z, W) &= (\mathcal{R}^f \cdot S)(X, Y, Z, W) \\ &\quad - S(\mathcal{R}^f(X, Y)Z, W) - S(Z, \mathcal{R}^f(X, Y)W). \end{aligned}$$

One simply finds the following algebraic properties of T

$$\begin{aligned} T(X, Y, Z, W) &= -T(Y, X, Z, W) = T(X, Y, W, Z), \\ T(JX, JY, Z, W) &= T(X, Y, JZ, JW) = -T(X, Y, Z, W), \\ \text{Trace}_g\{(Z, W) \mapsto T(X, Y, Z, W)\} &= 0. \end{aligned}$$

For the further use, using (10), we express $\mathcal{R}^f \cdot B$ with the help of $\mathcal{R}^f \cdot R$ and T ,

$$\begin{aligned} (13) \quad &(\mathcal{R}^f \cdot B)(U, V, X, Y, Z, W) \\ &= (\mathcal{R}^f \cdot R)(U, V, X, Y, Z, W) \\ &\quad - \frac{1}{n+4} (T(U, V, Y, Z)g(X, W) - T(U, V, X, Z)g(Y, W) \\ &\quad + T(U, V, X, W)g(Y, Z) - T(U, V, Y, W)g(X, Z) \\ &\quad - T(U, V, Y, JZ)g(X, JW) + T(U, V, X, JZ)g(Y, JW) \\ &\quad - T(U, V, X, JW)g(Y, JZ) + T(U, V, Y, JW)g(X, JZ) \\ &\quad + 2T(U, V, Z, JW)g(X, JY) + 2T(U, V, X, JY)g(Z, JW)). \end{aligned}$$

In the sequel, by (e_1, e_2, \dots, e_n) we denote a (local) frame which is adapted to the structure (J, g) . This means that it is a frame for which

$$\begin{aligned} g(e_\alpha, e_{\beta'}) &= g(e_{\alpha'}, e_\beta) = \delta_{\alpha\beta}, \quad g(e_\alpha, e_\beta) = g(e_{\alpha'}, e_{\beta'}) = 0, \\ J e_\alpha &= -e_\alpha, \quad J e_{\alpha'} = e_{\alpha'}, \end{aligned}$$

where the Greek indices run from 1 to $m = n/2$ and $\alpha' = m + \alpha$ for any α . Let g_{ij} and g^{ij} , $1 \leq i, j \leq n$, be the components of g and the inverse metric g^{-1} with respect to the adapted frame, that is, $g_{ij} = g(e_i, e_j)$, $[g^{ij}] = [g_{ij}]^{-1}$.

Regarding the algebraic properties of the geometric objects already introduced, we claim that the non-zero components of the tensor fields R , S , B , T , $\mathcal{R}^f \cdot R$, $\mathcal{R}^f \cdot B$, with respect to an adapted frame, are related to the following

$$R_{\alpha\beta'\gamma\delta'}, \quad S_{\alpha\beta'}, \quad B_{\alpha\beta'\gamma\delta'}, \quad T_{\alpha\beta'\gamma\delta'}, \quad (\mathcal{R}^f \cdot R)_{\lambda\mu'\alpha\beta'\gamma\delta'}, \quad (\mathcal{R}^f \cdot B)_{\lambda\mu'\alpha\beta'\gamma\delta'}.$$

For the Bochner tensor, we additionally have

$$B_{\alpha\beta'\gamma\delta'} = B_{\gamma\beta'\alpha\delta'} = B_{\alpha\delta'\gamma\beta'}, \quad \delta^{\lambda\mu} B_{\lambda\mu'\gamma\delta'} = 0, \quad \delta^{\lambda\mu} B_{\lambda\beta'\gamma\mu'} = 0,$$

and for the tensor T ,

$$\delta^{\lambda\mu} T_{\alpha\beta'\lambda\mu'} = 0.$$

[We always apply the Einstein summation rule with respect to the repeated indices.]

Theorem 2. *Let M be a para-Kählerian manifold.*

- (a) *If M is PH pseudosymmetric, then it is PH Bochner pseudosymmetric (with the same structure function f).*
- (b) *If M is PH Bochner pseudosymmetric and its paraholomorphic Bochner curvature tensor does not vanish on an open dense subset of M , then M is PH pseudosymmetric (with the same structure function f).*

Proof. (a) Assume (3). Then (4) also holds, which by (12) gives $T = 0$. Now, by applying (3) and $T = 0$ into (13), we get (11).

(b) Assume that $\mathcal{R}^f \cdot B = 0$ on M . Let M_1 be the open dense subset of M on which $B \neq 0$. In what follows, we restrict our considerations to the subset M_1 .

By $\mathcal{R}^f \cdot B = 0$, from (13), we obtain

$$\begin{aligned} & (\mathcal{R}^f \cdot R)(U, V, X, Y, Z, W) \\ &= \frac{1}{2m+4} (T(U, V, Y, Z)g(X, W) - T(U, V, X, Z)g(Y, W) \\ &\quad + T(U, V, X, W)g(Y, Z) - T(U, V, Y, W)g(X, Z) \\ &\quad - T(U, V, Y, JZ)g(X, JW) + T(U, V, X, JZ)g(Y, JW) \\ &\quad - T(U, V, X, JW)g(Y, JZ) + T(U, V, Y, JW)g(X, JZ) \\ &\quad + 2T(U, V, Z, JW)g(X, JY) + 2T(U, V, X, JY)g(Z, JW)). \end{aligned}$$

In the above formula, we suppose $U = e_\lambda$, $V = e_{\mu'}$, $X = e_\alpha$, $Y = e_{\beta'}$, $Z = e_\gamma$, $W = e_{\delta'}$, with (e_1, e_2, \dots, e_n) being a (local) adapted frame. Then we get

$$\begin{aligned} & (\mathcal{R}^f \cdot R)_{\lambda\mu'\alpha\beta'\gamma\delta'} \\ (14) \quad &= \frac{2}{n+4} (T_{\lambda\mu'\gamma\beta'}\delta_{\alpha\delta} + T_{\lambda\mu'\alpha\delta'}\delta_{\beta\gamma} + T_{\lambda\mu'\gamma\delta'}\delta_{\alpha\beta} + T_{\lambda\mu'\alpha\beta'}\delta_{\gamma\delta}). \end{aligned}$$

But by the generalized Walker's identity (5), for the components of $\mathcal{R}^f \cdot R$, it holds

$$(\mathcal{R}^f \cdot R)_{\lambda\mu'\alpha\beta'\gamma\delta'} + (\mathcal{R}^f \cdot R)_{\alpha\beta'\gamma\delta'\lambda\mu'} + (\mathcal{R}^f \cdot R)_{\gamma\delta'\lambda\mu'\alpha\beta'} = 0.$$

The relation (14) used in the above, gives the following

$$\begin{aligned} (15) \quad & T_{\lambda\mu'\gamma\beta'}\delta_{\alpha\delta} + T_{\lambda\mu'\alpha\delta'}\delta_{\beta\gamma} + T_{\lambda\mu'\gamma\delta'}\delta_{\alpha\beta} + T_{\lambda\mu'\alpha\beta'}\delta_{\gamma\delta} \\ &+ T_{\alpha\beta'\lambda\delta'}\delta_{\gamma\mu} + T_{\alpha\beta'\gamma\mu'}\delta_{\delta\lambda} + T_{\alpha\beta'\lambda\mu'}\delta_{\gamma\delta} + T_{\alpha\beta'\gamma\delta'}\delta_{\lambda\mu} \\ &+ T_{\gamma\delta'\alpha\mu'}\delta_{\lambda\beta} + T_{\gamma\delta'\lambda\beta'}\delta_{\mu\alpha} + T_{\gamma\delta'\alpha\beta'}\delta_{\lambda\mu} + T_{\gamma\delta'\lambda\mu'}\delta_{\alpha\beta} = 0. \end{aligned}$$

Contracting the last relation with $\delta^{\lambda\mu}$, we find

$$(16) \quad (m+2)(T_{\alpha\beta'\gamma\delta'} + T_{\gamma\delta'\alpha\beta'}) = -D_{\gamma\beta'}\delta_{\alpha\delta} - D_{\alpha\delta'}\delta_{\beta\gamma} - D_{\gamma\delta'}\delta_{\alpha\beta} - D_{\alpha\beta'}\delta_{\gamma\delta},$$

where $D_{\alpha\beta'} = \delta^{\lambda\mu}T_{\lambda\mu'\alpha\beta'}$. Next contracting (16) with $\delta^{\alpha\beta}$ and using $\delta^{\alpha\beta}D_{\alpha\beta'} = 0$, we get $D_{\lambda\mu'} = 0$. Therefore, the equation (16) gives the antisymmetry of $T_{\alpha\beta'\gamma\delta'}$ in pairs of indices, precisely, $T_{\alpha\beta'\gamma\delta'} + T_{\gamma\delta'\alpha\beta'} = 0$. On the other hand,

contracting (15) with $\delta^{\lambda\beta}$ and applying the just obtained antisymmetry, we obtain

$$(17) \quad mT_{\gamma\delta'\alpha\mu'} = -G_{\mu'\gamma}\delta_{\alpha\delta} - G_{\mu'\alpha}\delta_{\gamma\delta} - F_{\alpha\delta'}\delta_{\gamma\mu} - F_{\alpha\mu'}\delta_{\gamma\delta},$$

where $G_{\alpha'\beta} = \delta^{\lambda\mu}T_{\lambda\alpha'\beta\mu'}$, $F_{\alpha\beta'} = \delta^{\lambda\mu}T_{\alpha\lambda'\mu\beta'}$. Next, transvecting (17) with $\delta^{\gamma\delta}$ and applying $D_{\alpha\beta'} = 0$, we get $F_{\alpha\mu'} = -G_{\mu'\alpha}$, which substituted into (17) yields

$$(18) \quad mT_{\gamma\delta'\alpha\mu'} = G_{\delta'\alpha}\delta_{\gamma\mu} - G_{\mu'\gamma}\delta_{\alpha\delta}.$$

Hence, by transvection with $\delta^{\gamma\mu}$, we find additionally

$$(19) \quad \delta^{\gamma\mu}G_{\gamma\mu'} = 0.$$

On the other hand, by the assumption $\mathcal{R}^f \cdot B = 0$, we have

$$\begin{aligned} B(\mathcal{R}^f(U, V)X, Y, Z, W) + B(X, \mathcal{R}^f(U, V)Y, Z, W) \\ + B(X, Y, \mathcal{R}^f(U, V)Z, W) + B(X, Y, Z, \mathcal{R}^f(U, V)W) = 0. \end{aligned}$$

The substitution QU instead of U in the last equality leads to

$$(20) \quad \begin{aligned} B(\mathcal{R}^f(QU, V)X, Y, Z, W) + B(X, \mathcal{R}^f(QU, V)Y, Z, W) \\ + B(X, Y, \mathcal{R}^f(QU, V)Z, W) + B(X, Y, Z, \mathcal{R}^f(QU, V)W) = 0. \end{aligned}$$

Hence, interchanging U with V , we also have

$$(21) \quad \begin{aligned} B(\mathcal{R}^f(QV, U)X, Y, Z, W) + B(X, \mathcal{R}^f(QV, U)Y, Z, W) \\ + B(X, Y, \mathcal{R}^f(QV, U)Z, W) + B(X, Y, Z, \mathcal{R}^f(QV, U)W) = 0. \end{aligned}$$

Note that using (12), it can be derived

$$g(\mathcal{R}^f(QZ, W)X + \mathcal{R}^f(QW, Z)X, Y) = T(X, Y, Z, W),$$

and consequently,

$$(22) \quad \mathcal{R}^f(QZ, W)X + \mathcal{R}^f(QW, Z)X = -g^{rs}T(e_r, X, Z, W)e_s.$$

Adding the equalities (20) and (21) and taking into account (22), we obtain

$$\begin{aligned} g^{rs}(T(e_r, X, U, V)B(e_s, Y, Z, W) + T(e_r, Y, U, V)B(X, e_s, Z, W) \\ + T(e_r, Z, U, V)B(X, Y, e_s, W) + T(e_r, W, U, V)B(X, Y, Z, e_s)) = 0. \end{aligned}$$

For $U = e_\lambda$, $V = e_{\mu'}$, $X = e_\alpha$, $Y = e_{\beta'}$, $Z = e_\gamma$ and $W = e_{\delta'}$, the above equality gives

$$\delta^{\omega\eta}(-T_{\alpha\eta'\lambda\mu'}B_{\omega\beta'\gamma\delta'} + T_{\omega\beta'\lambda\mu'}B_{\alpha\eta'\gamma\delta'} - T_{\gamma\eta'\lambda\mu'}B_{\alpha\beta'\omega\delta'} + T_{\omega\delta'\lambda\mu'}B_{\alpha\beta'\gamma\eta'}) = 0.$$

By applying (18) into the last equation, we obtain

$$(23) \quad \begin{aligned} \delta^{\omega\eta}B_{\omega\beta'\gamma\delta'}G_{\eta'\lambda}\delta_{\alpha\mu} - G_{\mu'\alpha}B_{\lambda\beta'\gamma\delta'} + \delta^{\omega\eta}B_{\alpha\beta'\omega\delta'}G_{\eta'\lambda}\delta_{\gamma\mu} \\ - G_{\mu'\gamma}B_{\alpha\beta'\lambda\delta'} + \delta^{\omega\eta}B_{\alpha\eta'\gamma\delta'}G_{\mu'\omega}\delta_{\lambda\beta} - G_{\beta'\lambda}B_{\alpha\mu'\gamma\delta'} \\ + \delta^{\omega\eta}B_{\alpha\beta'\gamma\eta'}G_{\mu'\omega}\delta_{\lambda\delta} - G_{\delta'\lambda}B_{\alpha\beta'\gamma\mu'} = 0. \end{aligned}$$

Contracting (23) with $\delta^{\alpha\mu}$, we get

$$(24) \quad (m+1)\delta^{\omega\eta}B_{\omega\beta'\gamma\delta'}G_{\eta'\lambda} - \delta^{\omega\eta}B_{\omega\beta'\lambda\delta'}G_{\eta'\gamma} \\ + \delta^{\alpha\mu}\delta^{\omega\eta}B_{\alpha\eta'\gamma\delta'}G_{\mu'\omega}\delta_{\lambda\beta} + \delta^{\alpha\mu}\delta^{\omega\eta}B_{\alpha\beta'\gamma\eta'}G_{\mu'\omega}\delta_{\lambda\delta} = 0.$$

Furthermore, contracting (24) with $\delta^{\lambda\beta}$, we find $\delta^{\alpha\mu}\delta^{\omega\eta}B_{\alpha\eta'\gamma\delta'}G_{\mu'\omega} = 0$, which substituted into (24), gives

$$(m+1)\delta^{\omega\eta}B_{\omega\beta'\gamma\delta'}G_{\eta'\lambda} - \delta^{\omega\eta}B_{\omega\beta'\lambda\delta'}G_{\eta'\gamma} = 0.$$

Therefore, $\delta^{\omega\eta}B_{\omega\beta'\gamma\delta'}G_{\eta'\lambda} = 0$, which enables us to rewrite (23) in the following form

$$(25) \quad -G_{\mu'\alpha}B_{\lambda\beta'\gamma\delta'} - G_{\mu'\gamma}B_{\alpha\beta'\lambda\delta'} - G_{\beta'\lambda}B_{\alpha\mu'\gamma\delta'} - G_{\delta'\lambda}B_{\alpha\beta'\gamma\mu'} \\ + \delta^{\omega\eta}B_{\alpha\eta'\gamma\delta'}G_{\mu'\omega}\delta_{\lambda\beta} + \delta^{\omega\eta}B_{\alpha\beta'\gamma\eta'}G_{\mu'\omega}\delta_{\lambda\delta} = 0.$$

Transvection of (25) with $\delta^{\lambda\delta}$, gives $\delta^{\omega\eta}B_{\gamma\eta'\alpha\beta'}G_{\mu'\omega} = 0$, which turns (25) into

$$(26) \quad G_{\mu'\alpha}B_{\lambda\beta'\gamma\delta'} + G_{\mu'\gamma}B_{\alpha\beta'\lambda\delta'} + G_{\beta'\lambda}B_{\alpha\mu'\gamma\delta'} + G_{\delta'\lambda}B_{\alpha\beta'\gamma\mu'} = 0.$$

Next, alternating (26) in pairs of indices (λ, β) , (γ, μ) , we find

$$G_{\mu'\alpha}B_{\lambda\beta'\gamma\delta'} - G_{\beta'\alpha}B_{\gamma\mu'\lambda\delta'} + G_{\delta'\lambda}B_{\alpha\beta'\gamma\mu'} - G_{\delta'\gamma}B_{\alpha\beta'\lambda\mu'} = 0.$$

Replacing the pair of indices α, μ with λ, δ in the last relation, we obtain

$$G_{\delta'\lambda}B_{\alpha\beta'\gamma\mu'} - G_{\beta'\lambda}B_{\alpha\mu'\gamma\delta'} + G_{\mu'\alpha}B_{\lambda\beta'\gamma\delta'} - G_{\mu'\gamma}B_{\alpha\beta'\lambda\delta'} = 0,$$

which compared to (26) yields $G_{\mu'\alpha}B_{\lambda\beta'\gamma\delta'} + G_{\delta'\lambda}B_{\alpha\beta'\gamma\mu'} = 0$. Summing up the last equality cyclically in pairs of indices (α, μ) , (γ, β) , and (λ, δ) , we get

$$G_{\mu'\alpha}B_{\lambda\beta'\gamma\delta'} + G_{\beta'\gamma}B_{\alpha\delta'\lambda\mu'} + G_{\delta'\lambda}B_{\alpha\beta'\gamma\mu'} = 0.$$

Comparing the last two equalities, we find $G_{\beta'\gamma}B_{\alpha\delta'\lambda\mu'} = 0$. Since by our assumptions, B is non-zero everywhere on M_1 , it must be that $G_{\beta'\gamma} = 0$. This used in (18), leads to $T_{\gamma\delta'\alpha\mu'} = 0$. Therefore, from (14) it now follows that $(\mathcal{R}^f \cdot R)_{\lambda\mu'\alpha\beta'\gamma\delta'} = 0$. Consequently, $\mathcal{R}^f \cdot R = 0$ on M_1 . Since M_1 is open and dense, the condition $\mathcal{R}^f \cdot R = 0$ holds on the whole of the manifold M , which completes the proof. \square

6. A class of examples

Below, we give a generalization of the class of PH pseudosymmetric para-Kählerian spaces stated in [17].

Let $(x^\alpha, x^{\alpha'} = y^\alpha, x^{m+1} = z, x^{2m+2} = t)$ be the Cartesian coordinates in the Cartesian space \mathbb{R}^{2m+2} with $1 \leq \alpha \leq m$ and $\alpha' = \alpha + m$. Let (a, b) be an open interval and $M = \mathbb{R}^{2m+1} \times (a, b) \subset \mathbb{R}^{2m+2}$. Let A, B, C be three positive functions on (a, b) such that $A' = 2BC$. Define a pseudo-Riemannian metric g of signature $(m+1, m+1)$ on M by

$$g = A \sum_{\alpha=1}^m dx^\alpha \otimes dx^\alpha - A \sum_{\alpha=1}^m dy^\alpha \otimes dy^\alpha + B^2 \eta \otimes \eta - C^2 dt^2,$$

where the 1-form η is given by

$$\eta = -2 \sum_{\alpha=1}^m x^{\alpha} dy^{\alpha} + dz.$$

Moreover, define the $(1, 1)$ -tensor field J on M by

$$Jdx^{\alpha} = dy^{\alpha}, \quad Jdy^{\alpha} = dx^{\alpha}, \quad J\eta = \frac{C}{B} dt, \quad Jdt = \frac{B}{C} \eta.$$

By making certain hard but standard calculations, one can show that the structure (J, g) is para-Kählerian, and moreover

$$\mathcal{R} \cdot R \neq 0 \quad \text{and} \quad \mathcal{R}^f \cdot R = 0 \quad \text{with} \quad f = \frac{AB' - B^2C}{A^2C}.$$

which shows that the structure is PH pseudosymmetric and not semisymmetric in general.

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