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CONTINUITIES AND HOMEOMORPHISMS IN COMPUTER TOPOLOGY AND THEIR APPLICATIONS

Sang-Eon Han

Abstract. In this paper several continuities and homeomorphisms in computer topology are studied and their applications are investigated in relation to the classification of subspaces of Khalimsky n-dimensional space $(\mathbb{Z}^n, T^n)$. Precisely, the notions of K-(k_0, k_1)-,(k_0, k_1)-,KD-(k_0, k_1)-continuities, and Khalimsky continuity as well as those of K-(k_0, k_1)-, (k_0, k_1)-, KD-(k_0, k_1)-homeomorphisms, and Khalimsky homeomorphism are studied and further, their applications are investigated.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ be the sets of natural numbers, integers, and real numbers, respectively. Furthermore, by $\mathbb{Z}^n$ and $\mathbb{R}^n$ we denote the Cartesian product of n-tuples of $\mathbb{Z}$ and the n-dimensional real space, respectively. Even though several topologies for the study of a set $X \subset \mathbb{Z}^n$ have been studied in [3, 4, 5, 6, 11, 17, 20], in this paper two kinds of topological structures are used. One is the discrete (or digital) topology on $\mathbb{Z}^n$ [7, 8, 9, 10], denoted by $(\mathbb{Z}^n, D^n)$, and the other is the Khalimsky product topology on $\mathbb{Z}^n$, denoted by $(\mathbb{Z}^n, T^n)$ which is the product space of the Khalimsky line topology $(\mathbb{Z}, T)$ in [1] (see also [3, 6, 17, 19, 20, 22]). Consider two discrete topological spaces $X \subset \mathbb{Z}^{n_0}$ with k_0-adjacency, denoted by (X, k_0), and $Y \subset \mathbb{Z}^{n_1}$ with k_1-adjacency, denoted by (Y, k_1). Then, a continuous map $f : (X, k_0) \to (Y, k_1)$ need not preserve the k_0-connectivity of X into the k_1-connectivity of Y. Meanwhile, in order to study objects in $\mathbb{Z}^n$, the preservation of the k_0-connectivity into the k_1-connectivity by the map f is strongly required so that the digital (k_0, k_1)-continuity of f was developed in [21] (see also [2, 7, 8, 9, 10, 12, 13, 14, 15, 16]) and has been used in the study of a digital k-curve, a closed k-surface [9, 10, 12, 14], digital covering theory [7, 8, 15, 16], and so forth. Indeed, by using both digital covering theory and the k-homotopic thinning in [13, 16], we can calculate the

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digital fundamental groups of both a closed k-curve and a digital product of two simple closed k-curves [16].

For a set $X \subset \mathbb{Z}^n$, considering the subspace induced from the n-dimensional Khalimsky space $(\mathbb{Z}^n, T^n)$ [1] (see also [3, 19, 22]), we denote by $(X, T^n_X) \subset (\mathbb{Z}^n, T^n)$ the subspace. Furthermore, in this paper a topological space (X, T^n_X) with one of the k-adjacency relations of $\mathbb{Z}^n$ is called a (*computer topological*) *space with k-adjacency* and is denoted by (X, k, T^n_X). Hereafter, we briefly use the notation $(X, k, T^n_X) := X_{n,k}$ [6] (see also [11]). Indeed, by computer topology is usually meant the mathematical recognition of a space $X \subset \mathbb{Z}^n$, *e.g.*, a development of tools implementing topological concepts for use in computer science and information technology. Computer topology plays a significant role in computer graphics, image synthesis, image analysis, and so forth. It grew out of discrete geometry expanded into applications where significant topological issues arise and further, it may be of interest both for computer scientist who try to apply topological knowledge for investigating discrete spaces and for mathematicians who want to use computers to solve complicated topological problems. In this paper we can recognize some difference between *computer topology* and *digital topology*. Precisely, while computer topology needs some reasonable topological structure for the research of a space $X \subset \mathbb{Z}^n$ such as *Khalimsky product topology, Lawson topology, Alexandroff topology*, and so forth [1, 4], digital topology requires the discrete topological structure of a space $X \subset \mathbb{Z}^n$ with k-adjacency.

Indeed, for two sets $A, B \subset \mathbb{Z}$, a Khalimsky continuous map $f : A_{1,2} \to B_{1,2}$ obviously preserves the 2-connectivity [11]. But, for two k_0- and k_1-connected spaces X_{n_0,k_0} and Y_{n_1,k_1}, a Khalimsky continuous map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ need not preserve the k_0-connectivity of X into the k_1-connectivity of Y, where $n_0 \geq 2, n_1 \geq 1$ (see the points x_5 and x_6 in Fig.1). Thus, in computer topology we strongly need several continuities of the map f in relation to the preservation of the k_0-connectivity into the k_0-connectivity.

Up to now in computer topology several kinds of continuities such as K-(k_0, k_1)-, (k_0, k_1)-, KD-(k_0, k_1)-continuities, and Khalimsky continuity have been studied [6, 11]. Indeed, each of these has some advantages and disadvantages depending on the domain and the codomain of a given map. While Khalimsky continuity for a map $f : (X, T^n_X) \to (\mathbb{Z}, T)$ was partially studied in [11, 19], and a special kind of homeomorphism was studied in [3] the others have been recently studied from a computer topological point of view. Now we say that a property which when possessed by a space in $\mathbb{Z}^n$ is also possessed by each of K-(k_0, k_1)-, (k_0, k_1)-, KD-(k_0, k_1)-, and Khalimsky homeomorphisms in [10, 15] is called a *computer topological property*. Indeed, the research of the computer topological properties is also a field of interest in computer topology.

Recently, in [6] the comparison between (k_0, k_1)-continuity (see Definition 4) and digital (k_0, k_1)-continuity (see Definition 1) was done. Furthermore, in [11] the notion of K-(k_0, k_1)-continuity, which is stronger than the Khalimsky continuity, was established to study a computer topological space $X_{n,k}$. Besides, the notion of KD-(k_0, k_1)-continuity was developed and its applications were studied for several cases [11].

This paper is organized as follows. In Section 2 we provide some basic notions for computer topology and some properties of the Khalimsky product topology. In Section 3 we investigate some relations among four types of continuities in computer topology. In Section 4 four kinds of homeomorphisms such as K-(k_0, k_1)-, (k_0, k_1)-, KD-(k_0, k_1)-homeomorphisms, and Khalimsky homeomorphism are investigated and compared with each other. In Section 5 we conclude the paper with referring to four forgetful functors from each of several computer topological categories into the digital topological category.

Furthermore, Khalimsky topological subspace (X, T^n_X) in this paper can be recognized as a set of n-cells which are unit n-cubes centered at the points with integers coordinates.

2. Preliminaries

A set $X \subset \mathbb{Z}^n$ with k-adjacency, denoted by (X, k), has usually been considered in a quadruple $(\mathbb{Z}^n, k, \bar{k}, X)$ owing to the *digital k-connectivity paradox* and the *digital k-curve theorem* in [18], where $n \in \mathbf{N}$, k represents an adjacency relation for X, and $\bar{k}$ represents an adjacency relation for $\mathbb{Z}^n - X$ [18]. However, in this paper we are not concerned with adjacencies among n-xels of $\mathbb{Z}^n - X$. Precisely, we consider a set $X \subset \mathbb{Z}^n$ with k-adjacency and Khalimsky product topology.

As the generalization of the commonly used 4- and 8-adjacency of $\mathbb{Z}^2$ and further, 6-, 18- and 26-adjacency of $\mathbb{Z}^3$ in [18], the k-adjacency relations of $\mathbb{Z}^n$ have been used in [5] (see also [7, 8, 9, 16]). Precisely, for a positive integer m with $1 \le m \le n$, we say that two points $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$ are *adjacent* according to the number m if

- there are at most m indices i such that $|p_i - q_i| = 1$; and
- for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

In the following, this operator consisting of the above two statements is called $(CON\star)$ [5] (see also [6, 7, 8, 9]). Then, by $X_k(p)$ we denote the set of the points $q \in \mathbb{Z}^n$ which are adjacent to a given point p according to $(CON\star)$ and the number $k := k(m,n)$ is the cardinal number of $X_k(p)$ called the *k-neighbors of p*. Indeed, $X_k(p)$ is equal to the set $N^*_k(p) := \{x \in X | p \text{ is } k-\text{adjacent to } x\}$ which is the *k-neighbors* of the point p [18]. In addition, we recall $N_k(p) = N^*_k(p) \cup \{p\}$ [18]. Indeed, the number m in $(CON\star)$ determines one of the k-adjacency relations of $\mathbb{Z}^n$ [5] (see also [6, 7, 8]). Consequently, the following k-adjacency relations of $\mathbb{Z}^n$ are recently established in [5] (see also [6, 7, 8]).

$$k \in \{2n(n \ge 1), 3^n - 1(n \ge 2), 3^n - \sum_{t=0}^{r-2} C^n_t 2^{n-t} - 1(2 \le r \le n-1, n \ge 3)\}, \tag{2.1}$$

where $C^n_t = \frac{n!}{(n-t)!t!}$.

For example, $(n,m,k) \in \{(2,1,4),\ (2,2,8);\ (3,1,6),\ (3,2,18),\ (3,3,26);\ (4,1,8),\ (4,2,32),\ (4,3,64),\ (4,4,80);\ (5,1,10),\ (5,2,50),\ (5,3,130),\ (5,4,210),\ (5,5,242);\ (6,1,12),\ (6,2,72),\ (6,3,232),\ (6,4,472),\ (6,5,664),\ (6,6,728)\}$.

Indeed, the k-adjacency relations of (2.1) can be rewritten in a simpler and more generic form as follows.

Proposition 2.1 ([16]). *$k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n$, where $C_i^n = \frac{n!}{(n-i)!\ i!}$.*

In order to justify Khalimsky line topology, let us consider the partition $\cup_{n\in\mathbb{Z}}\{[2n-\frac{1}{2}, 2n+\frac{1}{2}], (2n+\frac{1}{2}, 2n+\frac{3}{2})\}$ of $\mathbb{R}$ which leads to an equivalence relation '$\sim$' on $\mathbb{R}$. Precisely, we obtain the set of equivalence classes $\{[x] : x \in \mathbb{R}\}$ from the equivalence relation set $(\mathbb{R}, \sim)$, where $x \sim y$ if and only if x and y are in the same set $[2n-\frac{1}{2}, 2n+\frac{1}{2}]$ or $(2n+\frac{1}{2}, 2n+\frac{3}{2})$. Then we have the equivalence class of $x \in \mathbb{R}$ as follows. $\mathbb{R}/\sim := \mathbb{Z}$. Furthermore, we consider the following quotient map

$$p : (\mathbb{R}, U) \to \mathbb{Z} \quad \text{for which} \quad p(x) = [x] \in \mathbb{Z}, x \in \mathbb{R}.$$

Then the quotient topology on $\mathbb{Z}$ is called the *Khalimsky line topology* on $\mathbb{Z}$.

Indeed, *Khalimsky line topology* on $\mathbb{Z}$ is induced from the subbase $\{[2n-1, 2n+1]_{\mathbb{Z}} : n \in \mathbb{Z}\}$ [1] (see also [3, 6, 11, 17, 22]). Namely, the family of the subset $\{\{2n+1\}, [2m-1, 2m+1]_{\mathbb{Z}} : m, n \in \mathbb{Z}\}$, which induces open sets for $(\mathbb{Z}, T)$, is a basis of the Khalimsky line topology on $\mathbb{Z}$. Furthermore, the *product topology* on $\mathbb{Z}^n, n \geq 2$, is derived from $(\mathbb{Z}, T)$. Then the typical product topology on $\mathbb{Z}^n$ induced from $(\mathbb{Z}, T)$ is called the *Khalimsky product topology* on $\mathbb{Z}^n$, denoted by $(\mathbb{Z}^n, T^n)$.

If the set $[a,b]_{\mathbb{Z}} = \{a \leq n \leq b : n \in \mathbb{Z}\}$ with 2-adjacency is considered with discrete topology, then it is called a *digital interval* [21] and further, if the set $[a,b]_{\mathbb{Z}}$ is considered as a subspace of $(\mathbb{Z}, T)$ with Khalimsky line topology, then it is called a *Khalimsky interval.*

Let us examine the structure of $(\mathbb{Z}^n, T^n), n \in \mathbb{N}$. A point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ is *open* if all coordinates are odd, and *closed* if each of the coordinates is even [17, 19]. These points are called *pure* and the other points in $\mathbb{Z}^n$ is called *mixed* [17, 19, 22]. For all subspaces of $(\mathbb{Z}^n, T^n), n \geq 2$, in Fig.1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 the symbols such as $\blacksquare$, $\bullet$, and a jumbo dot mean a pure closed point, a mixed point, and a pure open point, respectively.

For a set (X, k) in $\mathbb{Z}^n$, a pair of two points $x, y \in X$ are called *k-connected* if there is a sequence $(x_0, x_1, \ldots, x_m) \subset X$ such that $x_0 = x$, $x_m = y$ and further, x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbb{Z}}, m \geq 1$ [18]. The number m is called the *length* of this k-path [18]. For an adjacency relation k, a *simple k-path* in X is a sequence $(x_i)_{i\in[0,m]_{\mathbb{Z}}} \subset X$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$ [18].

For a set (X, k) in $\mathbb{Z}^n$, let us recall a (digital) k-neighborhood of a point $x_0 \in X$ as follows. The *k-neighborhood* of $x_0 \in X$ with radius ε, $\varepsilon \in \mathbb{N}$, is defined to be the following subset of X [5] (see also [6, 7, 8, 9, 12, 13, 14])

$$N_k(x_0, \varepsilon) := \{x \in X \mid \ l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \tag{2.2}$$

where $l_k(x_0, x)$ is the length of a *shortest simple k-path* from x_0 to x in X.

Hereafter, for a set $X \subset \mathbb{Z}^n$, considering a subspace $(X, T_X^n) \subset (\mathbb{Z}^n, T^n)$ with one of the k-adjacency relations in Proposition 2.1, we remind the notation $(X, k, T_X^n) := X_{n,k}$.

For a space $X_{n,k}$ and $x \in X$, by the *neighborhood* V of the point x is typically meant the existence of some open set $O_x \in T_X^n$ such that $x \in O_x \subset V$.

Furthermore, if the set $N_k(x_0, \varepsilon)$ in (2.2) is a *topological neighborhood* of x_0 in (X, T_X^n), then this set is called a (*computer topological*) *k-neighborhood of x with radius* $\varepsilon \in \mathbb{N}$ and is denoted by

$$N_k^*(x_0, \varepsilon). \tag{2.3}$$

For example, consider a space $X_{2,8}$, where $X = \{x_i | i \in [0,7]_{\mathbb{Z}}\}$ in Fig.1. Then we see that $N_8^*(x_0, 1) = \{x_i | i \in [0,4]_{\mathbb{Z}}\}$, $N_8^*(x_0, 2) = \{x_i | i \in [0,5]_{\mathbb{Z}}\}$, $N_8^*(x_0, 3) = X - \{x_7\}$, and $N_8^*(x_0, 4) = X$.

Let us now consider the set X with $X_{2,4}$ instead of $X_{2,8}$. Then we cannot have $N_4^*(x_0, \varepsilon)$, $\varepsilon \in \mathbb{N}$, because every open set $O_{x_0} \in T_X^2$ containing the point x_0 cannot have an open set O_{x_0} containing the point x_0 such that $O_{x_0} \subset N_4(x_0, \varepsilon)$, $\varepsilon \in \mathbb{N}$. But $N_4^*(x_1, 1) = \{x_0, x_1, x_2\}$ and $N_4^*(x_1, 2) = \{x_0, x_1, x_2, x_4, x_5\}$.

Therefore, we see that the current k-neighborhood in (2.3) is different from the (digital) k-neighborhood in (2.2) owing to the computer topological structure of $X_{n,k}$. Furthermore, the digital k-neighborhood in (2.2) and the k-neighborhood in (2.3) are useful to define the digital (k_0, k_1)-continuity in Definition 1 and three kinds of computer topological continuities (see Definitions 3, 4, and 5).

3. Four types of continuities in computer topology and their comparisons

In order to study a digital k-curve, a digital k-fundamental group, a closed k-surface, a digital connected sum, Euler characteristic of an object in $\mathbb{Z}^n$, relative digital homotopy, digital covering theory in [7, 8, 9, 10, 12, 13, 14, 15, 16], and so forth, we have used the following digital (k_0, k_1)-continuity in a fashion which is a generalization of digital continuities of [2, 21].

Definition 1 ([5, Digital (k_0, k_1)-continuity]). (see also [6, 7, 8, 9, 12, 13, 14]) For two discrete topological spaces with k_i-adjacency, $i \in \{0, 1\}$, (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a function $f : (X, k_0) \to (Y, k_1)$ is said to be digitally (k_0, k_1)-continuous at a point $x \in X$ if for any $N_{k_1}(f(x), \varepsilon) \subset Y$, there is $N_{k_0}(x, \delta) \subset X$ such that $f(N_{k_0}(x, \delta)) \subset N_{k_1}(f(x), \varepsilon)$.

Furthermore, we say that a map $f : (X, k_0) \to (Y, k_1)$ is digitally (k_0, k_1)-continuous if the map f is digitally (k_0, k_1)-continuous at every point $x \in X$.

Indeed, the digital (k_0, k_1)-continuity from (X, k_0) to (Y, k_1) implies the preservation of the k_0-connectivity of (X, k_0) into the k_1-connectivity of (Y, k_1). Furthermore, it turns out that digital (k_0, k_1)-continuity has the uniform (k_0, k_1)-continuity in [15].

Remark 3.1 ([15]). The digital (k_0, k_1)-continuity in Definition 1 is equivalent to the following: $f(N_{k_0}(x,1)) \subset N_{k_1}(f(x),1)$, which means that if x_1 and x_2 are k_0-connected, then the images $f(x_1)$ and $f(x_2)$ are k_1-connected or equal to each other [2].

Let us now recall four kinds of continuities in computer topology as follows.

Definition 2 ([1, Khalimsky continuity]). (see also [3, 19, 22]) For two spaces $(X, T_X^{n_0})$ and $(Y, T_Y^{n_1})$, a function $f : X \to Y$ is said to be Khalimsky continuous at a point $x \in X$ if f is continuous at the point x with the Khalimsky product topology. Furthermore, we say that a map $f : X \to Y$ is Khalimsky continuous if it is Khalimsky continuous at every point $x \in X$.

The current Khalimsky continuity need not preserve the preservation of the k_0-connectivity of $X_{n_0,k}$ into the k_1-connectivity of Y_{n_1,k_1} which is one of the essential requirements in digital and computer topology (see the map in Fig.1 or Example 3.2). Precisely, consider the map $f : X_{2,8} \to Y_{2,2}$ in Fig.1, where $Y = [1,4]_{\mathbb{Z}}$. Then, while the map f is a Khalimsky continuous map, it cannot be digitally $(8,2)$-continuous at the points x_5 and x_6. Let us consider the following example showing the importance of the notions of several continuities in computer topology such as *(k_0, k_1)-continuity, KD-(k_0, k_1)-continuity, and K-(k_0, k_1)-continuity.*

Example 3.2. Consider two spaces $(\mathbb{Z}, 2, T)$, $(\mathbb{Z}^2, 4, T^2)$, and the diagonal map $f : \mathbb{Z} \to \mathbb{Z}^2$ given by $f(x) = (x, x)$. Then,

(1) while the map is obviously Khalimsky continuous, it cannot preserve the 2-connectivity into the 4-connectivity.
(2) If we consider the map f with codomain $(\mathbb{Z}^2, 8, T^2)$ instead of $(\mathbb{Z}^2, 4, T^2)$, then f obviously transforms the 2-connectivity into the 8-connectivity. Namely, $f(N_2(x,1)) \subset N_8(f(x),1)$.
(3) $f(N_2^*(x,1)) \subset N_8^*(f(x),1)$.
(4) For any $\varepsilon \in \mathbb{N}$, there is no $N_4^*(f(0),\varepsilon)$ in $f(\mathbb{Z}) \subset \mathbb{Z}^2$ so that $f(N_2^*(0,1)) \not\subset N_4^*(f(0),\varepsilon)$ because there is no digital 4-neighborhood of $f(0)$ containing the smallest open set including the point $f(0)$ in $f(\mathbb{Z})_{2,4}$.

Motivated by Example 3.2, the following three continuities have been developed in computer topology. In terms of Definitions 1, 2 and Remark 3.1 we have established the following.

Definition 3 ([11, Khalimsky digital (k_0, k_1)-continuity]). For two spaces X_{n_0,k_0} and Y_{n_1,k_1}, a function $f : X \to Y$ is said to be KD-(k_0, k_1)-continuous at a point $x \in X$ if

(1) f is (Khalimsky) continuous at the point x; and
(2) f is digitally (k_0, k_1)-continuous at the point $x \in X$.

Furthermore, we say that a map $f : X \to Y$ is KD-(k_0, k_1)-continuous if the map f is KD-(k_0, k_1)-continuous at every point $x \in X$.

For example, the diagonal map f in Example 3.2 is KD-$(2,8)$-continuous.

Meanwhile, none of the conditions (1) and (2) of Definition 3 implies the other [11]. More precisely, consider the spaces $X_{2,8}$, $Y_{1,2}$, and the map $f: X_{2,8} \to Y_{1,2}$ in Fig.1, where $X = \{x_i | i \in [0,7]_{\mathbb{Z}}\}$ and $Y = [1,4]_{\mathbb{Z}}$. Then, while the map f is Khalimsky continuous, it cannot be KD-$(8,2)$-continuous at the points x_5 and x_6 in Fig.1. Furthermore, even though the condition (2) of Definition 3 is not related to the Khalimsky topology, the preservation of the k_0-connectivity of X_{n_0,k_0} into the k_1-connectivity of Y_{n_0,k_1} should be required for the study of a map $f: X_{n_0,k_0} \to Y_{n_0,k_1}$.

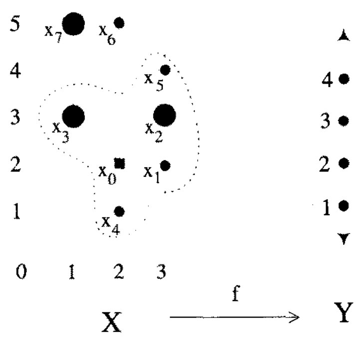

FIGURE 1

Now, by the use of the computer topological k-neighborhood in (2.3), the following continuity in a fashion was established in [11].

Definition 4 ([6, 11, (k_0,k_1)-continuity]). For two spaces X_{n_0,k_0} and Y_{n_1,k_1}, a function $f: X \to Y$ is said to be (k_0,k_1)-continuous at a point $x \in X$ if for any $N^*_{k_1}(f(x),\varepsilon) \subset Y$, there is $N^*_{k_0}(x,\delta) \subset X$ such that

$$f(N^*_{k_0}(x,\delta)) \subset N^*_{k_1}(f(x),\varepsilon),$$

where for some $\varepsilon \in \mathbb{N}, N^*_{k_1}(f(x),\varepsilon)$ is assumed to be existed. Furthermore, we say that a map $f: X \to Y$ is (k_0,k_1)-continuous if the map f is (k_0,k_1)-continuous at every point $x \in X$.

In Definition 4, if such the neighborhood $N^*_{k_1}(f(x),\varepsilon)$ does not exist, then we obviously say that f cannot be (k_0,k_1)-continuous at the point x.

As a generalization of $(k,2)$-continuity in [11] (see Remark 3.2 in [11]), we obtain the following.

Remark 3.3. The (k_0,k_1)-continuity of Definition 4 is equivalent to the following:

$$f(N^*_{k_0}(x,r)) \subset N^*_{k_1}(f(x),s),$$

where the number r is the least element of $\mathbb{N}$ such that $N^*_{k_0}(x,r)$ contains an open set including the point x (so $N^*_{k_0}(x,r) = N_{k_0}(x,r)$) and s is the least element of $\mathbb{N}$ such that $N^*_{k_1}(f(x), s)$ contains an open set including the point $f(x)$ (so $N^*_{k_1}(f(x), s) = N_{k_1}(f(x), s)$).

In Example 3.2, we observe that the map f is not $(2, 4)$- but $(2, 8)$-continuous. The current (k_0, k_1)-continuity is useful to develop essential tools in computer topology such as (k_0, k_1)-homotopy, (k_0, k_1)-homotopy equivalence, (k_0, k_1)-covering theory, and so forth.

Using Definitions 2 and 4, and Remark 3.3, we obtain the following.

Definition 5 ([11, Khalimsky (k_0, k_1)-continuity]). For two spaces X_{n_0,k_0} and Y_{n_1,k_1}, a function $f : X \to Y$ is said to be K-(k_0, k_1)-continuous at a point $x \in X$ if

(1) f is (Khalimsky) continuous at the point x; and

(2) f is (k_0, k_1)-continuous at the point x.

Furthermore, we say that a map $f : X \to Y$ is K-(k_0, k_1)-continuous if f is K-(k_0, k_1)-continuous at every point $x \in X$.

We see that none of the conditions (1) and (2) of Definition 5 implies the other [11]. More precisely, see the map f in Fig.1 and the points x_5 and x_6 in X. Then, while the map f is Khalimsky continuous, it cannot be an $(8, 2)$-continuous map at the points x_5 and x_6.

For a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$, the notions of Khalimsky continuity, (k_0, k_1)-continuity, KD-(k_0, k_1)-continuity, and K-(k_0, k_1)-continuity are different from each other and further, their usages depend on the spaces X_{n_0,k_0} and Y_{n_1,k_1}:

Remark 3.4. Consider a map $f : X_{1,2} \to Y_{n,k}$. Then we obtain the following.

(1) K-$(2, k)$-continuity of f implies KD-$(2, k)$-continuity of f, but the converse does not hold.

(2) None of $(2, k)$-continuity of f and KD-$(2, k)$-continuity of f implies the other.

For example, consider the map $f : A_{2,4} \to Y_{1,2}$ given by $f(a_1) = 1$ and $f(a_2) = 2$, where $A = \{a_1 = (0, 1), a_2 = (1, 1)\}$ and $Y = \{1, 2\} \subset \mathbb{Z}$. Then, while the map f is $(4, 2)$-continuous, it cannot be Khalimsky continuous at the point a_1 because $\{1\} \in T_Y$ and $\{a_1\} \notin T^2_A$.

Hereafter, we investigate some relations among Khalimsky continuity, KD-(k_0, k_1)-, (k_0, k_1)-, and K-(k_0, k_1)-continuities of a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$, $1 \leq n_i \leq 3, i \in \{0, 1\}$ (see Theorem 3.5 and Corollary 3.6). In [11], for a map $f : X_{n,k} \to Y_{1,2}$, various properties of the map f were studied. As a generalization of this property we obtain the following.

Theorem 3.5. *Let $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ be a map, $1 \leq n_i \leq 3, i \in \{0, 1\}$. Then K-$(k_0, k_1)$-continuity of f implies* KD-(k_0, k_1)*-continuity of f.*

Proof. First, in case $f : X_{1,2} \to Y_{1,2}$, it is obvious that K-$(2, 2)$-continuity of f is equivalent to KD-$(2, 2)$-continuity of f (see Corollary 3.5 in [11]) because for any points $x \in X$ and $y \in Y$, we obtain

$$N_2(x, 1) = N^*_2(x, 1) \quad \text{and} \quad N_2(y, 1) = N^*_2(y, 1).$$

Second, in case $f : X_{n,k} \to Y_{1,2}$, $2 \le n \le 3$, for any point $y \in Y$, we obtain $N_2(y,1) = N_2^*(y,1)$. Therefore, K-$(k,2)$-continuity of f obviously implies KD-$(k,2)$-continuity of f because the existence of $N_k^*(x,r)$ in Remark 3.3 implies $N_k(x,1) \subset N_k^*(x,r)$.

Third, in case $f : X_{1,2} \to Y_{n,k}$, $2 \le n \le 3$, the assertion that K-(k_0,k_1)-continuity of f implies KD-(k_0,k_1)-continuity of f is also successful because for any point $x \in X$, $N_2^*(x,1) = N_2(x,1)$ and further, for any point $y \in Y$ the existence of $N_k^*(y,s)$ in Remark 3.3 implies $N_k(y,1) \subset N_k^*(y,s)$ in Remark 3.3. Let us consider the following case. In Fig.2, consider a map $f : X_{1,2} \to Y_{2,4}$ for which $f(0) = y_0 = (0,0) \in Y$ and $f(1) = y_6$. Then, in $Y_{2,4}$ and $X_{1,2}$, we obtain the smallest 4- and 2-neighborhoods of $f(0)$ and 0 such as $N_4^*(f(0),6) = Y$ and $N_2^*(0,1) = X$, respectively. While $f(N_2^*(0,1)) \subset N_4^*(f(0),6)$ and further, f is Khalimsky continuous at the point $0 \in X$, we obtain $f(N_2(0,1)) \not\subseteq N_4(f(0),1)$, which means that K-$(k_0,k_1)$-continuity of f at the point $0 \in X$ cannot imply KD-$(2,4)$-continuity of f at the point $0 \in X$. Meanwhile, this map f cannot be K-$(2,4)$-continuous at the point $1 \in X$ either. Thus this map f cannot be a suitable counterexample of the assertion of Theorem 3.5.

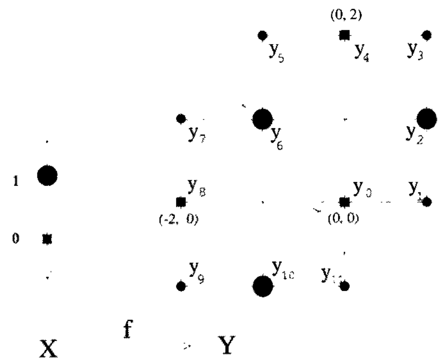

FIGURE 2

Fourth, let us now prove Theorem 3.5 for the case $2 \le n_i \le 3, i \in \{0,1\}$. Precisely, we now prove that K-(k_0,k_1)-continuity of f implies KD-(k_0,k_1)-continuity of f at each point $x_1 \in X$. If not, for some point $x_1 \in X$, we can find some K-(k_0,k_1)-continuous map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ such that $f(x_2) \notin N_{k_1}(f(x_1),1)$, where $x_2 \in N_{k_0}(x_1,1)$. Thus, motivated by Remarks 3.3 and 3.4, considering a pair of distinct points $x_1, x_2 \in X$ such that x_1 and x_2 are k_0-adjacent, we investigate the following nine cases according to the locations of both $x_1 \in X$ and $f(x_1) \in Y$ (see Fig.3).

(Case 1) Assume that the two points $x_1 \in X$ and $f(x_1) \in Y$ are *pure open points*. Since both $\{x_1\}$ and $\{f(x_1)\}$ are the smallest open sets containing

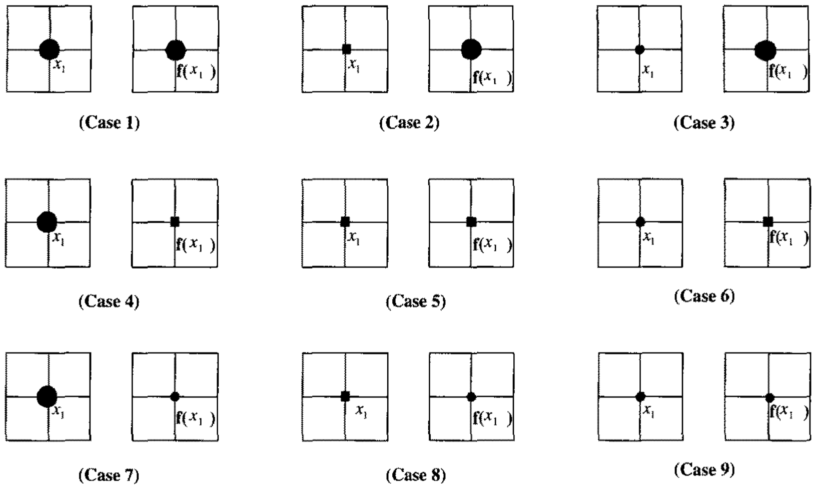

FIGURE 3

the points x_1 and $f(x_1)$, respectively, we see that $N_{k_0}(x_1,1) = N^*_{k_0}(x_1,1)$ and $N_{k_1}(f(x_1),1) = N^*_{k_1}(f(x_1),1)$. By the K-$(k_0,k_1)$-continuity of f at the point x_1, we obtain

$$f(N_{k_0}(x_1,1)) = f(N^*_{k_0}(x_1,1)) \subset N^*_{k_1}(f(x_1),1) = N_{k_1}(f(x_1),1),$$

which means that f is a KD-(k_0,k_1)-continuous map at the point x_1 by Remark 3.1.

(Case 2) Assume that x_1 is a *pure closed point* and $f(x_1)$ is a *pure open point.* Then, owing to the K-(k_0,k_1)-continuity of f at the point x_1, there is a smallest open set containing the point x_1 is $N^*_{3^{n_0}-1}(x_1,1) = N_{3^{n_0}-1}(x_1,1)$ if $k_0 = 3^{n_0}-1$ and O_{x_1} such that $O_{x_1} \subset N^*_{k_0}(x_1,r)$ for some $r \in \mathbb{N}$ in Remark 3.3 according to the k_0-adjacency of X_{n_0,k_0} and further, $N^*_{k_1}(f(x_1),1) = N_{k_1}(f(x_1),1)$ because $\{f(x_1)\} \in T^{n_1}_Y$. Due to the K-(k_0,k_1)-continuity of f at the point x_1, and

$$f(O_{x_1}) = \{f(x_1)\} \quad \text{and} \quad f(N^*_{k_0}(x_1,r)) = N^*_{k_1}(f(x_1),1).$$

Thus, $f(x_1)$ and $f(x_2)$ are equal to each other or k_1-adjacent, where $x_2 \in N_{k_0}(x_1,1)$. Therefore,

$$f(N_{k_0}(x_1,1)) \subset f(N^*_{k_0}(x_1,r)) \subset N^*_{k_1}(f(x_1),1) = N_{k_1}(f(x_1),1)$$

for any k_0-adjacency of X_{n_0,k_0}, which implies that f is KD-(k_0,k_1)-continuous at the point x_1.

(Case 3) Assume that x_1 is a *mixed point* and $f(x_1)$ is a *pure open point.* Then, due to the K-(k_0,k_1)-continuity of f, via Remark 3.3, there is $N^*_{k_0}(x_1,r)$ such that

$$f(N_{k_0}(x_1,1)) \subset f(N^*_{k_0}(x_1,r)) \subset N^*_{k_1}(f(x_1),1) = N_{k_1}(f(x_1),1)$$

because $\{f(x_1)\} \in T_Y^{n_1}$, where the number r is assumed in Remark 3.3. Thus the map f is KD-(k_0, k_1)-continuous at the point x_1 by Remark 3.1.

(Case 4) Assume that x_1 is a *pure open point* and $f(x_1)$ is a *pure closed point.* By the hypothesis of K-(k_0, k_1)-continuity of f at the point x_1 and Remark 3.3, take $N_{k_1}^*(f(x_1), s) \subset Y$, where s is assumed in Remark 3.3. Then, since the singleton $\{x_1\}$ is an open set, there is $N_{k_0}^*(x_1, 1) = N_{k_0}(x_1, 1) \subset X$ such that

$$f(N_{k_0}^*(x_1, 1)) \subset N_{k_1}^*(f(x_1), s). \tag{3.1}$$

Meanwhile, the formula in (3.1) need not imply KD-(k_0, k_1)-continuity of f at the point x_1. In other word, the number s in (3.1) need not be equal to $1 \in \mathbb{N}$. Thus, by the use of (3.1) and Remarks 3.1 and 3.3 we now strongly need to examine that K-(k_0, k_1)-continuity of f implies KD-(k_0, k_1)-continuity of f at the point x_1 for any k_0-adjacency of X_{n_0,k_0} and k_1-adjacency of Y_{n_1,k_1}. Namely, it suffices to prove that the K-(k_0, k_1)-continuous map f at any point $x_1 \in X$ is digitally (k_0, k_1)-continuous at the point x_1.

(Case 4-1) If $k_1 = 3^{n_1} - 1$ and $2 \leq n_1 \leq 3$ in (3.1), due to the K-$(k_0, 3^{n_1} - 1)$-continuity of f, then we obtain

$$f(N_{k_0}(x_1, 1)) \subset N_{3^{n_1}-1}(f(x_1), 1)$$

because $N_{3^{n_1}-1}^*(f(x_1), 1) = N_{3^{n_1}-1}(f(x_1), 1)$ and $N_{k_0}^*(x_1, 1) = N_{k_0}(x_1, 1)$ so that we may take $s = 1$ in (3.1), which means that the map f is KD-(k_0, k_1)-continuous at the point x_1 for any k_0-adjacency of X.

(Case 4-2) If $k_1 \neq 3^{n_1} - 1$ and $2 \leq n_1 \leq 3$ in (3.1), consider a point x_2 k_0-adjacent to x_1, i.e., $x_2 \in N_{k_0}(x_1, 1)$ and $x_1 \neq x_2$. Even though the point x_2 is a mixed point or a pure closed point depending on the k_0-adjacency of X_{n_0,k_0}, $f(x_2)$ should be a pure closed point in $N_{3^{n_1}-1}(f(x_1))$, i.e., $f(x_2)$ can be neither a pure open point nor a mixed point in $N_{3^{n_1}-1}(f(x_1))$. If not, first suppose that $f(x_2) \notin N_{3^{n_1}-1}(f(x_1))$, then the map f cannot be a K-(k_0, k_1)-continuous map at the point x_2. Second, suppose that $f(x_2)$ is a pure open point in $N_{3^{n_1}-1}(f(x_1))$. Then we have a contradiction to the K-(k_0, k_1)-continuity of f at the point x_2. Precisely, by the K-(k_0, k_1)-continuity of f at the points x_1 and x_2, $f^{-1}(\{f(x_2)\})$ contains the set $\{x_1, x_2\}$ because $\{f(x_2)\} \in T_Y^{n_1}$ and a smallest open set in $T_X^{n_0}$ containing the point $x_2 \in N_{k_0}(x_1, 1)$ with $x_1 \neq x_2$ should include the pure open point x_1 (see Fig.4). Thus we obtain $f(x_1) \in \{f(x_2)\}$, which leads to a contradiction to the hypothesis that $f(x_1) \notin \{f(x_2)\}$. Third, suppose that $f(x_2)$ is a mixed point in $N_{3^{n_1}-1}(f(x_1))$. Then there is always a smallest open set $O_{f(x_2)} \in T_Y^{n_1}$ such that $f(x_2) \in O_{f(x_2)}$ and $f(x_1) \notin O_{f(x_2)}$ by the topological property of Y_{n_1,k_1} because $f(x_1)$ is a pure closed point in Y (see Fig.4). By the K-(k_0, k_1)-continuity of f at the points x_1 and x_2, $f^{-1}(O_{f(x_2)})$ should contain the set $\{x_1, x_2\}$ because $x_2 \in N_{k_0}(x_1, 1)$ with $x_1 \neq x_2$, and the fact that any smallest open set containing the point x_2 should include the point x_1, which contradicts $f(x_1) \notin O_{f(x_2)}$.

Thus $f(x_2)$ should be a pure closed point in $N_{3^{n_1}-1}(f(x_1))$. Precisely, $f(x_2) = f(x_1) \in N_{k_1}(f(x_1),1)$, which means that the map f is KD-(k_0,k_1)-continuous at the point x_1.

Now, if we intensively investigate several examples in $\mathbb{Z}^2$ and $\mathbb{Z}^3$ testifying Case 4-2, then it is helpful to apply Case 4-2 in discrete geometry and its applications in relation to coding theory, computer science, informatics and further, prove Cases 5-2, 6-2, 7-2, 8-2, and 9-2 below.

Precisely, let us examine the cases X_{n_0,k_0} and Y_{n_1,k_1} in Case 4-2, where $(n_0,k_0) \in \{(2,4),(2,8)\}$ and $(n_1,k_1) \in \{(2,4),(3,6),(3,18)\}$ (see Fig.4). Similarly, we can examine the following cases $(n_0,k_0) \in \{(3,6),(3,18),(3,26)\}$ and $(n_1,k_1) \in \{(2,4),(3,6),(3,18)\}$:

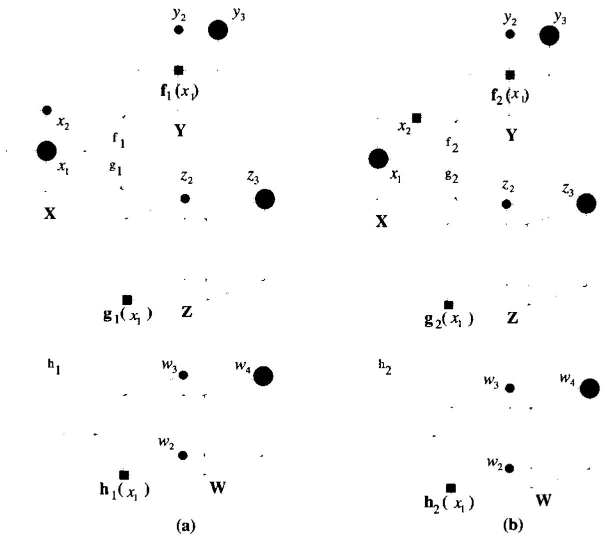

FIGURE 4

(Case 4-2-1) Let us investigate the case $(n_1,k_1) = (2,4)$ and $(n_0,k_0) = (2,4)$ with K-$(4,4)$-continuity. To be specific, assume a K-$(4,4)$-continuous map $f_1 : X_{2,4} \to Y_{2,4}$ in Fig.4(a), where $X = \{x_1,x_2\}$ in which x_1 and x_2 are 4-adjacent and $Y = \{y_1 := f_1(x_1), y_2, y_3\}$ is 4-connected. Let us now examine $f_1(x_2)$ with K-$(4,4)$-continuity of f_1 at the points x_1 and x_2. By the K-$(4,4)$-continuity of f_1 at the pure open point x_1 and a mixed point x_2, we obtain

$$(3.2) \qquad \left\{ \begin{array}{r} f_1(N_4(x_1,1)) = f_1(N_4^*(x_1,1)) \subset N_4^*(f_1(x_1),s); \\ f_1(N_4^*(x_1,1)) \subset N_8(f_1(x_1)) \end{array} \right\}$$

because $f_1(x_1)$ is a pure closed point and $\{x_1\} \in T_X^2$, where the number s is assumed in Remark 3.3. Then, since the point $x_2 \in N_4(x_1,1)$ with $x_1 \neq x_2$ is a mixed point (see Fig.4(a)), $f_1(x_2)$ should be a pure closed point in $N_8(f_1(x_1))$.

If not, first suppose $f_1(x_2) \notin N_8(f_1(x_1))$, then the map f cannot be a K-$(4,4)$-continuous map at the point x_2 because the smallest open set containing the point x_2 is the set X and for a smallest open set containing the point $f_1(x_2)$, denoted by $O_{f_1(x_2)} \in T_Y^2$, we obtain $X \subset f_1^{-1}(O_{f_1(x_2)})$, which contradicts to the Khalimsky continuity of f_1 at the points x_1 and x_2. Second, suppose that $f_1(x_2)$ is a pure open point in $N_8(f_1(x_1))$ such as $f_1(x_2) = y_3 \in Y$ in Fig.4(a). Then we also have a contradiction to the K-$(4,4)$-continuity of f_1 at the point x_2. Precisely, by the K-$(4,4)$-continuity of f_1 at the points x_1 and x_2, we obtained $f_1^{-1}(\{y_3\}) = \{x_1, x_2\}$. To be specific, $\{y_3\}$ is an open set and for $x_2 \in N_4^*(x_1, 1) = N_4(x_1, 1)$, any smallest open set containing the point x_2 should include the point x_1 from the topological property of $X_{2,4}$ because the point x_1 is a pure open point in X (see Fig.4(a)), which contradicts that $f_1(x_1) = y_1 \notin \{y_3\}$. Third, suppose that $f_1(x_2)$ is a mixed point such as $f_1(x_2) = y_2 \in Y$ in Fig.4(a). Then we also have a contradiction to the K-$(4,4)$-continuity of f_1 at the point x_2. Precisely, by the K-$(4,4)$-continuity of f_1 at the points x_1 and x_2, $f_1^{-1}(\{y_2, y_3\}) = \{x_1, x_2\}$ because $\{y_2, y_3\}$ is the smallest open set containing the point y_2 and further, $\{x_1, x_2\}$ is also the smallest open set containing the point x_2, which contradicts that $f_1(x_1) = y_1 \notin \{y_2, y_3\}$. Thus $f_1(x_2)$ should be a pure closed point satisfying (3.2). Therefore the map f_1 is KD-$(4,4)$-continuous at the point x_1 because $f_1(x_2) = f_1(x_1) \in N_4(f_1(x_1), 1)$.

(Case 4-2-2) Let us investigate the following case $(n_1, k_1) = (3, 18)$, $(n_0, k_0) = (2, 4)$ with K-$(4,18)$-continuity (see Fig.4(a)). For instance, consider the spaces $X_{2,4}$, $Z_{3,18}$, and the map $g_1 : X_{2,4} \to Z_{3,18}$ in Fig.4(a), where $Z = \{z_1 := g_1(x_1), z_2, z_3\}$ is 18-connected. Let us now examine $g_1(x_2)$. Then, the point $g_1(x_2)$ should be a pure closed point in $N_{26}(g_1(x_1))$ satisfying (3.3) below.

$$(3.3) \qquad \left\{ \begin{array}{r} g_1(N_4(x_1,1)) = g_1(N_4^*(x_1,1)) \subset N_{18}^*(g_1(x_1), s); \\ g_1(N_4^*(x_1,1)) \subset N_{26}(g_1(x_1)) \end{array} \right\}$$

because the smallest open set containing the point $g_1(x_1)$ is the set Z, where the number s is assumed in Remark 3.3. If not, first suppose $g_1(x_2) \notin N_{26}(g_1(x_1))$, then the map f cannot be a K-*(4, 18)-continuous map at the point* x_2. Second, suppose that the point $g_1(x_2) = z_3$ which is a pure open point. Then we must take $g_1^{-1}(\{z_3\}) = \{x_1, x_2\}$ owing to the K-$(4,18)$-continuity of g_1 at the point x_2, because $\{z_3\} = \{g_1(x_2)\} \in T_Z^3$, which contradicts that $g_1(x_1) = z_1 \notin \{z_3\}$. Thus $g_1(x_2)$ cannot be a pure open point. Third, suppose that $g_1(x_2) = z_2$ is a mixed point. Then, since the smallest open set contains the point x_2 is the set $\{x_1, x_2\}$, we also have a contradiction to the K-$(4,18)$-continuity of g_1 at the point x_2 by the same method as the above because $g_1^{-1}(\{z_2, z_3\}) = \{x_1, x_2\}$ and $\{z_2, z_3\}$ is also the smallest open set containing the point z_2. Thus $g_1(x_2)$ cannot be a mixed point in $N_{26}(g_1(x_1))$. Therefore, $g_1(x_1) = g_1(x_2)$ by (3.3), which means that the map g_1 is KD-$(4,18)$-continuous at the point x_1 because $g_1(x_2) \in N_{18}(g_1(x_1), 1)$.

(Case 4-2-3) Let us consider $(n_1, k_1) = (3,6)$, $(n_0, k_0) = (2,4)$ with K-$(4,6)$-continuity. For instance, consider the spaces $X_{2,4}$, $W_{3,6}$, and the map $h_1 : X_{2,4} \to W_{3,6}$ in Fig.4(a), where $W = \{w_1 := h_1(x_1), w_2, w_3, w_4\}$ is 6-connected. Let us now examine $h_1(x_2)$. Then, the point $h_1(x_2)$ should be a pure closed point in $N_{26}(h_1(x_1))$. If not, first suppose $h_1(x_2) \notin N_{26}(h_1(x_1))$. Then the map h_1 cannot be a K-$(4,6)$-continuous map at the point x_2. Second, suppose that $h_1(x_2) = w_4$ which is a pure open point. Then we must take $h_1^{-1}(\{w_4\}) = \{x_1, x_2\}$ owing to the K-$(4,6)$-continuity of h_1 at the point x_2 because the smallest open set containing the point x_2 is exactly the set $\{x_1, x_2\}$ and $\{w_4\} \in T_W^3$, which contradicts that $h_1(x_1) = w_1 \notin \{w_4\}$. Third, suppose $h_1(x_2) \in \{w_2, w_3\}$ which is a mixed point in $N_{26}(h_1(x_1))$. Then, due to the K-$(4,6)$-continuity of h_1 at the point x_2, we must take $h_1^{-1}(\{w_2, w_3, w_4\}) = \{x_1, x_2\}$ because the smallest open set containing the point w_2 or w_3 is exactly the set $\{w_2, w_3, w_4\}$, which contradicts that $h_1(x_1) = w_1 \notin \{w_2, w_3, w_4\}$. Thus $h_1(x_2)$ should be the pure closed point satisfying (3.4) below.

$$
\left\{
\begin{array}{r}
h_1(N_4(x_1,1)) = h_1(N_4^*(x_1,1)) \subset N_{18}^*(h_1(x_1), s); \\
h_1(N_4^*(x_1,1)) \subset N_{26}(f_1(x_1))
\end{array}
\right\}
\tag{3.4}
$$

because the smallest open set containing the point $h_1(x_1)$ is the set W, where the number s is assumed in Remark 3.3.

Thus $h_1(x_1) = h_1(x_2)$ in (3.4), which makes the map h_1 be KD-$(4,6)$-continuous at the point x_1 because $h_1(x_2) \in N_6(h_1(x_1), 1)$.

Furthermore, let us examine further the case $(n_0, k_0) = (2,8)$ instead of $(2,4)$ and $(n_1, k_1) \in \{(2,4), (3,6), (3,18)\}$. Precisely, consider the spaces $X_{2,8}$, $Y_{2,4}$, $Z_{3,18}$, and $W_{3,6}$ in Fig.4(b), and the maps

$$
f_2 : X \to Y,\ g_2 : X \to Z,\ h_2 : X \to W,
$$

where $X = \{x_1, x_2\}$ in which x_1 and x_2 are 8-adjacent,

$Y = \{y_1 := f_2(x_1), y_2, y_3\}$ is 4-connected,
$Z = \{z_1 := g_2(x_1), z_2, z_3\}$ is 18-connected, and
$W = \{w_1 := h_2(x_1), w_2, w_3, w_4, w_5\}$ is 6-connected.

Then, we now prove that K-$(8,4)$-, K-$(8,18)$-, and K-$(8,6)$-continuities of f_2, g_2, and h_2 imply KD-$(8,4)$-, KD-$(8,18)$-, and KD-$(8,6)$-continuities of f_2, g_2, and h_2, respectively.

(Case 4-2-4) Let us consider the two spaces $X_{2,8}$ and $Y_{2,4}$ in Fig.4(b). By the hypothesis of K-$(8,4)$-continuity of f_2 at the point x_1, we see that

$$
\left\{
\begin{array}{r}
f_2(N_8(x_1,1)) = f_2(N_8^*(x_1,1)) \subset N_4^*(f_2(x_1), s); \\
f_2(N_8^*(x_1,1)) \subset N_8(f_2(x_1))
\end{array}
\right\}
\tag{3.5}
$$

because the smallest open set containing the point $f_2(x_1)$ is the set Y and $\{x_1\} \in T_X^2$, where the number s is assumed in Remark 3.3. Then, $f_2(x_2)$ should be a pure closed point satisfying (3.5) by the same method as the case

$(n_1, k_1) = (2,4) = (n_0, k_0)$ above. Thus the map f_2 satisfies digital $(8,4)$-continuity at the point x_1 because $f_2(x_2) = f_2(x_1) \in N_4(f_2(x_1), 1)$, which means that f_2 is KD-$(8,4)$-continuous at the point x_1.

(Case 4-2-5) Consider the two spaces $X_{2,8}$, $Z_{3,18}$, and the map $g_2 : X \to Z$ in Fig.4(b). By the hypothesis of K-$(8,18)$-continuity of g_2 at the points x_1 and x_2, we obtain

$$\left\{ \begin{array}{r} g_2(N_8(x_1,1)) = g_2(N_8^*(x_1,1)) \subset N_{18}^*(g_2(x_1), s); \\ g_2(N_8^*(x_1,1)) \subset N_{26}(g_2(x_1)) \end{array} \right\} \tag{3.6}$$

because the smallest open set containing the point $g_2(x_1)$ is the set Z, where the number s is assumed in Remark 3.3. Then $g_2(x_2)$ should be a pure closed point satisfying (3.6) by the same method as the case $(n_1, k_1) = (3, 18)$ and $(n_0, k_0) = (2,4)$ above. Thus the map g_2 satisfies KD-$(8,18)$-continuity at the point x_1 because $g_2(x_2) \in N_{18}(g_1(x_1), 1)$.

(Case 4-2-6) Consider the two spaces $X_{2,8}$, $W_{3,6}$, and the map $h_2 : X \to W$ in Fig.4(b). By the hypothesis of K-$(8,6)$-continuity of h_2 at the points x_1 and x_2, we obtain

$$\left\{ \begin{array}{r} h_2(N_8(x_1,1)) = h_2(N_8^*(x_1,1)) \subset N_6^*(h_2(x_1), s); \\ h_2(N_8^*(x_1,1)) \subset N_{26}(h_2(x_1)) \end{array} \right\} \tag{3.7}$$

because the smallest open set containing the point $h_2(x_1)$ is the set W, where the number s is assumed in Remark 3.3. Then $h_2(x_2)$ should be a pure closed point satisfying (3.7) by the same method as the case $(n_1, k_1) = (3,6)$ and $(n_0, k_0) = (2,4)$ above. Thus $h_2(x_1) = h_2(x_2)$, which means that the map h_2 satisfies KD-$(8,6)$-continuity at the point x_1 because $h_2(x_2) \in N_6(h_2(x_1), 1)$.

(Case 5) Consider the case that x_1 is a *pure closed point* and $f(x_1)$ is a *pure closed point.*

(Case 5-1) If $k_1 = 3^{n_1} - 1$, then K-(k_0, k_1)-continuity of $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ implies KD-(k_0, k_1)-continuity of f for any X_{n_0,k_0} because

$$N_{3^{n_1}-1}(f(x_1), 1) = N_{3^{n_1}-1}^*(f(x_1), 1) \quad \text{and}$$

for some $N_{k_0}^*(x, r) \subset X_{n_0,k_0}$,

$$f(N_{k_0}^*(x, r)) \subset N_{3^{n_1}-1}(f(x_1), 1) = N_{3^{n_1}-1}^*(f(x_1), 1).$$

Thus, for any point $x_1 \in X$, $f(N_{k_0}(x_1, 1)) \subset N_{3^{n_1}-1}(f(x_1), 1))$ because if there is $N_{k_0}^*(x_1, r)$, then $N_{k_0}(x_1, 1) \subset N_{k_0}^*(x_1, r)$, where the number r is assumed in Remark 3.3.

(Case 5-2) If $k_1 \neq 3^{n_1} - 1$, consider a pure closed point $x_1 \in X$. Then the point $x_2 \in N_{k_0}(x_1, 1)$ should be a mixed point or a pure open point depending on the k_0-adjacency of X_{n_0,k_0}. By Remark 3.3 and the K-(k_0, k_1)-continuity of f at the two distinct points x_1 and $x_2 \in N_{k_0}(x_1, 1)$, we obtain

$$f(x_2) \in N_{k_1}(f(x_1), 1). \tag{3.8}$$

If not, suppose that there is a point $x_2 \in N_{k_0}(x_1, 1)$ with $x_1 \neq x_2$ which does not satisfy (3.8). Then we have a contradiction to the K-(k_0, k_1)-continuity of f at the point x_2 by the similar method as Case 4-2.

Precisely, for the point x_2 which is a mixed point or a pure open point, $f(x_2)$ should be a pure closed point or a mixed point in $N_{k_1}(f(x_1))$. If not, suppose that $f(x_2)$ is a pure open point in $N_{k_1}(f(x_1))$. Then we have a contradiction to the K-(k_0, k_1)-continuity of f at the point x_2 by the same method as Case 4-2 because $k_1 \neq 3^{n_1} - 1$. Next, suppose that $f(x_2)$ is a mixed point such that $f(x_2) \notin N_{k_1}(f(x_1))$. Then we also have a contradiction to the K-(k_0, k_1)-continuity of f at the point x_2. Thus $f(x_2)$ should be a pure closed point or a mixed point in $N_{k_1}(f(x_1))$. Therefore the map f is digitally (k_0, k_1)-continuous at the point x_1, which means that the map f is KD-(k_0, k_1)-continuous at the point x_1.

As an example related to Case 5-2, in cases $(n_1, k_1) \in \{(2,4),(3,18),(3,6)\}$ and $(n_0, k_0) \in \{(2,4),(2,8)\}$, see Fig.5(a) and (b) with the same method as Case 4-2. Furthermore, even if we can examine the cases such as $(n_0, k_0) \in \{(3,6),(3,18),(3,26)\}$ and $(n_1, k_1) \in \{(2,4),(3,6),(3,18)\}$, we now investigate the cases $(n_1, k_1) \in \{(2,4),(3,6)\}$ and $(n_0, k_0) \in \{(3,18),(3,26)\}$ in Fig.5(c), where $X = \{x_1, x_2\}$ in which x_1 and x_2 are 18-adjacent and $Z = \{z_1 := g_3(x_1), z_2, z_3, z_4\}$ is 6-connected. Assume a K-$(18,6)$-continuous map $g_3 : X_{3,18} \to Z_{3,6}$ (see Fig.5(c)). Let us examine $g_3(x_2)$. By the K-$(18,6)$-continuity of g_3 at the pure closed point x_1 and a mixed point $x_2 \in N_{18}(x_1, 1)$, we obtain

$$(3.9) \qquad \left\{ \begin{array}{r} g_3(N_{18}(x_1,1)) = g_3(N^*_{18}(x_1,1)) \subset N^*_6(g_3(x_1), s); \\ g_3(N^*_{18}(x_1,1)) \subset N_{26}(g_3(x_1)) \end{array} \right\}$$

because the smallest open set containing the point $g_3(x_1)$ is the set Z, where the number s is considered in Remark 3.3. Then $g_3(x_2)$ should be a pure closed point or a certain mixed point in $N_6(g_3(x_1))$.

If not, suppose that $g_3(x_2)$ is a pure open point in $N_{26}(g_3(x_1))$ such as $g_3(x_2) = z_4 \in Z$ in Fig.5(c). Then we have a contradiction to the K-$(18,6)$-continuity of g_3 at the point x_2 because $g_3(N^*_{18}(x_2,1)) \not\subseteq N^*_6(g_3(x_2),1))$, where $N^*_6(g_3(x_2),1) = \{z_3, z_4\}$ because $\{g_3(x_2)\} \in T^3_Z$.

Besides, the mapping $g_3(x_2) = z_3 \in Z$ in Fig.5(c) cannot be successful either owing to the K-$(18,6)$-continuity of g_3 at the point x_2 because $g_3(N^*_{18}(x_2,1)) \not\subseteq N^*_6(g_3(x_2),1)$, where $\{z_2, z_3, z_4\} = N^*_6(g_3(x_2),1)$ because the set $\{z_3, z_4\}$ is the smallest open set in T^3_Z containing the point z_3.

Besides, the smallest open set containing the point z_2 is the set $\{z_2, z_3, z_4\}$. Thus $N^*_6(z_2, 2)$ contains the point $g_3(x_1)$. Thus, by the hypothesis of the K-$(18,6)$-continuity of g_3 at the points x_1 and x_2, we obtain $g_3(N^*_{18}(x_1,1)) \subset N^*_6(z_2, 2)$. Thus $g_3(x_2)$ should be equal to the point $g_3(x_1)$ or $z_2 \in N_6(g_3(x_1))$ by (3.9), which means that g_3 is a KD-$(18,6)$-continuous map at the point x_1. The other cases in Fig.5(c) can be also examined to prove that K-(k_0, k_1)-continuity implies KD-(k_0, k_1)-continuity by the same method as the above (see

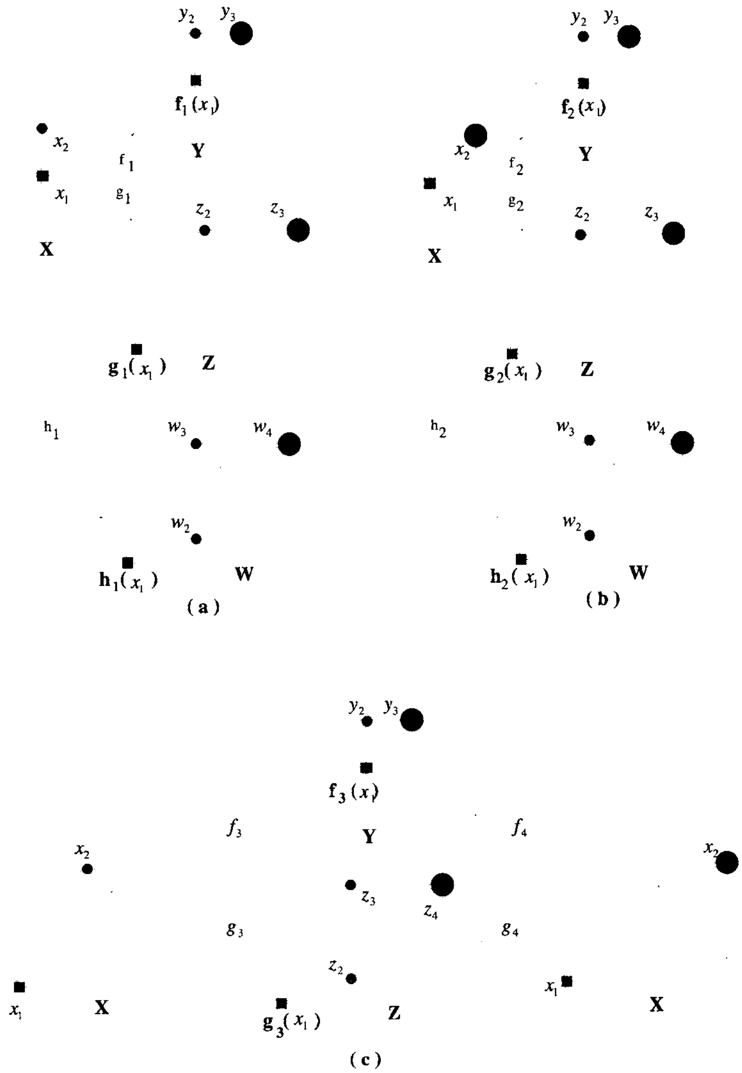

FIGURE 5

Fig.5(c)), where $(k_0, k_1) \in \{(6,4), (6,6), (6,18), (18,4), (18,18), (26,4), (26,6), (26,18)\}$.

(Case 6) Assume that x_1 is a *mixed point* and $f(x_1)$ is a *pure closed point.*

(Case 6-1) If $k_1 = 3^{n_1} - 1$, then K-(k_0, k_1)-continuity of f leads to KD-(k_0, k_1)-continuity of f for any X_{n_0,k_0} by the same method as Cases 4-1 and 5-1.

(Case 6-2) Let us consider the case that $k_1 \neq 3^{n_1} - 1$. Then the point $x_2 \in N_{k_0}(x_1, 1)$ (see Fig.6(a)) should be a mixed point, a pure open point, or a pure closed point depending on the k_0-adjacency of X_{n_0,k_0} (see Fig.6).

First, if the point x_2 is a pure closed point in $N_{k_0}(x_1, 1)$, then $f(x_2)$ should be a pure closed point in $N_{k_1}(f(x_1))$ and finally $f(x_2) = f(x_1)$ owing to the K-(k_0, k_1)-continuity of f at the points x_1 and x_2 by the same method as Case 5-2. Second, if the point x_2 is a pure open point in $N_{k_0}(x_1, 1)$ (see Fig.6(a)), then $f(x_2)$ should be a pure closed point or a mixed point in $N_{k_1}(f(x_1))$ owing to the K-(k_0, k_1)-continuity of f at the points x_1 and x_2 by the same method as Case 5-2.

Third, if the point x_2 is a mixed point in $N_{k_0}(x_1, 1)$, then $f(x_2)$ can be mapped into the point $f(x_1)$ or a mixed point in $N_{k_1}(f(x_1))$, depending on the k_1-adjacency of Y_{n_1,k_1}, owing to the K-(k_0, k_1)-continuity of f at the points x_1 and x_2.

Thus it turns out that K-(k_0, k_1)-continuity of $f : X_{k_0,n_0} \to Y_{k_1,n_1}$ implies KD-(k_0, k_1)-continuity of f.

For example, a K-$(4, 4)$-continuous map $f_1 : X_{2,4} \to Y_{2,4}$ can be assumed in two fashions depending on the location of the point $x_2 \in N_4(x_1)$ in Fig.6(a) and $Y = \{y_1 := f_1(x_1), y_2, y_3\}$ is 4-connected. Then the point $x_2 \in N_4(x_1, 1)$ with $x_1 \neq x_2$ can be assumed to be a pure closed point or a pure open point (see Fig.6(a)). Let us now examine $f_1(x_2)$. By the K-$(4, 4)$-continuity of f_1 at the mixed point x_1, we obtain

$$\left\{ \begin{array}{r} f_1(N_4(x_1, 1)) = f_1(N_4^*(x_1, 1)) \subset N_4^*(f_1(x_1), s); \\ f_1(N_4^*(x_1, 1)) \subset N_8(f_1(x_1)) \end{array} \right\} \tag{3.10}$$

because $N_4^*(x_1, 1) = X = N_4(x_1, 1)$ and the smallest open set containing the point $f_1(x_1)$ is the set Y, where the number s is considered in Remark 3.3. If the point $x_2 \in X$ is a pure closed point, then $f_1(x_2)$ should be a pure closed point in $N_4(f_1(x_1))$, *i.e.*, $f_1(x_2) \notin \{y_2, y_3\}$ owing to the K-$(4, 4)$-continuity of f_1 at the points x_1 and x_2 and finally, $f_1(x_1) = f_1(x_2)$ by (3.10). If the point x_2 is a pure open point (see Fig.6(a)), then $f_1(x_2)$ should be a mixed point or a pure closed point in $N_4(f_1(x_1))$ owing to the K-$(4, 4)$-continuity of f_1 at the points x_1 and x_2. Besides, assume a K-$(8, 4)$-continuous map $f_2 : X \to Y$ with the same hypothesis above (see Fig.6(b)). Now for a point $x_2 \in N_8(x_1, 1) - N_4(x_1, 1)$, let us examine $f_2(x_2)$. Since the point x_2 is a mixed point (see Fig.6(b)), $f_2(x_2)$ should be equal to the point $f_2(x_1)$ or a mixed point $y_2 \in N_4(f_2(x_1))$ owing to the K-$(8, 4)$-continuity of f_2 at the points x_1 and x_2, which means that the map f_2 is a KD-$(8, 4)$-continuous map at the point x_1. The other cases in Fig.6(c) such as

$$f_3 : Z_{3,18} \to Y_{2,4} \quad \text{and} \quad g_3 : Z_{3,18} \to W_{3,6}$$

$$f_4 : Z_{3,26} \to Y_{2,4} \quad \text{and} \quad g_4 : Z_{3,26} \to W_{3,6}$$

can be also examined to prove the assertion by the same method above.

(Case 7) Assume that x_1 is a *pure open point* and $f(x_1)$ is a *mixed point.*

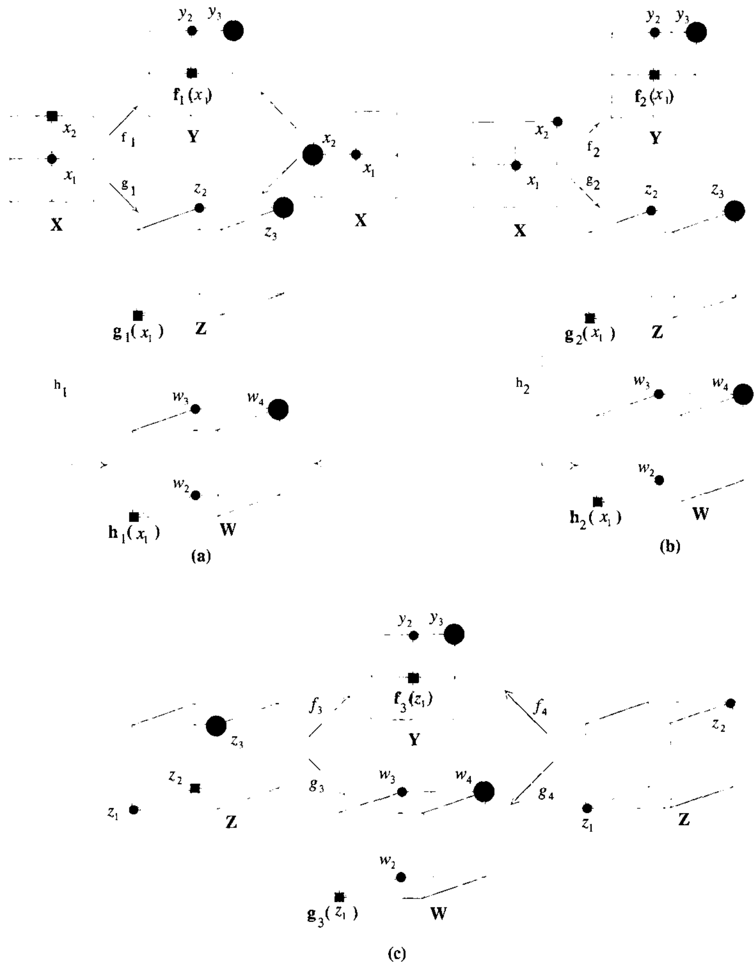

FIGURE 6

(Case 7-1) If $k_1 = 3^{n_1} - 1$, then K-(k_0, k_1)-continuity of f leads to KD-(k_0, k_1)-continuity of f for any k_0-adjacency of X_{n_0,k_0} by the same method as Case 6-1.

(Case 7-2) Let us consider the case $k_1 \neq 3^{n_1} - 1$. Then the point $x_2 \in N_{k_0}(x_1, 1)$ should be a mixed point or a pure closed point in $N_{k_1}(f(x_1), 1)$ depending on the k_0-adjacency of X_{n_0,k_0}. If the point x_2 is a mixed point in $N_{k_0}(x_1, 1)$, then $f(x_2)$ should be equal to the point $f(x_1)$ or a pure closed point in $N_{k_1}(f(x_1))$ because $k_1 \neq 3^{n_1} - 1$. Precisely, if $f(x_2) \notin N_{k_1}(f(x_1))$, then we have a contradiction to the K-(k_0, k_1)-continuity of f at the point x_2 by the similar methods as Case 4-2. Next, suppose that $f(x_2)$ is a mixed point in $N_{k_1}(f(x_1), 1)$ and $f(x_1) \neq f(x_2)$, then we also have a contradiction to the K-(k_0, k_1)-continuity of f at the point x_2. Besides, if $x_2 \in N_{k_0}(x_1, 1)$ is a pure closed point, then $f(x_2)$ should be a pure closed point in $N_{k_1}(f(x_1))$ or the point $f(x_1)$ owing to the K-(k_0, k_1)-continuity of f at the points x_1 and x_2.

Consequently, K-(k_0, k_1)-continuity of $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is proved to be KD-(k_0, k_1)-continuity of f.

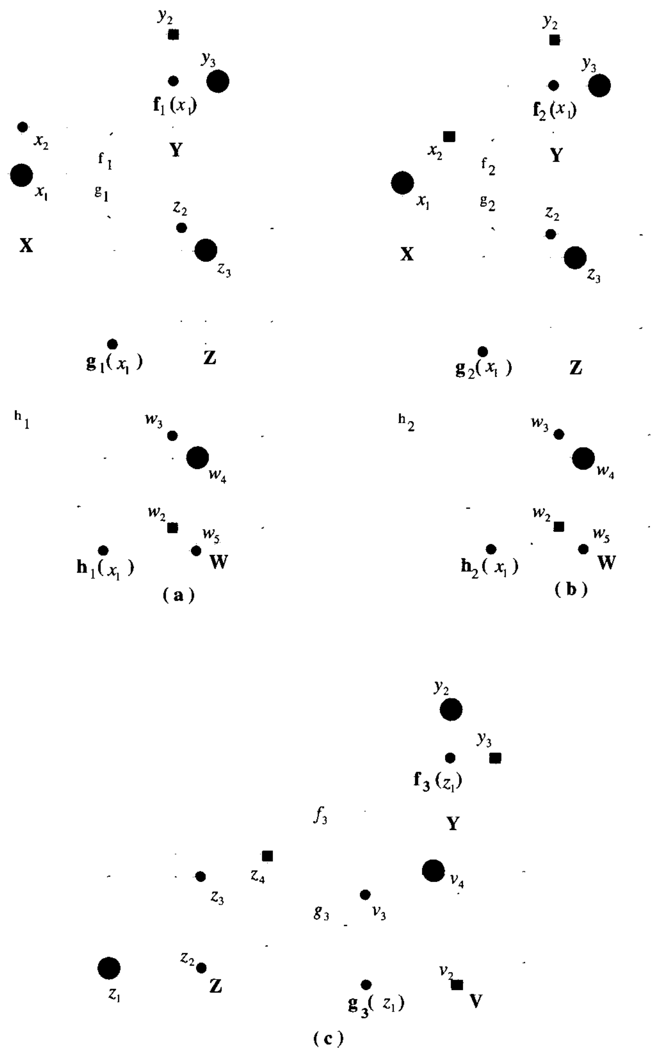

FIGURE 7

For example, assume a K-$(4,4)$-continuous map $f_1 : X_{2,4} \to Y_{2,4}$ in Fig.7(a), where $X = \{x_1, x_2\}$ in which x_1 and x_2 are 4-adjacent and $Y = \{y_1 := f_1(x_1), y_2, y_3\}$ is 4-connected. Precisely, let us examine $f_1(x_2)$. By the K-$(4,4)$-continuity of f_1 at the pure open point x_1 in Fig.7(a) we obtain

$$
(3.11) \qquad \left\{ \begin{array}{r} f_1(N_4(x_1,1)) = f_1(N_4^*(x_1,1)) \subset N_4^*(f_1(x_1),s); \\ f_1(N_4^*(x_1,1)) \subset N_8(f_1(x_1)) \end{array} \right\}
$$

because $N_4^*(f_1(x_1),1) = Y$ and $\{x_1\} \in T_X^2$, where the number s is considered in Remark 3.3. Thus, for the mixed point x_2, $f_1(x_2)$ should be equal to the point $f(x_1)$ or the point $y_2 \in N_4(f_1(x_1))$ owing to the K-$(4,4)$-continuity of f_1 at the points x_1 and x_2 by (3.11). Besides, assume a K-$(8,4)$-continuous map $f_2 : X_{2,8} \to Y_{2,4}$, where $X = \{x_1, x_2\}$ with 8-adjacency and $Y = \{f_2(x_1), y_2, y_3\}$ above (see Fig.7(b)). Let us now examine $f_2(x_2)$. Then, for the pure closed point $x_2 \in N_8(x_1,1) - N_4(x_1,1)$ (see Fig.7(b)), $f_2(x_2)$ should be either a pure closed point in $N_8(f_2(x_1))$ such as y_2 or $f_2(x_1)$ owing to the K-$(8,4)$-continuity of f_2 at the points x_1 and x_2, which means that the map f_2 is a KD-$(8,4)$-continuous map at the point x_1.

Similarly, both a K-$(6,4)$-continuous map $f_3 : Z_{3,6} \to Y_{2,4}$ and a K-$(6,6)$-continuous map $g_3 : Z_{3,6} \to V_{3,6}$ in Fig.7(c) can be proved to be KD-$(6,4)$- and KD-$(6,6)$-continuous maps, respectively, where $V = \{g_3(z_1), v_2, v_3, v_4\}$ and $Z = \{z_1, z_2, z_3, z_4\}$ are 6-connected, respectively.

(Case 8) Assume that x_1 is a *pure closed point* and $f(x_1)$ is a *mixed point.*

(Case 8-1) If $k_1 = 3^{n_1} - 1$, then K-(k_0, k_1)-continuity of f leads to KD-(k_0, k_1)-continuity of f for any X_{n_0,k_0} by the same method as Case 5-1.

(Case 8-2) Let us consider the case $k_1 \neq 3^{n_1} - 1$. Then the point $x_2 \in N_{k_0}(x_1, 1)$ should be a pure open point or a mixed point depending on the k_0-adjacency of X_{n_0,k_0}. If the point x_2 is a mixed point in $N_{k_0}(x_1,1)$, then $f(x_2)$ can be mapped into the point $f(x_1)$, a mixed point in $N_{k_1}(f(x_1))$, or a pure open point in $N_{k_1}(f(x_1))$ depending on the k_1-adjacency of Y_{n_1,k_1} owing to the K-(k_0,k_1)-continuity of f at the points x_1 and x_2. Precisely, $f(x_2)$ cannot be mapped into a pure closed point in $N_{k_1}(f(x_1))$ owing to the K-(k_0,k_1)-continuity of f at the points x_1 and x_2. Besides, if the point x_2 is a pure open point in $N_{k_0}(x_1,1)$, then $f(x_2)$ should be equal to the point $f(x_1)$ or a certain point in $N_{k_1}(f(x_1))$ owing to the K-(k_0,k_1)-continuity of f at the points x_1 and x_2.

For example, assume a K-$(4,4)$-continuous map $f_1 : X_{2,4} \to Y_{2,4}$ in Fig.8(a), where $X = \{x_1, x_2\}$ and $Y = \{y_1 := f_1(x_1), y_2, y_3\}$. Let us now examine $f_1(x_2)$.

If x_2 is a mixed point, then $f_1(x_1) = f_1(x_2)$ or $f_1(x_1) = y_3$.

If not, suppose that $f_1(x_2) = y_2$, then we have a contradiction to the K-$(4,4)$-continuity of f_1 at the points x_1 and x_2 because $f_1^{-1}(\{f_1(x_1), y_3\}) = \{x_1\} \notin T_X^2$, where $\{f_1(x_1), y_3\} \in T_Y^2$.

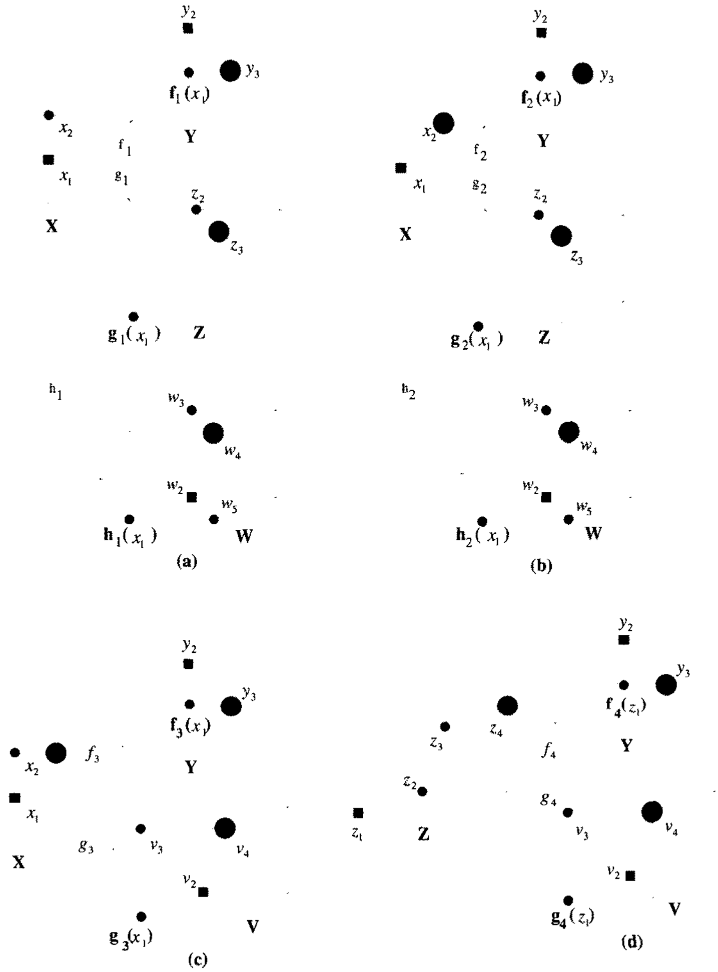

FIGURE 8

Besides, assume a K-$(8,4)$-continuous map $f_2 : X_{2,8} \to Y_{2,4}$, where $X = \{x_1, x_2\}$ with 8-adjacency and $\{f_2(x_1), y_2, y_3\}$ with 4-adjacency in Fig.8(b). Let us now examine $f_2(x_2)$. For the point $x_2 \in N_8(x_1, 1) - N_4(x_1, 1)$ which is a pure open point (see Fig.8(b)), we see that $f_2(x_2) \in \{f_2(x_1), y_3\}$ owing to the K-$(8,4)$-continuity of f_2 at the points x_1 and x_2 and further, because the smallest open set containing the point $x_1 \in X$ is the set $\{x_1, x_2\}$ and $T_Y^2 = \{Y, \phi, \{f_2(x_1), y_3\}, \{y_3\}\}$, which means that the map f_2 is a KD-$(8,4)$-continuous map at the point x_1.

Similarly, by the same method as Case 7-2, the cases in Fig.8(c) also show that a K-$(18,4)$-continuous map $f_4 : Z_{3,18} \to Y_{2,4}$ and a K-$(18,6)$-continuous map $g_4 : Z_{3,18} \to V_{3,6}$ are also KD-$(18, 4)$- and KD-$(18, 6)$-continuous maps, respectively.

(Case 9) Assume that both x_1 and $f(x_1)$ are *mixed points.*

(Case 9-1) If $k_1 = 3^{n_1} - 1$, then K-(k_0, k_1)-continuity of f leads to KD-(k_0, k_1)-continuity of f for any X_{n_0,k_0} by the same method as Case 5-1.

(Case 9-2) Let us consider the case $k_1 \neq 3^{n_1} - 1$. Then the point $x_2 \in N_{k_0}(x_1, 1)$ should be a mixed point, a pure open point, or a pure closed point depending on the k_0-adjacency of X_{n_0,k_0}.

First, if the point x_2 is a pure closed point, then $f(x_2)$ should be a pure closed point in $N_{k_1}(f(x_1))$ or $f(x_1)$ depending on the k_1-adjacency of Y_{n_1,k_1} owing to the K-(k_0, k_1)-continuity of f at the points x_1 and x_2. Second, if the point x_2 is a pure open point in $N_{k_0}(x_1, 1)$, then $f(x_2)$ should be the point $f(x_1)$, a mixed point in $N_{k_1}(f(x_1))$, or a pure open point in $N_{k_1}(f(x_1))$ depending on the k_1-adjacency of Y_{n_1,k_1} owing to the K-(k_0, k_1)-continuity of f at the points x_1 and x_2. Third, if the point x_2 is a mixed point in $N_{k_0}(x_1, 1)$, then $f(x_2) \in N_{k_1}(f(x_1))$ can be mapped into the point $f(x_1)$, a mixed point, a pure closed point, or a pure open point, depending on the k_1-adjacency of $(Y, k_1, T_Y^{n_1})$, owing to the K-(k_0, k_1)-continuity of f at the points x_1 and x_2.

For example, assume a K-$(4, 6)$-continuous map $h_1 : X_{2,4} \to W_{3,6}$ (see Fig.9(a)), where $X = \{x_1, x_2\}$ can be considered in two fashions according to the location of $x_2 \in N_4(x_2, 1)$ and $W = \{w_1 := h_1(x_1), w_2, w_3, w_4, w_5\}$ with 6-adjacency. Let us now examine $h_1(x_2)$. By the K-$(4, 6)$-continuity of h_1 at the points x_1 and x_2, we obtain

$$(3.12) \qquad \left\{ \begin{array}{r} h_1(N_4(x_1, 1)) = h_1(N_4^*(x_1, 1)) \subset N_6^*(h_1(x_1), s); \\ h_1(N_4^*(x_1, 1)) \subset N_{26}(h_1(x_1)) \end{array} \right\}$$

because $N_4(x_1, 1) = X$, $N_6^*(h_1(x_1), 2) = W$ and further, there is no $N_6^*(h_1(x_1), 1)$ because the smallest open set containing the point $h_1(x_1)$ is the set $\{h_1(x_1), w_4, w_5\}$, where the number s is considered in Remark 3.3.

Then, the point $x_2 \in N_4(x_1, 1)$ with $x_1 \neq x_2$ is a pure closed point or a pure open point (see Fig.9(a)). In case the point x_2 is a pure closed point, $h_1(x_2)$ should be a pure closed point in $N_4(h_1(x_1))$ or $h_1(x_1)$ owing to the K-$(4, 6)$-continuity of h_1 at the points x_1 and x_2 and (3.12), which means that h_1 is a KD-$(4, 6)$-continuous map.

In case the point x_2 is a pure open point, $h_1(x_2)$ should be mapped into $\{h_1(x_1)\}$ or $h_1(x_2) = w_5$ because $\{w_4, w_5\} \in T_W^3$. If not, suppose $h_1(x_2) = w_2$, then we have a contradiction to the $(4, 6)$-continuity of h_1 at the points x_1 and x_2, which means that h_1 is a KD-$(4, 6)$-continuous map.

Besides, assume a K-$(8, 6)$-continuous map $h_2 : X_{2,8} \to W_{3,6}$ with the same hypothesis above (see Fig.9(b)). Let us now examine $h_2(x_2)$. Then, since the point $x_2 \in N_8(x_1, 1) - N_4(x_1, 1)$ is a mixed point (see Fig.9(b)), $h_2(x_2)$ should be a pure closed point w_2, w_5, or $h_1(x_1)$, which means that h_2 is a KD-$(8, 6)$-continuous map.

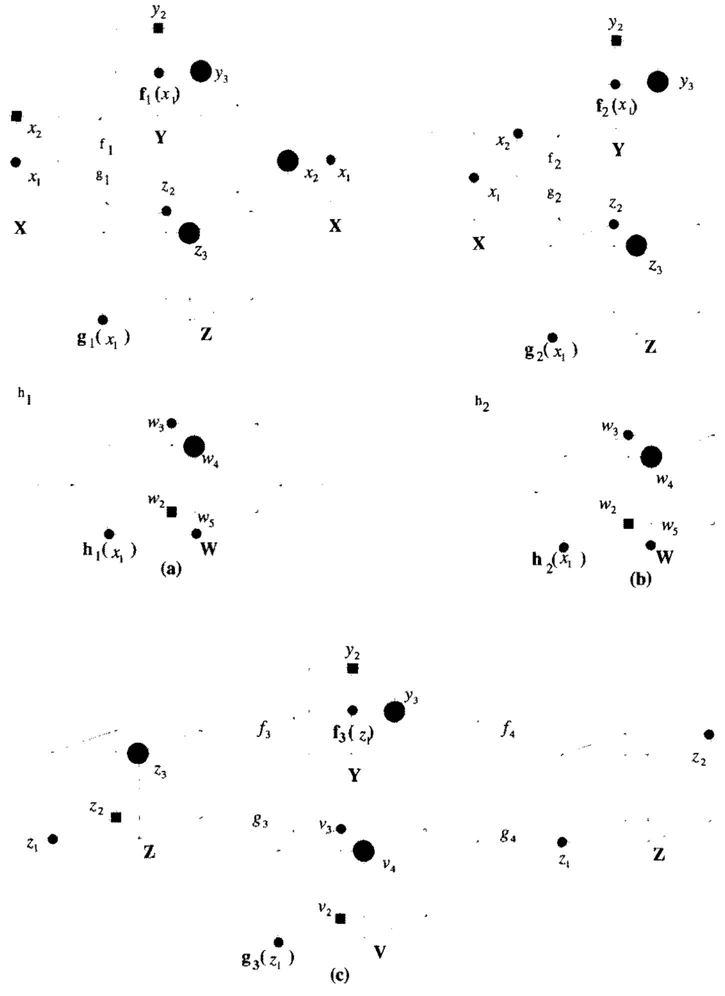

FIGURE 9

Furthermore, in Fig.9(c) for the other cases $(n_0, k_0) \in \{(3,6),(3,18),(3,26)\}$ and $(n_1, k_1) \in \{(2,4),(3,6),(3,18)\}$, K-$(k_0, k_1)$-continuity implies KD-$(k_0, k_1)$-continuity. $\square$

Corollary 3.6. (1) *The Cases* (1), (2), *and* (3) *of Theorem 3.5 can be extended into the following. For a map* $f : X_{n_0,k_0} \to Y_{n_1,k_1}, n_i \in \mathbb{N}, i \in \{0,1\}$, *any* K-$(k_0, k_1)$*-continuity implies* KD-(k_0, k_1)*-continuity.*

(2) *As we referred in the proof of Theorem 3.5, for a map* $f : X_{n_0,k_0} \to Y_{n_1,k_1}$, K-$(k_0, k_1)$*-continuity of* f *at some point* $x \in X$ *need not imply* KD-(k_0, k_1)*-continuity of* f *at the point* $x \in X$. *Namely,* K-(k_0, k_1)*-continuity of* f

need not imply KD-(k_0, k_1)*-continuity of* f *from a local point of view contrary to a global point of view.*

Indeed, the converse of Theorem 3.5 does not hold. Precisely, KD-(k_0, k_1)-continuity of $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ need not imply K-(k_0, k_1)-continuity of f because for some point $x \in X$ or $y \in Y$ we may not have $N^*_{k_0}(x, r)$ or $N^*_{k_1}(f(x), s)$ in Remark 3.3.

Motivated by Theorem 3.5, we obtain the following because for every point $x \in X$ with $X_{n,3^n-1}$, $N^*_{3^n-1}(x, 1) = N_{3^n-1}(x, 1)$ [11].

Theorem 3.7. *For a map* $f : X_{n_0,k_0} \to Y_{n_1,k_1}$*, if* $(k_0, k_1) = (3^{n_0} - 1, 3^{n_1} - 1)$*, then the notions of* KD-(k_0, k_1)*-continuity and* K-(k_0, k_1)*-continuity are equivalent each other.*

In terms of Remark 3.1, Theorems 3.5 and 3.7, and Corollary 3.6, four kinds of continuities in computer topology have been investigated and compared with each other. Then we have the following query:

Under what condition are the above-mentioned three kinds of continuities in Definitions 2, 3, and 5 equivalent to each other?

We now have an answer to the question as follows.

Theorem 3.8. *For a map* $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ *and* $1 \le n_i \le 3, i \in \{0, 1\}$*, assume that*

(1) *any points* $x \in X$ *and* $f(x) \in Y$ *have* $N^*_{k_0}(x, 1) \subset X$ *and* $N^*_{k_1}(f(x), 1) \subset Y$*, respectively,*

(2) $f(X^*)$ *is* k_1*-connected, where* X^* *is any* k_0*-connected subset of* X*.*

Then the three kinds of continuities in Definition 2, 3, and 5 are equivalent to each other.

Proof. With the hypothesis it suffice to prove that Khalimsky continuity implies K-(k_0, k_1)-continuity via Definitions 2, 3, and 5, Remark 3.4, and Theorem 3.5 because in general

K-(k_0, k_1)-continuity $\Rightarrow$ KD-(k_0, k_1)-continuity $\Rightarrow$ Khalimsky continuity.

Due to the Khalimsky continuity of f and the existence of both $N^*_{k_0}(x, 1)$ and $N^*_{k_1}(f(x), 1)$, we obtain both a smallest open set $O_x \in T^{n_0}_X$ containing the point x such that $O_x \subset N^*_{k_0}(x, 1)$ and a smallest open set $O_{f(x)} \in T^{n_1}_Y$ containing the point $f(x)$ such that $O_{f(x)} \subset N^*_{k_1}(f(x), 1)$. Furthermore, for an open set $O_{f(x)} \in T^{n_1}_Y$, we have $f(O_x) \subset O_{f(x)}$.

Thus we obtain

$$f(O_x) \subset f(N^*_{k_0}(x, 1)) \subset N^*_{k_1}(f(x), 1) \subset N^*_{k_1}(f(x), s), \tag{3.13}$$

where s is the number in Remark 3.3.

If not, in (3.13), suppose that $f(N^*_{k_0}(x, 1))$ is not a subset of $N^*_{k_1}(f(x), 1)$. Then there is a point $x' \in N^*_{k_0}(x, 1)$ such that $f(x') \notin N^*_{k_1}(f(x), 1)$ so that $f(x)$ is not k_1-adjacent to $f(x')$, which contradicts to the condition (2) of the hypothesis. □

Example 3.9. Consider an $(18,2)$-continuous map in Fig.10

$$f : A_{3,18} \to [0,2]_{\mathbb{Z}}$$

for which $f(a_1) = 0, f(a_2) = 2, f(\{a_3, a_4\}) = \{1\}$ in Fig.10, where $A = \{a_1, a_2, a_3, a_4\}$. Then we observe that T_A^3 is established with the following base $\{\{a_1\}, \{a_2\}, \{a_3, a_4\}, \{a_4\}\}$ and further, $T_{[0,2]_{\mathbb{Z}}} = \{[0,2]_{\mathbb{Z}}, \phi, \{1\}, \{1,2\}, \{0,1\}\}$. Since the map $f : A_{3,18} \to [0,2]_{\mathbb{Z}}$ satisfies the hypothesis of Theorem 3.8, we see that the three kinds of continuities of f in Definitions 2, 3, 5 are equivalent to each other.

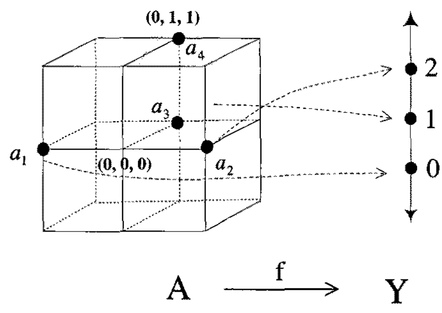

FIGURE 10

We have used *a digital topological category*, briefly DTC, consisting of three things:

(1) A class of objects (X, k) in $\mathbb{Z}^n$;
(2) For every ordered pair of objects (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$ as morphisms, all (k_0, k_1)-continuous maps $f : (X, k_0) \to (Y, k_1)$;
(3) For every ordered triple of objects (X, k_0) in $\mathbb{Z}^{n_0}$, (Y, k_1) in $\mathbb{Z}^{n_1}$, and (Z, k_2) in $\mathbb{Z}^{n_2}$ and a function associating to a pair of morphisms $f : (X, k_0) \to (Y, k_1)$ which is a digitally (k_0, k_1)-continuous map and $g : (Y, k_1) \to (Z, k_2)$ which is a digitally (k_1, k_2)-continuous map, their composite $g \circ f : (X, k_0) \to (Z, k_2)$ which is a digitally (k_0, k_2)-continuous map.

Then we easily see that DTC satisfies the following axioms: *Associativity* and *identity*.

On the basis of Definitions 3, 4, 5 and by the same method as the establishment of DTC, three kinds of categories motivated by the Khalimsky continuity, K-(k_0, k_1)-continuity, the (k_0, k_1)-continuity, and the KD-(k_0, k_1)-continuity, respectively. These are denoted by KTC, KCTC, CTC, KDTC, respectively.

4. Classification of computer topological spaces up to each of K-(k_0, k_1)-, (k_0, k_1)-, KD-(k_0, k_1)-homeomorphisms, and Khalimsky homeomorphism

In DTC, in order to classify digital images (X, k) up to digital k-homeomorphism we have used the following.

Definition 6 ([6, Digital (k_0, k_1)-homeomorphism]). (see also [7, 8, 9, 12, 13, 14]) For two spaces (X, k_0) and (Y, k_1), a function $f : X \to Y$ is said to be a digital (k_0, k_1)-homeomorphism if

(1) the map f is bijective, and
(2) the map f is a digitally (k_0, k_1)-continuous map and further, f^{-1} is a digitally (k_1, k_0)-continuous map.

Then we say that the space X is digitally (k_0, k_1)-homeomorphic to Y.

In CTC, in order to classify computer topological spaces $X_{n,k}$ up to Khalimsky-homeomorphism, KD-k-homeomorphism, k-homeomorphism, or K-k-homeomorphism, we have used the following.

Definition 7 (Khalimsky homeomorphism). For two spaces $(X, T_X^{n_0})$ and $(Y, T_Y^{n_1})$, a map $h : X \to Y$ is called a Khalimsky homeomorphism if h is a Khalimsky continuous bijection and further, $h^{-1} : Y \to X$ is Khalimsky continuous.

Definition 8 ([15, KD-(k_0, k_1)-homeomorphism]). For two spaces X_{n_0,k_0} and Y_{n_1,k_1}, a function $f : X \to Y$ is said to be a KD-(k_0, k_1)-homeomorphism if

(1) the map f is bijective, and
(2) the map f is a KD-(k_0, k_1)-continuous map and further, f^{-1} is a KD-(k_1, k_0)-continuous map.

Then we say that the space X is KD-(k_0, k_1)-homeomorphic to Y.

Definition 9 ((k_0, k_1)-homeomorphism). For two spaces X_{n_0,k_0} and Y_{n_1,k_1}, a map $h : X \to Y$ is called a (k_0, k_1)-homeomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous.

Then we say that the space X is (k_0, k_1)-homeomorphic to Y.

Definition 10 ([10, K-(k_0, k_1)-homeomorphism]). For two spaces X_{n_0,k_0} and Y_{n_1,k_1}, a map $h : X \to Y$ is called a K-(k_0, k_1)-homeomorphism if

(1) h is a K-(k_0, k_1)-continuous bijection, and
(2) $h^{-1} : Y \to X$ is K-(k_1, k_0)-continuous.

Then we say that the space X is K-(k_0, k_1)-homeomorphic to Y.

By $X \approx_{\mathrm{K}\cdot(k_0,k_1)} Y$, $X \approx_{(k_0,k_1)} Y$, $X \approx_{\mathrm{KD}\cdot(k_0,k_1)} Y$, and $X \approx_{\mathrm{K}h} Y$ we denote K-(k_0, k_1)-, (k_0, k_1)-, KD-(k_0, k_1)-homeomorphisms and Khalimsky homeomorphism from X_{n_0,k_0} to Y_{n_1,k_1}, respectively. If $n_0 = n_1$ and $k_0 = k_1$, we use the notation '$\approx_{k_0}$' instead of '$\approx_{(k_0,k_0)}$'.

By Remark 3.4 and Theorem 3.5 we obtain the following.

Remark 4.1. We see that $X \approx_{\mathrm{K}\cdot(k_0,k_1)} Y \Rightarrow X \approx_{\mathrm{KD}\cdot(k_0,k_1)} Y \Rightarrow X \approx_{\mathrm{K}h} Y$ by Theorem 3.5. Meanwhile, by Remark 3.4 we see that none of $X \approx_{(k_0,k_1)} Y$ and $X \approx_{\mathrm{KD}\cdot(k_0,k_1)} Y$ implies the other. Moreover, we see that K-k-, k-, KD-k-homeomorphisms instead of Khalimsky homeomorphism are so meaningful to classify computer topological spaces $X_{n,k}$.

Motivated by Definitions 1, 2, 3, 5 and Theorem 3.7 and Remarks 3.1 and 4.1, we obtain the following because $N_{3^n-1}(x,1) = N^*_{3^n-1}(x,1)$ in $X_{n,3^n-1}$.

Theorem 4.2. *For two spaces X_{n_0,k_0}, Y_{n_1,k_1}, and a function $f : X \to Y$, the notions of* K-(k_0,k_1)- *and* KD-(k_0,k_1)-*homeomorphisms are equivalent to each other if $(k_0,k_1) = (3^{n_0}-1, 3^{n_1}-1)$.*

Remark 4.3. In Theorem 4.2 if $(k_0,k_1) \neq (3^{n_0}-1, 3^{n_1}-1)$, then the assertion does not hold.

By Theorem 3.8 we obtain the following.

Theorem 4.4. *For a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$, assume that*

(1) *any points $x \in X$ and $f(x) \in Y$ have $N^*_{k_0}(x,1) \subset X$ and $N^*_{k_1}(f(x),1) \subset Y$, respectively and*

(2) *$f(X^*)$ is k_1-connected, where X^* is any k_0-connected subset of X.*

Then the three kinds of homeomorphisms in Definitions 7, 8, and 10 are equivalent to each other.

Since the five kinds of homeomorphisms in Definitions 6, 7, 8, 9, and 10 are different from each other and their usages depend on the given topological categories such as KTC, KCTC, KDTC, CTC, and DTC.

5. Summary and concluding remark

We have studied several continuities and homeomorphisms in computer and digital topology. These have advantages and disadvantages, their usages depend on the situation.

We now adopt a forgetful functor from each of KCTC, KDTC, and CTC into DTC, denoted by

$$F^* : \mathrm{KCTC} \to \mathrm{DTC}, \quad F^* : \mathrm{KDTC} \to \mathrm{DTC}, \quad F^* : \mathrm{CTC} \to \mathrm{DTC}.$$

By the use of the forgetful functors F^* a computer topological space $X_{n,k}$ can be transformed into a discrete topological space (or digital image) with k-adjacency [5, 6, 7, 8, 9]. Furthermore, the current K-(k_0,k_1)-, (k_0,k_1)-, and KD-(k_0,k_1)-continuities are also transformed into the digital (k_0,k_1)-continuity in DTC.

In particular, in relation to the computer topological morphology the four kinds of digital and computer topological homeomorphisms come from Definitions 6, 8, 9, and 10 can be useful to classify digital topological spaces with k-adjacency and computer topological spaces with k-adjacency, respectively. Finally, by Remark 3.4, Theorem 3.5, Corollary 3.6, Definitions 2, 3, 4, and 5, we now show a distribution diagram among several continuities between computer topological spaces in $\mathbb{Z}^n, 1 \leq n \leq 3$ (see Fig.11). For $n \geq 4$, the statement of Theorem 3.5 at Case 4-2 need to examine as an open problem.

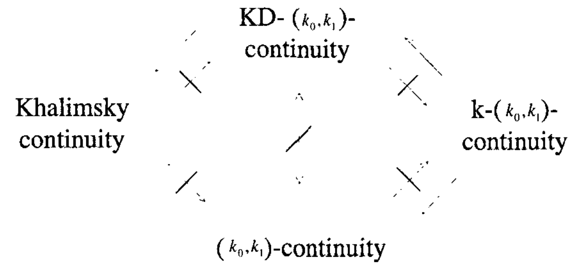

FIGURE 11. Distribution diagram of several continuities in computer topology

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College of Environmental Science and Engineering
Honam University
Gwangju 506-714, Korea
E-mail address: sehan@honam.ac.kr