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LINEAR OPERATORS THAT PRESERVE PERIMETERS OF BOOLEAN MATRICES

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Abstract. For a Boolean rank 1 matrix $A = \mathbf{a}\mathbf{b}^t$, we define the perimeter of A as the number of nonzero entries in both $\mathbf{a}$ and $\mathbf{b}$. The perimeter of an $m \times n$ Boolean matrix A is the minimum of the perimeters of the rank-1 decompositions of A. In this article we characterize the linear operators that preserve the perimeters of Boolean matrices.

1. Introduction and preliminaries

The *Boolean algebra* consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

There are many papers on linear operators that preserve the rank of matrices over several semirings. Boolean matrices also have been the subject of research by many authors ([3], [4] and [5]). Beasley and Pullman ([1]) obtained characterizations of rank-preserving operators of Boolean matrices. Song et al. ([6]) characterized the Boolean linear operators that preserve rank and perimeter of Boolean rank-1 matrices only.

In this article we extend the results in [6] to matrices of arbitrary Boolean rank. That is, we characterize the linear operators that preserve the perimeters of Boolean matrices of arbitrary rank.

Let $\mathbb{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in the Boolean algebra $\mathbb{B}$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well. Throughout this paper, we shall adopt the convention that $2 \leq m \leq n$ unless otherwise specified.

A Boolean matrix in $\mathbb{M}_{m,n}(\mathbb{B})$ is called a *cell* if it has exactly one 1. We denote the cell whose one 1 is in the $(i,j)^{th}$ position by $E_{i,j}$. Let $\mathbb{E} = \{E_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

An $n \times n$ Boolean matrix A is said to be *invertible* if for some X, $AX = XA = I_n$, where I_n is the $n \times n$ identity matrix. This matrix X is necessarily

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unique when it exists. It is well known that the permutation matrices are the only invertible Boolean matrices (see [1]).

If an $m \times n$ Boolean matrix A is not zero, then its *Boolean rank*, $b(A)$, is the least k for which there exist $m \times k$ and $k \times n$ Boolean matrices B and C with $A = BC$. The Boolean rank of the zero matrix is 0. It is well known that $b(A)$ is the least k such that A is the sum of k matrices of Boolean rank 1 ([1], [6]). If $A = \sum_{i=1}^{k} A_i$ is a sum of Boolean rank 1 matrices A_i, then this sum $\sum_{i=1}^{k} A_i$ is called a *Boolean rank* 1 *decomposition* of A. Further, the zero matrix is the only matrix with 0 rank 1 decomposition.

If A and B are in $\mathbb{M}_{m,n}(\mathbb{B})$, we say A *dominates* B (written $B \le A$ or $A \ge B$) if $a_{ij} = 0$ implies $b_{ij} = 0$ for all i, j. Equivalently, $B \le A$ if and only if $A + B = A$.

Also lowercase, boldface letters will represent vectors, all vectors $\mathbf{u}$ are column vectors ($\mathbf{u}^t$ is a row vector) for $\mathbf{u} \in \mathbb{B}^m$ $[= \mathbb{M}_{m,1}(\mathbb{B})]$.

It is easy to verify that the Boolean rank of $A \in \mathbb{M}_{m,n}(\mathbb{B})$ is 1 if and only if there exist nonzero (Boolean) vectors $\mathbf{a} \in \mathbb{M}_{m,1}(\mathbb{B})$ and $\mathbf{b} \in \mathbb{M}_{n,1}(\mathbb{B})$ such that $A = \mathbf{a}\mathbf{b}^t$. And these vectors $\mathbf{a}$ and $\mathbf{b}$ are uniquely determined by A. Therefore there are exactly $(2^m - 1)(2^n - 1)$ rank 1 $m \times n$ Boolean matrices.

The *perimeter* of a Boolean rank 1 matrix $A = \mathbf{a}\mathbf{b}^t \in \mathbb{M}_{m,n}(\mathbb{B})$, $p(A)$, is $|\,\mathbf{a}\,| + |\,\mathbf{b}\,|$ where $|\,\mathbf{a}\,|$ denotes the number of nonzero entries in $\mathbf{a}$. Since the factorization of A as $\mathbf{a}\mathbf{b}^t$ is unique, the perimeter of A is also unique (see [1]).

For $A \in \mathbb{M}_{m,n}(\mathbb{B})$, the *perimeter* of A, $p(A)$, is defined as

$$\min\left\{\sum_{i=1}^{k} p(A_i) \mid A = \sum_{i=1}^{k} A_i \text{ is a Boolean rank 1 decomposition of } A\right\}.$$

Example 1.1. In $\mathbb{M}_{2,2}(\mathbb{B})$, consider these matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Then $b(A) = b(B) = b(E) = 1$ but $b(C) = b(D) = 2$. And $p(A) = 2, p(B) = 3, p(C) = 4, p(D) = 5$ and $p(E) = 4$.

Let $\mathbb{P}_k$, $k = 2, \ldots, mn$, denote the set of matrices in $\mathbb{M}_{m,n}(\mathbb{B})$ whose perimeter is k. An easy observation is that every matrix in $\mathbb{M}_{m,n}(\mathbb{B})$ whose perimeter is either 2 or 3 has Boolean rank 1.

A *line* of a matrix is defined to be a row or column of the matrix. The *term rank* of a matrix A, $t(A)$, is the minimum number of lines that contain all the nonzero entries of the matrix (see [2]). A *generalized diagonal* of an $m \times n$ $(m \leq n)$ Boolean matrix A is a submatrix consisting of the sum of m cells no two in any one line. So, the set of permutation matrices is the set of all generalized diagonals of J_n, the $n \times n$ matrix of all ones.

A mapping $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{m,n}(\mathbb{B})$ is called a *Boolean linear operator* if $T(A+B) = T(A)+T(B)$ for all $A, B \in \mathbb{M}_{m,n}(\mathbb{B})$ and $T(0) = 0$ for zero matrix $0 \in \mathbb{M}_{m,n}(\mathbb{B})$. A linear operator $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{m,n}(\mathbb{B})$ is called a *(U, V)-operator* (see [6]) if there exist invertible matrices U and V of appropriate orders, such that $T(A) = UAV$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$, or, $m = n$ and $T(A) = PA^tQ$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$ where A^t denotes the transpose of A. A linear operator T is said to *preserve* a set $\mathbb{Q}$ if $A \in \mathbb{Q}$ implies $T(A) \in \mathbb{Q}$.

There have been many articles on linear preserver problems. For an excellent survey see [4, 5]. In [2], the linear operators that preserve the term rank and other combinatorial properties of matrices were characterized and in [6] the linear operators $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{m,n}(\mathbb{B})$ which preserve both the set of matrices of Boolean rank one and the set $\mathbb{P}_k$ for some k were studied.

In the followings, each linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ means the Boolean linear operator.

2. Perimeter preservers of Boolean matrices

In this section, we characterize the linear operators that preserve the perimeter of Boolean matrices in $\mathbb{M}_{m,n}(\mathbb{B})$.

The following example shows that not all perimeter-2 preserving operators T are of the form $T(A) = UAV$ for some invertible Boolean matrices U, V.

Example 2.1. Let $T : \mathbb{M}_{2,2}(\mathbb{B}) \to \mathbb{M}_{2,2}(\mathbb{B})$ be defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+b+c+d)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then T is a linear operator and preserves $\mathbb{P}_2$. But T is not a (U, V)-operator: If there existed invertible matrices U and V such that $T(A) = UAV$ for all $A \in \mathbb{M}_{2,2}(\mathbb{B})$, then for $j = 1, 2$, we have $T(E_{1,j}) = U\mathbf{e}_1\mathbf{e}_j{}^tV = \mathbf{u}_1\mathbf{v}_j{}^t$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $\mathbf{u}_1$ is the first column of U and $\mathbf{v}_j$ is the jth column of V. But

$$T(E_{1,1}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1, 0]$$

and

$$T(E_{1,2}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1, 0]$$

and hence

$$V = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

which is not invertible. This contraction implies that T is not a (U, V)-operator.

Moreover, T does not preserve $\mathbb{P}_3, \mathbb{P}_4$ and $\mathbb{P}_5$ since $T(B) = T(C) = T(D) = A$ for the matrices in Example 1.1.

Theorem 2.1. *If T is a (U, V)-operator on $\mathbb{M}_{m,n}(\mathbb{B})$, then T preserves the perimeter of each Boolean matrix.*

Proof. **Case 1.** Let A_i be a matrix in $\mathbb{M}_{m,n}(\mathbb{B})$ with $b(A_i) = 1$ and $A_i = \mathbf{a}\mathbf{b}^t$ be the factorization of A_i with $p(A_i) = |\mathbf{a}| + |\mathbf{b}|$. Since $T(A_i) = UA_iV = (U\mathbf{a})(\mathbf{b}^tV) = (U\mathbf{a})(V^t\mathbf{b})^t$ and U, V are invertible matrices (and hence permutations), we have

$$b(T(A_i)) = b\left((U\mathbf{a})(V^t\mathbf{b})^t\right) = 1,$$

and

$$p(T(A_i)) = |U\mathbf{a}| + |V^t\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| = p(A_i).$$

If $m = n$ and $T(A_i) = UA_i{}^tV$, then we can show that $b(T(A_i)) = 1$ and $p(T(A_i)) = |\mathbf{a}| + |\mathbf{b}|$ by the similar method as above.

Hence (U, V)-operator preserves the perimeter of each Boolean matrix of Boolean rank 1.

Case 2. Now, consider arbitrary matrix $A \in \mathbb{M}_{m,n}(\mathbb{B})$ and its rank 1 decompositions of A such that $A = \sum_{i=1}^{k} A_i$. Since

$$T(A) = UAV = U\left(\sum_{i=1}^{k} A_i\right)V = \sum_{i=1}^{k} UA_iV$$

is a rank 1 decomposition of $T(A)$ and $p(UA_iV) = p(A_i)$ from Case 1, we have

$$\begin{aligned}
&p(T(A))\\
&= p(UAV)\\
&= \min\left\{\sum_{i=1}^{k} p(UA_iV) \mid \sum_{i=1}^{k} UA_iV \text{ is a Boolean rank 1decomposition of } UAV\right\}\\
&= \min\left\{\sum_{i=1}^{k} p(A_i) \mid \sum_{i=1}^{k} A_i \text{ is a Boolean rank 1 decomposition of } A\right\}\\
&= p(A).
\end{aligned}$$

Thus (U, V)-operators preserve the perimeter of each Boolean matrix in $\mathbb{M}_{m,n}(\mathbb{B})$. □

Lemma 2.1. *If $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{m,n}(\mathbb{B})$ is a linear operator which preserves $\mathbb{P}_2$, then there exists a mapping $f : \mathbb{E} \to \mathbb{E}$ such that for $A = (a_{i,j})$, $T(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} f(E_{i,j})$.*

Proof. Since $\mathbb{P}_2$ is the set of all cells in $\mathbb{M}_{m,n}(\mathbb{B})$, the lemma holds. □

Lemma 2.2. *If A is a Boolean matrix of perimeter 4, then one of the followings holds:*

(1) $t(A) = 1$ *and A is a sum of three collinear cells,*
(2) $t(A) = 2$ *and A is a sum of two non collinear cells,*
(3) *A is a sum of four cells lying in the intersection of two rows and two columns.*

Proof. It is obvious from the definition of perimeter. □

Lemma 2.3. *For the elements of $\mathbb{P}_k$, we have the following structures:*

(1) *The elements of $\mathbb{P}_3$ are the sum of two cells in one line.*
(2) *The elements of $\mathbb{P}_5$ with fewest number of nonzero entries are the sum of two cells in one line and another one cell not in that line.*
(3) *If $k = 2l$ for any $l \le m$, then the elements of $\mathbb{P}_k$ with fewest number of nonzero entries are the sum of l cells which lie on a generalized diagonal. The sum of these l nonzero entries has term rank l.*
(4) *If $k \ge 7$, $k = 2l + 1$ for any $l \le m$, then the elements of $\mathbb{P}_k$ with fewest number of nonzero entries are the sum of $l + 1$ cells, l of which have nonzero entries only on a generalized diagonal and the other cell collinear with at least one of the first l cells. The sum of these $l + 1$ nonzero entries has term rank l.*

Proof. It is trivial from the form of Boolean matrices with fewest number of nonzero entries in $\mathbb{P}_k$. □

Lemma 2.4. *Let A be a Boolean matrix with term rank l. Then*

(1) *if A is the sum of l cells, then $p(A) = 2l$;*
(2) *if A is the sum of $l + 1$ cells, then $p(A) = 2l + 1$;*
(3) *if A is the sum of $l + 2$ cells, then $p(A) = 2l$ or $2l + 2$.*

Proof. For (1) and (2), they are obvious.

For (3), let A be the sum of $l + 2$ cells, and $t(A) = l$. Without loss of generality we may assume that $a_{i,i} \neq 0$ for $i = 1, \ldots, l$. Then it suffices to consider three cases:

$$A_1 = I_{l-1} \oplus \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \oplus 0_{(m-l),(n-l-2)},$$

$$A_2 = I_{l-2} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \oplus 0_{(m-l),(n-l)},$$

$$A_3 = I_{l-2} \oplus \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \oplus 0_{(m-l),(n-l-1)},$$

where the entries of empty positions are zeros, and $0_{s,t}$ is the $s \times t$ zero matrix. Then $p(A_1) = 2(l-1) + 4 = 2l + 2$, $p(A_2) = 2(l-2) + 4 = 2l$ and $p(A_3) = 2(l-2) + 2 \cdot 3 = 2l + 2$. □

Lemma 2.5. *Suppose* $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{m,n}(\mathbb{B})$ *is a linear operator defined by* $T(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} f(E_{i,j})$ *for some function* $f : \mathbb{E} \to \mathbb{E}$. *If* T *preserves* $\mathbb{P}_k$ *for a* k *with* $4 \le k \le 2m+1$, *then* f *is bijective.*

Proof. **Case 1)** $k = 2l$ for a l with $2 \le l \le m$. Suppose that f is not bijective. Then there are two cells $E_{i,j}$ and $E_{r,s}$ such that $f(E_{i,j}) = f(E_{r,s})$. If these cells are not collinear then there is an element, X, of $\mathbb{P}_k$ with fewest entries which dominate $E_{i,j}$ and $E_{r,s}$. But then, $T(A)$ cannot be an element of $\mathbb{P}_k$ since it has fewer nonzero entries than those of A. Thus, $E_{i,j}$ and $E_{r,s}$ are collinear. Without loss of generality we may assume that $f(E_{1,1}) = f(E_{1,2})$. Now, for any $q > 2$, $E_{1,1} + E_{1,2} + E_{1,q} + E_{2,2} + \cdots + E_{l-1,l-1}$ has perimeter k and its image has at most l cells. Since T preserves $\mathbb{P}_k$, $T(E_{1,1} + E_{1,2} + E_{1,q} + E_{2,2} + \cdots + E_{l-1,l-1})$ has perimeter k and must have at least l cells by Lemma 2.3(3). Thus, $T(E_{1,1} + E_{1,2} + E_{1,q} + E_{2,2} + \cdots + E_{l-1,l-1})$ has exactly l cells and has perimeter $k = 2l$. By Lemma 2.3(3), $T(E_{1,1} + E_{1,2} + E_{1,q} + E_{2,2} + \cdots + E_{l-1,l-1})$ has term rank l. Without loss of generality we may assume that $f(E_{1,1}) = E_{1,1}, f(E_{2,2}) = E_{2,2}, \ldots, f(E_{l-1,l-1}) = E_{l-1,l-1}$ and that for each $q \ge 3$, $f(E_{1,q}) = E_{u,v}$ for some $u, v \ge l$. Further the set of all cells $\{f(E_{1,q}) | q \ge 3\}$ lie in one line. If not, say, $f(E_{1,3}) + f(E_{1,4})$ has term rank 2, then $T(E_{1,1} + E_{1,3} + E_{1,4} + E_{2,2} + \cdots + E_{l-1,l-1})$ is a sum of $l+1$ cells of term rank $l+1$, whose perimeter must be $k+2$ by Lemma 2.3(3), a contradiction since $p(E_{1,1} + E_{1,3} + E_{1,4} + E_{2,2} + \cdots + E_{l-1,l-1}) = k$. But then, $T(E_{1,1} + E_{1,3} + E_{1,4} + E_{2,2} + \cdots + E_{l-1,l-1})$ dominates $E_{1,1}, E_{2,2}, \ldots, E_{l-1,l-1}$ and two cells $E_{u,v_1} + E_{u,v_2}$ or $E_{u_1,v} + E_{u_2,v}$ with $u, u_1, u_2, v, v_1, v_2 \ge l$. This must have perimeter $k = 2l$, thus $v_1 = v_2$ or $u_1 = u_2$. It follows that for some fixed u, v, $f(E_{1,q}) = E_{u,v}$ for all $q \ge 3$. But then, $T(E_{1,3} + E_{1,4} + E_{1,5} + E_{2,1} + E_{3,2} + E_{4,4} + \cdots + E_{l-1,l-1})$ dominates at most $l-1$ cells and hence has perimeter less than k, while $p(E_{1,3} + E_{1,4} + E_{1,5} + E_{2,1} + E_{3,2} + E_{4,4} + \cdots + E_{l-1,l-1}) = k$, a contradiction. Therefore f is bijective.

Case 2) $k = 2l+1$ for a l with $2 \le l \le m$. Suppose that f is not bijective. Then there are two cells $E_{i,j}$ and $E_{r,s}$ such that $f(E_{i,j}) = f(E_{r,s})$. By Lemma 2.3 there is an element, A, of $\mathbb{P}_k$ of minimum nonzero entries which dominates $E_{i,j}$ and $E_{r,s}$. But then, $T(A)$ cannot be an element of $\mathbb{P}_k$ since it has fewer nonzero entries than those of A. Thus f is bijective. □

Lemma 2.6. *If* $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{m,n}(\mathbb{B})$ *is a linear operator which preserves* $\mathbb{P}_2$ *and* $\mathbb{P}_k$ *for a* k *with* $4 \le k \le 2m+1$, *then* T *maps lines to lines.*

Proof. **Case 1)** $k = 2l$ for a l with $2 \le l \le m$. Since T preserves perimeter 2, by Lemma 2.1, there is a mapping $f : \mathbb{E} \to \mathbb{E}$ such that for $A = (a_{i,j})$, $T(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} f(E_{i,j})$. By Lemma 2.5, f is bijective. If T does not preserve lines, since f is bijective, there are two cells not lying on one line and whose images do lie on one line. Without loss of generality, we may assume $T(E_{1,1} + E_{2,2})$ lie on the same line. But then, $E_{1,1} + E_{2,2} + \cdots + E_{l,l}$ has perimeter $2l = k$ while $T(E_{1,1} + E_{2,2} + \cdots + E_{l,l})$ is the sum of l cells of term rank at most $l-1$ and consequently has perimeter less than $2l$, a contradiction. Thus T maps lines to lines.

Case 2) $k = 2l+1$ for a l with $2 \leq l \leq m$. Since T preserves perimeter 2, by Lemma 2.1, there is a mapping $f : \mathbb{E} \to \mathbb{E}$ such that for $A = (a_{i,j})$, $T(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} f(E_{i,j})$. By Lemma 2.5, f is bijective.

Suppose that T does not preserve lines. Then without loss of generality we may assume that $T(E_{1,1}+E_{1,2})$ has term rank 2. But then $T(E_{1,1} + E_{1,2} + E_{2,2} + \cdots + E_{l,l})$ is a sum of $l+1$ cells and must have perimeter $2l+1$ since $p(E_{1,1} + E_{1,2} + E_{2,2} + \cdots + E_{l,l}) = 2l+1$. That can only happen if $T(E_{1,1} + E_{1,2} + E_{2,2} + \cdots + E_{l,l})$ has term rank l. Let A be the sum of the l cells in $\{E_{1,1}, E_{1,2}, E_{2,2}, \ldots, E_{l,l}\}$ which includes $E_{1,1}$ and $E_{1,2}$ whose image has term rank l. Then $T(A + E_{1,3} + E_{1,4})$ is the sum of $l+2$ cells of term rank l or more. By Lemma 2.4 the perimeter of $T(A + E_{1,3} + E_{1,4})$ is either $2l$, $2l+2$, $2l+3$ or $2l+4$, while the perimeter of $A + E_{1,3} + E_{1,4}$ is $2l+1 = k$, a contradiction. Thus T maps lines to lines. □

Theorem 2.2. *If $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{m,n}(\mathbb{B})$ is a linear operator which preserves $\mathbb{P}_2$ and $\mathbb{P}_k$ for a k with $4 \leq k \leq 2m+1$, then T is a (U,V)-operator.*

Proof. By Lemmas 2.1 and 2.5, T induces a bijection on the set of cells. By Lemma 2.6, T preserves lines. These two facts imply that T maps rows to rows and columns to columns or rows to columns and columns to rows (when $m = n$). It easily follows that there exist invertible matrices U and V of orders m and n respectively, such that $T(A) = UAV$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$ or $m = n$ and $T(A) = UA^tV$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$. Therefore T is a (U,V)-operator. □

Theorem 2.3. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{m,n}(\mathbb{B})$ be a linear operator. Then the following are equivalent:*

(1) *T preserves the perimeter of Boolean matrices;*
(2) *T preserves P_2 and P_k $(4 \leq k \leq 2m+1)$ of Boolean matrices;*
(3) *T is a (U,V)-operator.*

Proof. It is immediate that (1) implies (2). By Theorem 2.2, (2) implies (3). By Theorem 2.1, (3) implies (1). □

Now, we characterize those linear operators which preserve P_2 and P_3. In the following theorem we shall use the notation R_i to denote $\sum_{j=1}^{n} E_{i,j}$ and C_j to denote $\sum_{i=1}^{m} E_{i,j}$, that is, R_i is the matrix with ones in the i^{th} row and zeros elsewhere, and C_j is the matrix with ones in the j^{th} column and zeros elsewhere.

Example 2.2. Consider a linear map T on $\mathbb{M}_{m,n}(\mathbb{B})$ with $3 \leq m \leq n$ such that

$$T\left(\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}\right) = \begin{bmatrix} \sum_{i=1}^{m} b_{i,i} & \sum_{i=1}^{m} b_{i,i+1} & \cdots & \sum_{i=1}^{m} b_{i,i+(n-1)} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $b_{i,j} = a_{i,k}$ with $j \equiv k \pmod{n}$ and $1 \leq k \leq n$. Then T maps each row into a row (the first row) and each column into a row (the first row). Moreover, T preserves perimeters $2, 3, \ldots, n+1$ of Boolean matrices. But T is not a (U, V)-operator. Moreover the image of T is a line and when restricted to any line is bijective.

Theorem 2.4. *If $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{m,n}(\mathbb{B})$ is a linear operator which preserves $\mathbb{P}_2$ and $\mathbb{P}_3$, then either T is a (U, V)-operator, or the image of T is dominated by a single row (or column if $m = n$) and T restricted to any line is injective.*

Proof. By Lemma 2.1, $T(E_{i,j}) = E_{r,s}$. Now, if T does not map lines to lines, then there is a matrix which is the sum of a pair of cells on one line and whose image has term rank 2. But the sum of two collinear cells has perimeter 3 while the sum of two non collinear cells has perimeter 4. Thus lines are mapped to lines. Further, the sum of two collinear cells, which has perimeter 3, cannot be mapped to a single cell, which has perimeter 2, hence, T restricted to any line is injective.

Suppose T is not a (U, V)-operator. Then, since lines are mapped to lines, either a row and a column are mapped into the same row, a row and a column are mapped into the same column, two rows are mapped to the same row (or column if $m = n$), or two columns are mapped to the same row(or column if $m = n$).

If two rows are mapped into the same row, we may assume that $T(R_1) \leq R_1$ and $T(R_2) \leq R_1$. Then, since $T(E_{1,j} + E_{2,j}) \leq R_1$ we may assume that $T(E_{1,j}) = E_{1,1}$ and $T(E_{2,j}) = E_{1,2}$. Now, since $T(E_{1,j} + E_{i,j})$ must have perimeter 3 for any $2 \leq i \leq n$, $T(E_{i,j}) = E_{1,k}$ or $T(E_{i,j}) = E_{k,1}$ for some k. If $T(E_{i,j}) = E_{k,1}$, then $T(E_{i,j} + E_{2,j}) = E_{k,1} + E_{1,2}$ for some $2 \leq k \leq m$, but $E_{k,1} + E_{1,2}$ has perimeter 4, a contradiction. Therefore, we have that $T(C_j) \leq R_1$ for all j, that is $T(E_{i,j}) \leq R_1$ for all i, j. Now, since no two cells dominated by a line can be mapped to the same cell, we have that T is injective if T is restricted to any line.

If a row is mapped to a column, we must have that $m = n$, so that, if $m = n$ and two columns are mapped to a column, then the same arguments apply as for rows above.

If a row and a column are mapped to a row, say, $T(R_1) \leq R_1$ and $T(C_1) \leq R_1$, then $T(R_i)$ and $T(C_i)$ are dominated by the same line since $T(E_{1,i} + E_{i,i})$ is dominated by the same line as $T(E_{i,1} + E_{i,i})$. Again we can conclude that the image of T is dominated by R_1. As above, the mapping T is one of those described.

The other cases are similar. □

Thus we have characterized the linear operators that preserve the perimeters of Boolean matrices.

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