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EIGENVALUES ESTIMATES FOR THE DIRAC OPERATOR IN TERMS OF CODAZZI TENSORS

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ABSTRACT. We prove a lower bound for the first eigenvalue of the Dirac operator on a compact Riemannian spin manifold depending on the scalar curvature as well as a chosen Codazzi tensor. The inequality generalizes the classical estimate from [2].

1. Introduction

The first author proved in [2] that the smallest eigenvalue λ_1 of the Dirac operator D of a compact Riemannian spin manifold (M^n, g) satisfies

$$\lambda_1^2 \geq \frac{n}{4(n-1)} \cdot S_{\min} , \tag{1}$$

where $S_{\min}$ denotes the minimum of the scalar curvature. The limiting case of (1) occurs if and only if (M^n, g) admits a nontrivial spinor field ψ_1 satisfying

$$\nabla_X \psi_1 = -\frac{\lambda_1}{n} X \cdot \psi_1,$$

where X is an arbitrary vector field on M^n and the dot "$\cdot$" indicates Clifford multiplication [3]. Improvements of this estimate do typically depend on additional geometric structures on the considered manifold (M^n, g) [4, 8, 9]. The aim of this paper is to show that inequality (1) can be improved in case that a Codazzi tensor exists.

A symmetric $(0,2)$-tensor field β on (M^n, g) is called a *nondegenerate Codazzi tensor* [1] if β is nondegenerate at all points of M^n and satisfies

$$(\nabla_X \beta)(Y, Z) = (\nabla_Y \beta)(X, Z)$$

for all vector fields X, Y, Z. We identify β with the induced (1,1)-tensor β via $\beta(X, Y) = g(X, \beta(Y))$. Let $(E_1, \ldots, E_n)$ be a local orthonormal frame field on (M^n, g). Then the spin derivative ∇ and the Dirac operator D, acting on

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sections $\psi \in \Gamma(\Sigma(M^n))$ of the spinor bundle $\Sigma(M^n)$ over (M^n, g), are locally expressed as [3]

$$\nabla_X \psi = X(\psi) + \frac{1}{4}\sum_{i=1}^{n} E_i \cdot \nabla_X E_i \cdot \psi \ , \quad D\psi = \sum_{i=1}^{n} E_i \cdot \nabla_{E_i}\psi,$$

respectively. Moreover, we define the *β-twist* D_β of the Dirac operator D by

$$D_\beta \psi = \sum_{i=1}^{n} \beta^{-1}(E_i) \cdot \nabla_{E_i}\psi = \sum_{i=1}^{n} E_i \cdot \nabla_{\beta^{-1}(E_i)}\psi.$$

Theorem 1.1. *Let (M^n, g) be an n-dimensional closed Riemannian spin manifold and consider a nondegenerate Codazzi tensor β. Denote by $\overline{g}$ the metric induced by β via $\overline{g}(X,Y) = g(\beta(X), \beta(Y))$. Let $\lambda_1 \in \mathbb{R}$ and $\overline{\lambda}_1 \in \mathbb{R}$ be the smallest nonzero eigenvalue of the Dirac operators D and $\overline{D}$, respectively. Then we have*

$$\lambda_1^2 \geq \inf_M \Big\{ \frac{S}{4(p+1)} - \frac{q\overline{\lambda}_1^2}{p+1} + \frac{\Delta F}{2(p+1)F} \Big\}, \tag{2}$$

where $F : M^n \longrightarrow \mathbb{R}$ is a real-valued function defined by

$$F = -\frac{|\det(\beta^{-1})|}{q}, \tag{3}$$

$\Delta F := -(\mathrm{div} \circ \mathrm{grad})(F)$, and $p, q : M^n \longrightarrow \mathbb{R}$ are bounded real-valued functions satisfying

$$-\frac{1}{n} < p \leq 0, \quad -\frac{1}{|\beta^{-1}|^2} \leq q < 0, \tag{4}$$

that solve the system of two linear equations

$$np + c(\mathrm{tr}\beta^{-1})q = -1, \qquad (\mathrm{tr}\beta^{-1})p + c|\beta^{-1}|^2 q = -c \tag{5}$$

for some nonzero constant $c \neq 0 \in \mathbb{R}$.

The limiting case of (2) occurs if and only if there exists a spinor field ψ_1 on (M^n, g) such that

$$D\psi_1 = \lambda_1 \psi_1, \qquad D_\beta \psi_1 = \overline{\lambda}_1 \psi_1 \tag{6}$$

and

$$\nabla_X \psi_1 = \lambda_1 p X \cdot \psi_1 + \overline{\lambda}_1 q \beta^{-1}(X) \cdot \psi_1 \tag{7}$$

hold for all vector fields X. In the limiting case, the parameter $c = \overline{\lambda}_1/\lambda_1$ is the ratio of the two eigenvalues.

If $\beta = I$ is the identity map and $p + q = -1/n$, then (2) reduces to the inequality (1). If the eigenvalues of $\beta \neq I$ are constant, but not equal, then the solutions p, q of the linear system are constant, too,

$$p(c) = \frac{|\beta^{-1}|^2 - c\,\mathrm{tr}\beta^{-1}}{(\mathrm{tr}\beta^{-1})^2 - n\,|\beta^{-1}|^2}\ , \quad q(c) = \frac{cn - \mathrm{tr}\beta^{-1}}{c((\mathrm{tr}\beta^{-1})^2 - n\,|\beta^{-1}|^2)}\ .$$

Consequently, we obtain a family of inequalities depending on a parameter $c \neq 0$ linking λ_1^2, $\overline{\lambda}_1^2$ and $S_{\min}$,

$$\lambda_1^2 \; + \; \frac{q(c)}{p(c)+1}\,\overline{\lambda}_1^2 \;\geq\; \frac{1}{4(p(c)+1)}\,S_{\min}.$$

The optimal parameter c is a solution of a quadratic equation, we omit the corresponding formulas. A universal though not optimal value for the parameter c is

$$c \;:=\; \frac{|\beta^{-1}|^2}{\mathrm{tr}\beta^{-1}}\;.$$

In this case we have $p = 0$ and $q = -1/|\beta^{-1}|^2$. This particular inequality generalizes (1):

Corollary 1.2. *If the eigenvalues of the Codazzi tensor are constant, then*

$$\lambda_1^2 \;\geq\; \frac{1}{4}\,S_{\min} + \frac{\overline{\lambda}_1^2}{|\beta^{-1}|^2} \;\geq\; \frac{1}{4}\,S_{\min} + \frac{1}{|\beta^{-1}|^2}\cdot\frac{n}{4(n-1)}\,\overline{S}_{\min}.$$

If $\mathrm{tr}\beta^{-1} = 0$, the functions p and q do not depend on the parameter c, i.e., we obtain a unique inequality. We will formulate the result separately.

Theorem 1.3. *Let (M^n, g) be an n-dimensional closed Riemannian spin manifold and consider a nondegenerate Codazzi tensor such that* $\mathrm{tr}(\beta^{-1}) = 0$ *vanishes identically. Let λ_1 and $\overline{\lambda}_1$ be the smallest nonzero eigenvalue of D and $\overline{D}$, respectively. Then, in the notations of Theorem 1.1, we have*

$$\lambda_1^2 \geq \inf_M\left\{\frac{nS}{4(n-1)} + \frac{n\,\overline{\lambda}_1^2}{(n-1)|\beta^{-1}|^2} + \frac{n\,\Delta F}{2(n-1)F}\right\}, \tag{8}$$

where the real-valued function $F : M^n \longrightarrow \mathbb{R}$ is defined by

$$F = |\det(\beta^{-1})|\cdot|\beta^{-1}|^2. \tag{9}$$

The limiting case of (8) *occurs if and only if there exists a spinor field ψ_1 on (M^n, g) such that*

$$\nabla_X\psi_1 = -\frac{\lambda_1}{n}\,X\cdot\psi_1 - \frac{\overline{\lambda}_1}{|\beta^{-1}|^2}\,\beta^{-1}(X)\cdot\psi_1 \tag{10}$$

hold for all vector fields X.

Let us discuss the 2-dimensional case in detail. Suppose that β is traceless with eigenvalues a, $-a$. Then we obtain

$$\det(\beta^{-1}) \;=\; -\frac{1}{a^2}\,, \quad |\beta^{-1}|^2 \;=\; \frac{2}{a^2}\,, \quad F \;=\; \frac{2}{a^4}\;.$$

In particular, the formula of the latter theorem simplifies:

Corollary 1.4. *Let (M^2, g, β) be a 2-dimensional closed Riemannian spin manifold with a nondegenerate traceless Codazzi tensor. Denote by $\pm a$ its eigenvalues. Then we have*

$$\lambda_1^2 \geq \inf_M \left\{ \frac{S}{2} + a^2 \overline{\lambda}_1^2 + a^4 \Delta(a^{-4}) \right\} .$$

We apply the corollary to minimal surfaces $M^2 \subset X^3(\kappa)$ in a 3-dimensional space of constant curvature κ. The second fundamental form is a Codazzi tensor. The Gauss equation $S = 2\kappa - 2a^2$ yields finally the result

$$\lambda_1^2 \;\geq\; \kappa + \inf_M \Big((\overline{\lambda}_1^2 - 1)a^2 + a^4 \Delta(a^{-4}) \Big).$$

2. Deformation of the metric via a Codazzi tensor

In this section we establish some lemmata that we will need later to prove Theorem 1.1 and 1.3. Consider a nondegenerate symmetric (0, 2)-tensor field β on (M^n, g) and define a new metric $\overline{g}$ by

$$\overline{g}(X, Y) = g(\beta(X), \beta(Y)). \tag{11}$$

The Levi-Civita connection $\overline{\nabla}$ of $(M^n, \overline{g})$ is related to the Levi-Civita connection ∇ of (M^n, g) by [6]

$$\overline{\nabla}_{\beta^{-1}(X)} \left(\beta^{-1}(Y)\right) \;=\; \beta^{-1}\left(\nabla_{\beta^{-1}(X)} Y\right) + \beta^{-1}\left(\Lambda(X, Y)\right), \tag{12}$$

where Λ is the (1,2)-tensor field defined by

$$\begin{aligned} 2\,g(\Lambda(X,Y), Z) \;&=\; g\left(Z,\, \beta\{(\nabla_{\beta^{-1}(X)}\beta^{-1})(Y)\} - \beta\{(\nabla_{\beta^{-1}(Y)}\beta^{-1})(X)\}\right) \\ &\quad + g\left(Y,\, \beta\{(\nabla_{\beta^{-1}(Z)}\beta^{-1})(X)\} - \beta\{(\nabla_{\beta^{-1}(X)}\beta^{-1})(Z)\}\right) \\ &\quad + g\left(X,\, \beta\{(\nabla_{\beta^{-1}(Z)}\beta^{-1})(Y)\} - \beta\{(\nabla_{\beta^{-1}(Y)}\beta^{-1})(Z)\}\right). \end{aligned} \tag{13}$$

Note that the tensor Λ satisfies

$$g(\Lambda(X, Z), Y) + g(\Lambda(X, Y), Z) = 0$$

for all vector fields X, Y, Z. Using formula (12) we can relate the Riemann curvature tensor $\overline{R}$ of $(M^n, \overline{g})$ to the one R of (M^n, g) by

$$\begin{aligned} &\overline{R}(\beta^{-1}X, \beta^{-1}Z)(\beta^{-1}Y) - \beta^{-1}\{R(\beta^{-1}X, \beta^{-1}Z)(Y)\} \\ =\;& \beta^{-1}\{(\nabla_{\beta^{-1}(X)}\Lambda)(Z, Y) - (\nabla_{\beta^{-1}(Z)}\Lambda)(X, Y)\} + \beta^{-1}\{\Lambda(X, \Lambda(Z, Y)) \\ &-\Lambda(Z, \Lambda(X, Y))\} + \beta^{-1}\{\Lambda(\Lambda(Z, X) - \Lambda(X, Z),\, Y)\}. \end{aligned}$$

Let $(E_1, \ldots, E_n)$ be a local g-orthonormal frame field on (M^n, g). Then the scalar curvature $\overline{S}$ of $(M^n, \overline{g})$ is expressed as [6]

$$\begin{aligned} \overline{S} - \sum_{i,j=1}^{n} g(E_i, R(\beta^{-1}E_i, \beta^{-1}E_j)(E_j)) \\ = \ 2\sum_{i,j=1}^{n} g(E_i, (\nabla_{\beta^{-1}(E_i)}\Lambda)(E_j, E_j)) \\ - \sum_{i,j,k=1}^{n} \Lambda_{iik}\Lambda_{jjk} - \sum_{i,j,k=1}^{n} \Lambda_{ijk}\Lambda_{jik}, \end{aligned} \tag{14}$$

where $\Lambda_{ijk} := g(\Lambda(E_i, E_j), E_k)$. We now review briefly the behavior of the Dirac operator under the deformation (11) of metrics. Let $\Sigma(M)_g$ and $\Sigma(M)_{\overline{g}}$ be the spinor bundles of (M^n, g) and $(M^n, \overline{g})$, respectively. There are natural isomorphisms $\beta^{-1} : T(M) \longrightarrow T(M)$ and $\widehat{\beta^{-1}} : \Sigma(M)_g \longrightarrow \Sigma(M)_{\overline{g}}$ preserving the inner products of vectors and spinors as well as the Clifford multiplication:

$$\overline{g}(\beta^{-1}X, \beta^{-1}Y) = g(X, Y), \ \ \langle\widehat{\beta^{-1}}\varphi, \widehat{\beta^{-1}}\psi\rangle_{\overline{g}} \ = \ \langle\varphi, \psi\rangle_g,$$

$$(\beta^{-1}X) \cdot (\widehat{\beta^{-1}}\psi) = \widehat{\beta^{-1}}(X \cdot \psi), \ \ X, Y \in \Gamma(T(M)), \ \ \varphi, \psi \in \Gamma(\Sigma(M)_g).$$

For each spinor field ψ on (M^n, g) we denote by $\overline{\psi} := \widehat{\beta^{-1}}(\psi)$ the corresponding spinor field on $(M^n, \overline{g})$. We will use the same notation for vector fields, $\overline{X} := \beta^{-1}(X)$. It follows from (12) that the spinor derivatives $\overline{\nabla}$, ∇ are related by

$$\overline{\nabla}_{\beta^{-1}(E_j)}\overline{\psi} = \overline{\nabla_{\beta^{-1}(E_j)}\psi} + \frac{1}{4}\sum_{k,l=1}^{n} \Lambda_{jkl}\overline{E_k} \cdot \overline{E_l} \cdot \overline{\psi}. \tag{15}$$

Let ω and Ω be a 1-form and a 3-form generated by the tensor Λ via

$$\omega = \sum_{j,k=1}^{n} \Lambda_{jjk}E^k, \qquad E^k := g(\cdot, E_k),$$

and

$$\Omega = \sum_{j<k<l} (\Lambda_{jkl} + \Lambda_{klj} + \Lambda_{ljk})E^j \wedge E^k \wedge E^l,$$

respectively. The Dirac operator $\overline{D}$ of $(M^n, \overline{g})$ can be expressed through the β-twist D_β of D as

$$\begin{aligned} \overline{D}\overline{\psi} \ &= \ \sum_{i=1}^{n} \overline{E_i} \cdot \overline{\nabla}_{\overline{E_i}}\overline{\psi} \\ &= \ \sum_{i=1}^{n} \overline{E_i \cdot \nabla_{\beta^{-1}(E_i)}\psi} + \frac{1}{4}\sum_{j,k,l=1}^{n} \Lambda_{jkl}\overline{E_j \cdot E_k \cdot E_l \cdot \psi} \end{aligned}$$

$$= \overline{D_\beta \psi} - \frac{1}{2}\overline{\omega \cdot \psi} + \frac{1}{2}\overline{\Omega \cdot \psi} \tag{16}$$

and the square $\overline{D}^2$ of the Dirac operator $\overline{D}$ as

$$\begin{aligned} \overline{D}^2\,\overline{\psi} &= \overline{(D_\beta \circ D_\beta)(\psi)} - \frac{1}{2}\overline{\omega \cdot D_\beta \psi} + \frac{1}{2}\overline{\Omega \cdot D_\beta \psi} \\ &\quad -\frac{1}{2}\overline{D_\beta(\omega \cdot \psi)} + \frac{1}{2}\overline{D_\beta(\Omega \cdot \psi)} + \frac{1}{4}\overline{\omega \cdot \omega \cdot \psi} \\ &\quad -\frac{1}{4}\overline{\Omega \cdot \omega \cdot \psi} - \frac{1}{4}\overline{\omega \cdot \Omega \cdot \psi} + \frac{1}{4}\overline{\Omega \cdot \Omega \cdot \psi}. \end{aligned} \tag{17}$$

In the paper we focus our attention on an interesting property of the tensor Λ.
Note that

$$\begin{aligned} \Lambda(X,Y) - \Lambda(Y,X) &= \beta\{(\nabla_{\beta^{-1}X}\beta^{-1})(Y)\} - \beta\{(\nabla_{\beta^{-1}Y}\beta^{-1})(X)\} \\ &= -(\nabla_{\beta^{-1}X}\beta)(\beta^{-1}Y) + (\nabla_{\beta^{-1}Y}\beta)(\beta^{-1}X). \end{aligned}$$

Therefore, if β is a Codazzi tensor, then $\Lambda \equiv 0$. Consequently, all the equations simplify remarkably when β is a Codazzi tensor.

Lemma 2.1. *Let β be a nondegenerate Codazzi tensor on (M^n, g). Then we have:*

$$\overline{S} = \sum_{i,j=1}^{n} g(E_i, R(\beta^{-1}E_i, \beta^{-1}E_j)(E_j)), \tag{18}$$

$$\overline{\nabla}_X \overline{\psi} = \overline{\nabla_X \psi}, \tag{19}$$

$$\overline{D}\,\overline{\psi} = \overline{D_\beta \psi}, \tag{20}$$

$$\overline{D}^2\,\overline{\psi} = \overline{(D_\beta \circ D_\beta)(\psi)}. \tag{21}$$

We close the section with some more lemmata needed in the next section.

Lemma 2.2. *Let β be a nondegenerate symmetric tensor field on (M^n, g). If there exists a nontrivial spinor field ψ on (M^n, g) such that*

$$\nabla_X \psi = p\,X \cdot D\psi + q\,\beta^{-1}(X) \cdot D_\beta \psi \tag{22}$$

holds for some real-valued functions $p, q : M^n \longrightarrow \mathbb{R}$ and for all vector fields X, then

$$(1 + np)D\psi = -q\,\mathrm{tr}(\beta^{-1})D_\beta \psi, \tag{23}$$

$$(1 + q|\beta^{-1}|^2)D_\beta \psi = -p\,\mathrm{tr}(\beta^{-1})D\psi. \tag{24}$$

Lemma 2.3. *Let $(\cdot,\cdot) := \mathrm{Re}\langle\cdot,\cdot\rangle$ denote the real part of the standard Hermitian product $\langle\cdot,\cdot\rangle$ on the spinor bundle $\Sigma(M)$ over M^n. Let ψ and F be a spinor field and a real-valued function on M^n, respectively. Then we have*

$$F \cdot \Delta(\psi,\psi) - (\psi,\psi) \cdot \Delta F = \mathrm{div}\{(\psi,\psi)\mathrm{grad}F - F\,\mathrm{grad}(\psi,\psi)\}.$$

3. Proof of the theorems

Note that the volume form $\overline{\mu}$ of $(M^n, \overline{g})$ is related to the one μ of (M^n, g) by

$$\overline{\mu} = |\det(\beta^{-1})|\mu. \tag{25}$$

Let $\mathcal{Q} : \Gamma(T(M)) \times \Gamma(\Sigma(M)_g) \longrightarrow \Gamma(\Sigma(M)_g)$ be a *twistor-like operator* defined by

$$\mathcal{Q}_X(\varphi) = \nabla_X \varphi - p\, X \cdot D\varphi - q\, \beta^{-1}(X) \cdot D_\beta \varphi,$$

where $p, q : M^n \longrightarrow \mathbb{R}$ are some real-valued functions. Then we have

$$\begin{aligned}
&\sum_{j=1}^{n} (\mathcal{Q}_{E_j}(\varphi),\, \mathcal{Q}_{E_j}(\varphi)) \\
= \;& \operatorname{div}\Big[\sum_{j=1}^{n} (\varphi,\, E_j \cdot D\varphi + \nabla_{E_j}\varphi) E_j\Big] + (np^2 + 2p + 1)(D\varphi, D\varphi) \\
&-\frac{1}{4} S(\varphi, \varphi) + \{q^2 |\beta^{-1}|^2 + 2q\}(D_\beta \varphi, D_\beta \varphi) + 2pq \operatorname{tr}(\beta^{-1})(D\varphi, D_\beta \varphi).
\end{aligned} \tag{26}$$

Now, let ψ be an eigenspinor of D with eigenvalue $\lambda \neq 0 \in \mathbb{R}$. By Lemma 2.3, we then see that

$$\int_{M^n} F \operatorname{div}\Big[\sum_{j=1}^{n} (\psi,\, E_j \cdot D\psi + \nabla_{E_j}\psi) E_j\Big] \mu = -\frac{1}{2} \int_{M^n} (\psi, \psi) \Delta(F) \mu \tag{27}$$

holds for any real-valued function $F : M^n \longrightarrow \mathbb{R}$, since $(\psi, E_j \cdot \psi) = 0$. Let $\overline{\lambda}_1 \neq 0$ be the smallest eigenvalue of $\overline{D}$. Making use of (20), (25), (26), (27) and introducing free functions $F, B, C : M^n \longrightarrow \mathbb{R}$ (We assume that F is a positive function.) to control the unnecessary terms, we compute

$$\begin{aligned}
H_1 \;:=\; & \int_{M^n} \Big[(\overline{D}\,\overline{\psi}, \overline{D}\,\overline{\psi}) - \overline{\lambda}_1^2 (\overline{\psi}, \overline{\psi}) \Big] \overline{\mu} \\
& + \int_{M^n} \left[F \sum_{j=1}^{n} (\mathcal{Q}_{E_j}(\psi), \mathcal{Q}_{E_j}(\psi)) + B^2 (D_\beta \psi - C D\psi, D_\beta \psi - C D\psi) \right] \mu \\
=\;& \int_{M^n} \Big[\lambda^2 \Big((np^2 + 2p + 1) F + B^2 C^2 \Big) \\
& -\frac{1}{4} F S - \overline{\lambda}_1^2 |\det(\beta^{-1})| - \frac{1}{2} (\Delta F) \Big] (\psi, \psi) \mu \\
& + \int_{M^n} \Big[2\lambda \Big(pqF \operatorname{tr}(\beta^{-1}) - B^2 C \Big) (\psi, D_\beta \psi) \\
& + \Big((q^2 |\beta^{-1}|^2 + 2q) F + B^2 + |\det(\beta^{-1})| \Big) (D_\beta \psi, D_\beta \psi) \Big] \mu.
\end{aligned} \tag{28}$$

We choose the functions B, C in such a way that the second integral of (28) vanishes and the equations (23), (24) are satisfied with $D_\beta\psi = CD\psi$. To this end, it is required that the relations

$$B^2 = -qF(1+q|\beta^{-1}|^2)\ , \quad C^2 = \frac{p(1+np)}{q(1+q|\beta^{-1}|^2)} \tag{29}$$

hold. Note that (29) implies, in particular, the restriction (4) :

$$-\frac{1}{|\beta^{-1}|^2} \le q < 0, \qquad -\frac{1}{n} < p \le 0.$$

Now choose

$$F = -\frac{|\det(\beta^{-1})|}{q}, \qquad B^2 = |\det(\beta^{-1})|(1+q|\beta^{-1}|^2) \tag{30}$$

so that the last line in the latter part of (28) vanishes. Then we obtain

$$H_1 = \int\limits_{M^n} \left[\lambda^2(p+1)F - \frac{1}{4}FS - \overline{\lambda}_1^2|\det(\beta^{-1})| - \frac{1}{2}(\Delta F)\right](\psi,\psi)\mu \ge 0,$$

which proves the inequality of Theorem 1.1. The functions optimal for p and q are to be found when considering the limiting case. The former part of (28) yields in the limiting case that

$$\overline{D}\,\overline{\psi_1} = \overline{\lambda}_1\overline{\psi_1} = \overline{D_\beta\psi_1} = C\lambda_1\,\overline{\psi_1}. \tag{31}$$

Since $\overline{\lambda}_1 = C\lambda_1$, we find that the function C must be a nonzero constant $C = c \neq 0 \in \mathbb{R}$. Then, from (31) and (23), (24), we obtain the two relations in (5) immediately. The condition (6), (7) for the limiting case is easy to check.

To prove Theorem 1.3 we consider the integral

$$\begin{aligned} H_2 \;&:= \int\limits_{M^n}\Big[(\overline{D}\,\overline{\psi}, \overline{D}\,\overline{\psi}) - \overline{\lambda}_1^2(\overline{\psi},\overline{\psi})\Big]\overline{\mu} + \int\limits_{M^n}\Big[F\sum_{j=1}^n(Q_{E_j}(\psi), Q_{E_j}(\psi))\Big]\mu \\ &= \int\limits_{M^n}\Big[\lambda^2(np^2+2p+1)F - \frac{1}{4}FS - \overline{\lambda}_1^2|\det(\beta^{-1})| - \frac{1}{2}(\Delta F)\Big](\psi,\psi)\mu \\ &\quad + \int\limits_{M^n}\Big[(q^2|\beta^{-1}|^2+2q)F + |\det(\beta^{-1})|\Big](D_\beta\psi, D_\beta\psi)\mu \end{aligned} \tag{32}$$

and choose the free parameters p, q, F as

$$p = -\frac{1}{n}, \qquad q = -\frac{1}{|\beta^{-1}|^2}, \qquad F = |\det(\beta^{-1})||\beta^{-1}|^2.$$

Then the last line in the latter part of (32) vanishes and we have

$$H_2 = \int\limits_{M^n}\left[\frac{(n-1)\lambda^2F}{n} - \frac{1}{4}FS - \overline{\lambda}_1^2|\det(\beta^{-1})| - \frac{1}{2}(\Delta F)\right](\psi,\psi)\mu \ge 0.$$

This proves the inequality (8). The condition (10) for the limiting case is clear.

Remark 3.1. Let $\overline{\lambda}_1 \neq 0 \in \mathbb{R}$ be the smallest eigenvalue of $\overline{D}$. Suppose that there exist a nonzero constant $\lambda \neq 0 \in \mathbb{R}$ and a spinor field ψ such that the following equations hold:

$$D\psi = \lambda\psi\,, \quad D_\beta\psi = \overline{\lambda}_1\psi\,, \quad \nabla_X\psi = \lambda\, p\, X\cdot\psi + \overline{\lambda}_1 q\, \beta^{-1}(X)\cdot\psi.$$

Then it turns out that $\lambda = \lambda_1$ is equal to the smallest eigenvalue of D and, in the limiting case, the constant c in (5) is related to λ_1, $\overline{\lambda}_1$ by $\overline{\lambda}_1 = c\,\lambda_1$.

References

[1] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.
[2] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten, Riemann-schen Mannigfaltigkeit nichtnegativer Skalarkrümmung*, Math. Nachr. **97** (1980), 117–146.
[3] ———, *Dirac Operators in Riemannian Geometry*, Graduate Studies in Mathematics, 25. American Mathematical Society, Providence, RI, 2000.
[4] Th. Friedrich and E. C. Kim, *Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors*, J. Geom. Phys. **37** (2001), no. 1-2, 1–14.
[5] Th. Friedrich and K.-D. Kirchberg, *Eigenvalue estimates of the Dirac operator depending on the Ricci tensor*, Math. Ann. **324** (2002), no. 4, 799–816.
[6] E. C. Kim, *A local existence theorem for the Einstein-Dirac equation*, J. Geom. Phys. **44** (2002), no. 2-3, 376–405.
[7] ———, *The $\widehat{A}$-genus and symmetry of the Dirac spectrum on Riemannian product manifolds*, Differential Geom. Appl. **25** (2007), no. 3, 309–321.
[8] K.-D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Global Anal. Geom. **4** (1986), no. 3, 291–325.
[9] W. Kramer, U. Semmelmann, and G. Weingart, *Eigenvalue estimates for the Dirac operator on quaternionic Kähler manifolds*, Math. Z. **230** (1999), no. 4, 727–751.

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