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ON THE STABILITY OF THE MONOMIAL FUNCTIONAL EQUATION

YANG-HI LEE

ABSTRACT. In this paper, we modify L.Cădariu and V. Radu's result for the stability of the monomial functional equation

$$\sum_{i=0}^{n} {}_nC_i(-1)^{n-i}f(ix+y) - n!f(x) = 0$$

in the sense of Th. M. Rassias. Also, we investigate the superstability of the monomial functional equation.

1. Introduction

Throughout this paper, let X be a vector space and Y a Banach space. Let n be a positive integer. For a given mapping $f : X \to Y$, define a mapping $D_nf : X \times X \to Y$ by

$$D_nf(x,y) := \sum_{i=0}^{n} {}_nC_i(-1)^{n-i}f(ix+y) - n!f(x)$$

for all $x, y \in X$, where ${}_nC_i = \frac{n!}{i!(n-i)!}$. A mapping $f : X \to Y$ is called a monomial function of degree n if f satisfies the monomial functional equation $D_nf(x,y) = 0$. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := ax^n$ is a particular solution of the functional equation $D_nf = 0$. In particular, a mapping $f : X \times X \to Y$ is called an additive (quadratic, cubic, quartic, respectively) mapping if f satisfies the functional equation $D_1f = 0$ ($D_2f = 0$, $D_3f = 0$, $D_4f = 0$, respectively).

In 1940, S. M. Ulam [27] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

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for all $x \in G_1$?

In 1941, D. H. Hyers [7] proved the stability theorem for additive functional equation $D_1 f = 0$ under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [20] provided an extension of D. H. Hyers's Theorem by proving the generalized Hyers-Ulam stability for the linear mapping subject to the unbounded Cauchy difference that he introduced in [20]. Th. M. Rassias's Theorem provided a lot of influence for the rapid development of stability theory of functional equations during the last three decades. This generalized concept of stability is known today with the term Hyers-Ulam-Rassias stability of the linear mapping or of functional equations. Further generalizations of the Hyers-Ulam-Rassias stability concept have been investigated by a number of mathematicians worldwide (cf. [5, 6, 8, 9, 11, 12, 14, 17-19, 21-25]). In 1983, the Hyers-Ulam-Rassias stability theorem for the quadratic functional equation $D_2 f = 0$ was proved by F. Skof [26] and a number of other mathematicians (cf. [2, 3, 4, 10, 13]). The Hyers-Ulam-Rassias stability Theorem for the functional equation $D_3 f = 0$ and $D_4 f = 0$ was proved by J. Rassias [15, 16].

In 2007, L. Cădariu and V. Radu [1] proved the stability of the monomial functional equation $D_n f = 0$.

In this paper, we modify L. Cădariu and V. Radu's result for the stability of the monomial functional equation $D_n f = 0$ in the sense of Th. M. Rassias and the superstability of the monomial functional equation $D_n f = 0$.

2. The stability of the monomial functional equation

Since the equalities

$$(1-x^2)^n = \sum_{i=0}^{n} {}_nC_i(-1)^i x^{2i},$$

$$(1-x)^n(x+1)^n = (\sum_{k=0}^{n} {}_nC_k(-1)^k x^k)(\sum_{j=0}^{n} {}_nC_j x^j)$$

$$= \sum_{i=0}^{n}\sum_{l=0}^{2i} {}_nC_l \cdot {}_nC_{2i-l}(-1)^l x^{2i}$$

hold for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the equality

$${}_nC_i(-1)^i = \sum_{l=0}^{2i} {}_nC_l \cdot {}_nC_{2i-l}(-1)^l$$

holds for all $n \in \mathbb{N}$.

Lemma 1. *Let $f : X \to Y$ be a mapping satisfying the functional equation*

$$D_n f(x,y) := \sum_{i=0}^{n} {}_nC_i(-1)^{n-i} f(ix+y) - n! f(x)$$

for all $x, y \in X$. Then equality

$$f(2x) = 2^n f(x)$$

holds for all $x \in X$.

Proof. Using the equalities

$$ {}_nC_i(-1)^i = \sum_{l=0}^{2i} {}_nC_l \cdot {}_nC_{2i-l}(-1)^l \quad \text{and} \quad \sum_{i=0}^{n} {}_nC_i(-1)^i = 0,$$

the equality

$$n!(f(2x) - 2^n f(x)) = D_n f(2x, (-k)x) - \sum_{j=0}^{n} {}_nC_j D_n f(x, (j-k)x) = 0$$

holds for all $x \in X$ and $k \in \mathbb{N}$ as we desired. □

Now, we prove the stability of the monomial functional equation in the sense of Th. M. Rassias.

Theorem 2. *Let p be a real number with $0 \le p < n$ and X a normed space. Let $f : X \to Y$ be a mapping such that*

$$\|D_n f(x,y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in X$. Then there exists a unique monomial function of degree n $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{1}{n!} \inf_{k\in\mathbb{N}} (2^p + k^p + 2^n + \sum_{j=0}^{n} {}_nC_j |j-k|^p) \frac{\varepsilon}{2^n - 2^p} \|x\|^p \tag{2}$$

holds for all $x \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x) := \lim_{s\to\infty} \frac{f(2^s x)}{2^{ns}}$$

for all $x \in X$.

Proof. By (1), we get

$$\begin{aligned} \|n!(f(2x) - 2^n f(x))\| &= \|D_n f(2x, (-k)x) - \sum_{j=0}^{n} {}_nC_j D_n f(x, (j-k)x)\| \\ &\le \varepsilon(\|2x\|^p + \|kx\|^p + \sum_{j=0}^{n} {}_nC_j(\|x\|^p + \|(j-k)x\|^p)) \\ &= (2^p + k^p + 2^n + \sum_{j=0}^{n} {}_nC_j |j-k|^p)\varepsilon\|x\|^p \end{aligned}$$

for all $x \in X$ and $k \in \mathbb{N}$. Hence

$$\|f(x) - \frac{f(2x)}{2^n}\| \le \frac{\varepsilon}{n! \cdot 2^n} \inf_{k\in\mathbb{N}} (2^p + k^p + 2^n + \sum_{j=0}^{n} {}_nC_j |j-k|^p)\|x\|^p \tag{3}$$

and
(4)

$$\|f(x) - \frac{f(2^m x)}{2^{nm}}\| \leq \sum_{s=0}^{m-1} \|\frac{f(2^s x)}{2^{sn}} - \frac{f(2^{s+1}x)}{2^{(s+1)n}}\|$$
$$\leq \frac{\varepsilon}{n! \cdot 2^n} \inf_{k\in\mathbb{N}}(2^p + k^p + 2^n + \sum_{j=0}^{n} {}_nC_j|j-k|^p) \sum_{s=0}^{m-1} \frac{2^{sp}}{2^{sn}}\|x\|^p$$

for all $x \in X$. The sequence $\{\frac{f(2^s x)}{2^{sn}}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{f(2^s x)}{2^{sn}}\}$ converges for all $x \in X$. Define $F : X \to Y$ by

$$F(x) := \lim_{s\to\infty} \frac{f(2^s x)}{2^{sn}}$$

for all $x \in X$. By (1) and the definition of F, we obtain

$$D_n F(x, y) = \lim_{s\to\infty} \frac{D_n f(2^s x, 2^s y)}{2^{ns}} = 0$$

for all $x, y \in X$. Taking $m \to \infty$ in (4), we can obtain the inequality (2) for all $x \in X$.

Now, let $F' : X \times X \to Y$ be another monomial function satisfying (2). By Lemma 1, we have

$$\|F(x) - F'(x)\| \leq \|\frac{1}{2^{ns}}(F - f)(2^s x)\| + \|\frac{1}{2^{ns}}(f - F')(2^s x)\|$$
$$\leq \frac{2^{np}}{2^{ns}} \frac{2}{n!} \inf_{k\in\mathbb{N}}(2^p + k^p + 2^n + \sum_{j=0}^{n} {}_nC_j|j-k|^p) \frac{\varepsilon}{2^n - 2^p}\|x\|^p$$

for all $x, y \in X$ and $s \in \mathbb{N}$. As $s \to \infty$, we may conclude that $F(x) = F'(x)$ for all x as desired. □

Theorem 3. *Let p be a real number with $p > n$ and X a normed space. Let $f : X \to Y$ be a mapping satisfying (1) for all $x, y \in X$. Then there exists a unique monomial function of degree n $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \leq \frac{1}{n!} \inf_{k\in\mathbb{N}}(2^p + k^p + 2^n + \sum_{j=0}^{n} {}_nC_j|j-k|^p) \frac{\varepsilon}{2^p - 2^n}\|x\|^p$$

holds for all $x \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x) := \lim_{s\to\infty} 2^{ns} f(2^{-s} x)$$

for all $x \in X$.

Proof. By (3), we get

$$\|f(x) - 2^n f(\frac{x}{2})\| \leq \frac{\varepsilon}{n! \cdot 2^p} \inf_{k\in\mathbb{N}}(2^p + k^p + 2^n + \sum_{j=0}^{n} {}_nC_j|j-k|^p)\|x\|^p$$

for all $x \in X$ and $k \in \mathbb{N}$. The rest of the proof is similar with the proof of Theorem 2. □

3. The superstability of the functional equation $D_n f = 0$

Lemma 4. *Let p be a real number with $p < 0$ and X a normed space. Let $f : X \to Y$ be a mapping satisfying* (1) *for all $x, y \in X \backslash \{0\}$. Then there exists a unique monomial function of degree n $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \leq \frac{2^p + 2^n}{n!(2^n - 2^p)} \varepsilon \|x\|^p \tag{5}$$

holds for all $x \in X \backslash \{0\}$.

Proof. As in the proof of Theorem 2, the inequality

$$\begin{aligned}\|f(x) - \frac{f(2^m x)}{2^{nm}}\| &\leq \sum_{s=0}^{m-1} \|\frac{f(2^s x)}{2^{sn}} - \frac{f(2^{s+1} x)}{2^{(s+1)n}}\| \\ &\leq \frac{\varepsilon}{n! \cdot 2^n} \inf_{k \geq n+1} (2^p + k^p + 2^n + \sum_{j=0}^{n} {}_nC_j |j - k|^p) \sum_{s=0}^{m-1} \frac{2^{sp}}{2^{sn}} \|x\|^p\end{aligned}$$

holds for all $x \in X \backslash \{0\}$. Since $p < 0$, we get

$$\inf_{k \geq n+1} (2^p + k^p + 2^n + \sum_{j=0}^{n} {}_nC_j |j - k|^p) = (2^p + 2^n)$$

for all $x \in X \backslash \{0\}$. The rest of the proof is the same to the proof of Theorem 2. □

Now, we prove the superstability of the monomial functional equation.

Theorem 5. *Let p be a real number with $p < 0$ and X a normed space. Let $f : X \to Y$ be a mapping satisfying* (1) *for all $x, y \in X \backslash \{0\}$. Then f is a monomial function of degree n.*

Proof. Let F be the monomial function of degree n satisfying (5). From (1), the inequality

$$\begin{aligned}&\|f(x) - F(x)\| \\ &\leq \frac{1}{n} \|D_n(f - F)((k+1)x, -kx) + (-1)^n (F - f)(-kx) \\ &\quad + \sum_{i=2}^{n} {}_nC_i (-1)^{n-i} (F - f)(i(k+1)x - kx) - n!(F - f)((k+1)x)\| \\ &\leq \frac{1}{n} [\frac{2^p + 2^n}{n!(2^n - 2^p)} (k^p + \sum_{i=1}^{n-1} {}_nC_{i+1} (ik + i + 1)^p + n!(k+1)^p) \\ &\quad + (k+1)^p + k^p] \varepsilon \|x\|^p\end{aligned}$$

holds for all $x \in X\backslash\{0\}$ and $k \in \mathbb{N}$. Since $\lim_{k\to\infty}(k^p + \sum_{i=1}^{n-1} C_{i+1}^n (ik+i+1)^p + n!(k+1)^p) = 0$ and $\lim_{k\to\infty}(k^p+(k+1)^p) = 0$ for $p < 0$, we get

$$f(x) = F(x)$$

for all $x \in X\backslash\{0\}$. Since $\lim_{k\to\infty} k^p = 0$ and the inequality

$$\begin{aligned}\|f(0)-F(0)\| &\le \frac{1}{n}\|D_n(f-F)(kx,-kx)+(-1)^n(F-f)(-kx)\\ &\quad + \sum_{i=2}^{n} {}_nC_i(-1)^{n-i}(F-f)((i-1)kx) - n!(F-f)(kx)\|\\ &\le \frac{1}{n}[2+\frac{2^p+2^n}{n!(2^n-2^p)}(1+\sum_{i=1}^{n-1} {}_nC_{i+1}i^p + n!)]k^p\varepsilon\|x\|^p\end{aligned}$$

holds for any $x \in X\backslash\{0\}$ and $k \in \mathbb{N}$, we get

$$f(0) = F(0).$$

□

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Department of Mathematics Education
Gongju National University of Education
Gongju 314-711, Korea
E-mail address: yanghi2@hanmail.net