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EXPLICIT SOBOLEV ESTIMATES FOR THE CAUCHY-RIEMANN EQUATION ON PARAMETERS

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ABSTRACT. Let $\overline{M}$ be a smoothly bounded pseudoconvex complex manifold with a family of almost complex structures $\{\mathcal{L}^\tau\}_{\tau\in I}$, $0\in I$, which extend smoothly up to bM, the boundary of M, and assume that there is $\lambda\in C^\infty(bM)$ which is strictly subharmonic with respect to the structure $\mathcal{L}^0|_{bM}$ in any direction where the Levi-form vanishes on bM. We obtain explicit estimates for the $\bar\partial$-Neumann problem in Sobolev spaces both in space and parameter variables. Also we get a similar result when $\overline{M}$ is strongly pseudoconvex.

1. Introduction

Let $\overline{M}$ be a compact complex manifold with C^∞ boundary and let $\{\mathcal{L}^\tau\}_{\tau\in I}$ be a family of complex structures on M where, in the sequel, we let $I\subset\mathbb{R}^1$ be an interval containing 0.

Definition 1.1. $\{\mathcal{L}^\tau\}_{\tau\in I}$ is said to be a smooth family of diffeomorphic complex structures on M if there is a family of diffeomorphisms $d_\tau : M \longrightarrow M$, $\tau\in I$ such that

(1) $d_0 =$ Identity,
(2) $(d_\tau)_*(\mathcal{L}^\tau|_p)=\mathcal{L}^0|_p$ for each $p\in\overline{M}$ and,
(3) $d_\tau \longrightarrow d_0$ as $\tau\to 0$ in C^∞-topology.

Similarly, we can define a smooth family of diffeomorphic (pseudoconvex) complex manifolds.

Remark 1.2. (1) If $(M_\tau,\mathcal{S}^\tau)_{\tau\in I}$ is a smooth family of diffeomorphic complex manifolds with diffeomorphisms $F_\tau : M_\tau\to M_0$, then $\{\mathcal{L}^\tau := dF_\tau(\mathcal{S}^\tau)\}_{\tau\in I}$ is a family of smoothly varying almost complex structures on fixed manifold M_0.

(2) In [4, 5, 7], the first author constructed a smooth family of diffeomorphic pseudoconvex complex manifolds when M_0 is a compact pseudoconvex complex manifold of finite 1-type.

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Let $\{\mathcal{L}^\tau\}_{\tau\in I}$ be a smooth family of diffeomorphic complex structures on $\overline{M}$ which extend smoothly to bM, the boundary of M, and assume that M is pseudoconvex with respect to each structure $\mathcal{L}^\tau$, $\tau \in I$, and that there is a function $\lambda_1 \in C^\infty(bM)$ which is strictly subharmonic with respect to the structure $\mathcal{L}^0|_{bM}$ in any direction where the Levi-form vanishes on bM. If we set $\lambda_2 = \lambda_1 + \tilde{s}r$, then it follows that $\partial\overline{\partial}\lambda_2(L,\overline{L}) > 0$ on bM provided $\tilde{s}$ is sufficiently large. Set $\lambda_\delta^s = s\lambda_2 + e^{r/\delta}$. Then λ_δ^s is strictly plurisubharmonic near bM with respect to the structure $\mathcal{L}^0$ provided $\delta > 0$ is sufficiently small and s is sufficiently large. Fix δ and s satisfying this condition and set $\lambda = \lambda_\delta^s$. Then λ is strictly plurisubharmonic near bM. Using a convex function, it is standard to modify λ so that λ is smooth plurisubharmonic on $\overline{M}$ and if we set $M_\mu = \{z \in \overline{M} : \lambda(z) \le \mu\}$, $\mu \in \mathbb{R}$, then there are $\mu_0 < \mu_1 < \mu_2$ such that $M_{\mu_0} \subset\subset M_{\mu_1} \subset\subset M_{\mu_2} \subset\subset M$ and λ is strictly plurisubharmonic on $\overline{M} - M_{\mu_0}^0$.

Let ψ be a smooth nondecreasing convex function such that $\psi(\tau) = 0$ for $\tau \le \mu_2$, $\psi(\tau) > 0$ for $\tau > \mu_2$. Set $\chi_{s,t} = t\lambda + s\psi(\lambda)$. We then define $L^2_{(p,q)}(M, \chi_{s,t})$ as the space of all measurable forms in M of type (p,q), satisfying

$$||f||_{s,t}^2 := \int_M |f|^2 e^{-\chi_{s,t}} dV < \infty,$$

where dV is the volume form induced by the structure $\mathcal{L}_0$. Then for a sufficiently large t and s, it follows from Proposition 2.2.5 in [1] that

$$||f||_{s,t} \le C\left(||T^*f||_{s,t} + ||Sf||_{s,t}\right), \tag{1.1}$$

if $f \in D_{T^*} \cap D_S$, where D_{T^*} and D_S denote the domain of T^* and S respectively, and where T and S are the Hilbert space extension of $\bar{\partial}$ on $(p, q-1)$ and (p,q) forms respectively, and T^* denotes the Hilbert space adjoint of T. In the estimates of higher order Sobolev space, s will be fixed while t varies. Let us fix $s = s_0$ so that (1.1) holds for all sufficiently large t and denote the Sobolev norm of order m with respect to the weight $e^{-\chi_{s_0,t}}$ by $||\cdot||_{m,t}$.

Let $\overline{\partial}_\tau$ be the Cauchy Riemann operator with respect to the structure $\mathcal{L}^\tau$, and assume that $\{\alpha_\tau\}_{\tau\in I}$ is a smooth family of $\overline{\partial}_\tau$-closed (p,q) forms on M with respect to $\mathcal{L}^\tau$, and $\alpha_\tau \in (\mathcal{H}_\tau^{p,q})^\perp$ where $\mathcal{H}_\tau^{p,q}$ is the $\overline{\partial}_\tau$-cohomology group for each $\tau \in I$. Then from the Kohn's solution of $\bar{\partial}$-Neumann problem with weights, for each $m \ge 0$, there are $T_m > 0$, and the corresponding canonical solution u_τ^m of $\overline{\partial}_\tau u_\tau = \alpha_\tau$, with respect to the structure $\mathcal{L}^\tau$, satisfying

$$||u_\tau^m||_{m,t} \le C_m(\tau,t)||\alpha_\tau||_{m,t} \tag{1.2}$$

for $t > T_m$.

However in many applications of the estimate (1.2), we need to know explicitly how $C_m(\tau,t)$ depends on m, τ and t, as well as the Sobolev estimates in parameter variable $\tau \in I$. For example, when we study local embedding of tangential Cauchy Riemann structures, we have to know the precise estimates for the solutions of Cauchy Riemann equation under the influence the deformation of (almost) complex structures. Also, one approach to solve the local

$\bar{\partial}_b$ equation is to construct one parameter family of domains and solve $\bar{\partial}$ equation on each of these domains and get a precise estimates including parameter variables.

In [6], the first author showed the continuity of the estimates in space and parameter variables. However the estimates in [6] are not explicit on t and loose some extra derivative on space variables as well as in parameter variable. In the present paper, we estimate explicitly the solutions of (1.2) in Sobolev spaces both in space and parameter variables. Also, we obtain a similar optimal result when $\overline{M}$ is strictly pseudoconvex.

Let $\{L_1, \ldots, L_n\} \subset \mathcal{L}^0$ be an orthonormal frame of $\mathcal{L}^0$ on M with a smooth hermitian metric $\langle\ ,\ \rangle_0$ on M. Let $d_\tau : M \to M$, $\tau \in I$, be the family of diffeomorphisms described in Definition 1.1, and set $L_k^\tau := (d_\tau^{-1})_* L_k \in \mathcal{L}^\tau$, $1 \leq k \leq n$, and define $\langle\ ,\ \rangle_\tau$ on $\mathcal{L}^\tau$ by

$$\langle L_k^\tau, L_j^\tau \rangle_\tau = \langle L_k, L_j \rangle_0, \quad 1 \leq j, k \leq n. \tag{1.3}$$

Then $\{\mathcal{L}^\tau, \langle \cdot, \cdot \rangle_\tau\}_{\tau \in I}$ is a smooth family of hermitian structures on M and $\{L_1^\tau, \ldots, L_n^\tau\}$ form an orthonormal frame of $\mathcal{L}^\tau$, for each $\tau \in I$. Assume that I is sufficiently small (so that Proposition 2.5 holds) and let l, m be nonnegative integers. We denote by $H_{l,m}(I \times M)$ the Sobolev space of order l and m in $\tau \in I$ and $x \in M$ variables respectively, and denote the norm with weight $e^{-\phi}$ by $\|f\|_{l,m,\phi}$ which is induced from the hermitian structure defined in (1.3). When $l = 0$, we simply denote the Sobolev space of order m on M by $H_m(M)$.

We may assume that $\mathcal{L}^\tau$ is smooth in τ on $\bar{I}$ by shrinking I if necessary. Let ζ_j, $j = 1, 2, \ldots, N$ be smooth real valued functions such that $\sum_{j=1}^N \zeta_j^2 = 1$, and the support of each function ζ_j is contained in a coordinate neighborhood of $\overline{M}$. In local coordinates $x_1, x_2, \ldots, x_{2n}$, we write

$$L_i^\tau = \sum_{j=1}^{2n} a_{ij}(x, \tau) \frac{\partial}{\partial x_j},$$

and define

$$|\bar{\partial}_\tau - \bar{\partial}_\nu|_m = \sup_{x \in \overline{M}} \sum_{l=1}^N \sum_{i=1}^n \sum_{j=1}^{2n} \sum_{|\alpha| \leq m} |D_x^\alpha \zeta_l (a_{ij}(x, \tau) - a_{ij}(x, \nu))|, \tag{1.4}$$

and

$$|\bar{\partial}|_{k,m} = \sup_{\tau \in I} \sup_{x \in \overline{M}} \sum_{l=1}^N \sum_{i=1}^n \sum_{j=1}^{2n} \sum_{s=0}^k \sum_{|\alpha| \leq m} |D_\tau^s D_x^\alpha (\zeta_l a_{ij})|, \tag{1.5}$$

and set $|\bar{\partial}|_{0,m} = |\bar{\partial}|_m$. Note that $|\bar{\partial}|_{1,4} \approx 1$, independent of $\tau \in I$.

We state our main theorems as follows:

Theorem 1.3. *Let $\{\overline{M}, \mathcal{L}^\tau\}_{\tau \in I}$ be a smooth family of diffeomorphic pseudoconvex complex structures on $\overline{M}$. Suppose that $\{\alpha_\tau\}_{\tau \in I}$ is a family of (p, q) forms on M such that $\alpha_\tau \in R(\bar{\partial}_\tau)$ for each $\tau \in I$. For each nonnegative integer*

$m \geq 0$, assume that $\alpha_\nu \in H_{m+3}(M)$ for some $\nu \in I$ and $\alpha_\tau \in H_m(M)$ for each $\tau \in I$. Then there are constants $C_m > 0$ and $T_m > 0$, independent of $\tau \in I$, such that if $t \geq T_m$, then the canonical solution u_τ of $\overline{\partial}_\tau u = \alpha_\tau$, $\tau \in I$, which is provided by the $\overline{\partial}$-Neumann problem with weight $e^{-\chi_{s_0,t}}$, satisfies

$$\begin{aligned} &\|u_\tau - u_\nu\|_{m,t} \\ &\leq C_m \left(\|\alpha_\tau - \alpha_\nu\|_{m,t} + |\overline{\partial}|_m \|\alpha_\tau - \alpha_\nu\|_{0,t} + t^m \|\alpha_\tau - \alpha_\nu\|_{0,t}\right) \\ &\quad + C_m |\overline{\partial}_\tau - \overline{\partial}_\nu|_0 \left(\|\alpha_\nu\|_{m+3,t} + |\overline{\partial}|_{m+4} \|\alpha_\nu\|_{0,t} + t^{m+3}\|\alpha_\nu\|_{0,t}\right) \\ &\quad + C_m |\overline{\partial}_\tau - \overline{\partial}_\nu|_{m+3} \|\alpha_\nu\|_{0,t}. \end{aligned} \tag{1.6}$$

Remark 1.4. (1) Theorem 1.3 says that for each $\nu \in I$, the canonical solution u_τ converges to u_ν in the Sobolev space $H_m(M)$ as $\tau \to \nu$ provided that $\alpha_\nu \in H_{m+3}(M)$ and $\alpha_\tau \in H_m(M)$ for each $\tau \in I$, and $\alpha_\tau \to \alpha_\nu$ in $H_m(M)$ as $\tau \to \nu$.

(2) Note that $t^{s_1}|\overline{\partial}|_{s_2}\|\alpha_\tau - \alpha_\nu\|_{s_3,t}$ in the first line of the right hand side of (1.6) has total order of $m = s_1 + s_2 + s_3$, while the remaining error terms (terms containing $|\overline{\partial}_\tau - \overline{\partial}_\nu|_s$) have total order of $m+4$ because of the error term E^ν_τ introduced in the proof of Theorem 1.3.

In the sequel, we set $U = U(\tau, x) = U_\tau(x)$ and $\alpha = \alpha(\tau, x) = \alpha_\tau(x)$. Also for each fixed $k, m \geq 0$, $\Box U = \alpha$ should be understood that $\Box^{k,m}_\tau U = \alpha_\tau$ for each $\tau \in I$, where $\Box^{k,m}_\tau$ denotes the complex Laplacian with respect to the structure $\mathcal{L}^\tau$ with appropriate weight depending on k and m. For each nonnegative integers k and m and for each $t > 0$, we set

$$\begin{aligned} &G^t(\alpha, k, m) \\ &= \sum_{r=0}^{k} |\overline{\partial}|_{k-r,0} \left(\|\alpha\|_{r,m+2k-2r,t} + \left(t^{m+2k-2r} + |\overline{\partial}|_{k,m+2k-2r}\right)\|\alpha\|_{r,0,t}\right). \end{aligned} \tag{1.7}$$

Then we get the derivative estimates of the solutions of $\Box U = \alpha$ with respect to space as well as parameter variables:

Theorem 1.5. *Let $\{M, \mathcal{L}^\tau\}_{\tau \in I}$ and $\{\alpha_\tau\}_{\tau \in I}$ be as in Theorem 1.3. For each integers $k, m \geq 0$, there are constants $T_m > 0$ and $C_{k,m}$, independent of $\tau \in I$, such that for all $t \geq T_m$, the Neumann solution U of $\Box U = \alpha$, and the canonical solution $u = \overline{\partial}^* U$ of $\overline{\partial} u = \alpha$ satisfy*

$$\|U\|_{k,m,t} + \|U\|_{k-1,m+2,t} + \|u\|_{k,m,t} + \|u\|_{k-1,m+2,t} \leq C_{k,m} G^t(\alpha, k, m),$$

provided $\alpha \in H_{r,m+2k-2r}(I \times M)$, $0 \leq r \leq k$.

If M is strongly pseudoconvex, we use the stability of the estimates for $\overline{\partial}$ by Green and Krantz [8, 9]. Then we gain some derivatives and (1.6) can be improved.

Theorem 1.6. *Let $\{M, \mathcal{L}^\tau\}_{\tau \in I}$ and $\{\alpha_\tau\}_{\tau \in I}$ be as in Theorem 1.3 and assume that $\overline{M}$ is strongly pseudoconvex with respect to $\mathcal{L}^0$. Assume that $\alpha_\nu \in$*

$H_{s+2}(M)$ for some $\nu \in I$ and $\alpha_\tau \in H_s(M)$ for each $\tau \in I$ where $s \geq -1/2$. Then there exist a constant $C_s > 0$, independent of $\tau \in I$, such that the Neumann solution U_τ of $\Box_\tau U_\tau = \alpha_\tau$, and the canonical solution $u_\tau = \overline{\partial}_\tau^* U_\tau$ of $\overline{\partial}_\tau u = \alpha_\tau$, $\tau \in I$ satisfy

$$\begin{aligned} &\|U_\tau - U_\nu\|_{s+1} + \|u_\tau - u_\nu\|_{s+\frac{1}{2}} \\ &\lesssim \|\alpha_\tau - \alpha_\nu\|_s + |\overline{\partial}|_s \|\alpha_\tau - \alpha_\nu\|_0 \\ &\quad + |\overline{\partial}_\tau - \overline{\partial}_\nu|_{s+2} \|\alpha_\nu\|_0 \\ &\quad + |\overline{\partial}_\tau - \overline{\partial}_\nu|_0 \left(\|\alpha_\nu\|_{s+2} + |\overline{\partial}|_{s+4} \|\alpha_\nu\|_0 \right). \end{aligned} \tag{1.8}$$

For higher order Sobolev estimates, we obtain:

Theorem 1.7. *Let $\{M, \mathcal{L}^\tau\}_{\tau \in I}$ and $\{\alpha_\tau\}_{\tau \in I}$ be as in Theorem 1.6. Then for each nonnegative integer k, and for each real number $s \geq -1/2$, there is $C_{k,s} > 0$ such that the Neumann solution U of $\Box U = \alpha$ and the canonical solution $u = \overline{\partial}^* U$ of $\overline{\partial} u = \alpha$ satisfy*

$$\|U\|_{k,s+1} + \|u\|_{k,s+\frac{1}{2}} \leq C_{k,s} H(\alpha, k, s), \tag{1.9}$$

provided that $\alpha \in H_{r,s+k-r}(I \times M)$, $0 \leq r \leq k$, where

$$\begin{aligned} &H(\alpha, k, s) \\ &= \sum_{r=0}^{k} |\overline{\partial}|_{k-r,0} \|\alpha\|_{r,s+k-r} + \left(|\overline{\partial}|_{k-r,0} |\overline{\partial}|_{0,s+2k+1} + |\overline{\partial}|_{k-r,s+2k+1} \right) \|\alpha\|_{r,k-r-\frac{1}{2}}. \end{aligned}$$

The annoying factors $\|\alpha\|_{r,m+2k-2r,t}$ and $\|\alpha\|_{r,s+k-r}$ in the right hand side of (1.7) and (1.9) come from by introducing some error terms to estimate the derivatives of U_τ with respect to $\tau \in I$ variable.

Theorem 1.3–1.7 could be used, for example, in the study of the stability as well as the continuity properties of the Bergman kernel and invariant metrics, and the weak extension problem of a given CR-forms on a hypersurface with the estimates in Sobolev spaces, as well as a local embedding problem of a given CR manifolds. These problems will be discussed in near future.

2. Stability of the estimates for $\overline{\partial}$

In this section we get estimates for $\overline{\partial}_\tau$ in Sobolev spaces with respect to the structure $\mathcal{L}^\tau$, $\tau \in I$, and study how these estimates depend on parameters. For this purpose, we need to know the estimates with commutators in weighted spaces. In [3], Catlin, and Cho proved the following interpolation theorems in weighted spaces.

Proposition 2.1 ([3, Corollary 4.4]). *Let $u, v \in C_0^\infty(\mathbb{R}^d)$. For any nonnegative integer m and for each small $\delta > 0$,*

$$\|uv\|_{m,t} \lesssim \|u\|_{m,t} |v|_0 + \delta^{1-m} \|u\|_{0,t} |v|_m + \delta t^m \|u\|_{0,t} |v|_0. \tag{2.1}$$

Proposition 2.2 ([3, Corollary 4.5]). *Let P be a partial differential operator of order $m \geq 1$ with smooth bounded coefficients. Then for all $u, v \in C_0^\infty(\mathbb{R}^d)$ and non-negative integer l,*

$$\|[P, v]u\|_{l,t} \lesssim \|u\|_{m+l-1,t}|v|_1 + \|u\|_{0,t}|v|_{m+l} + t^{m+l-1}\|u\|_{0,t}|v|_1. \tag{2.2}$$

Proposition 2.3 ([3, Proposition 4.2]). *Let $u \in C_0^\infty(\mathbb{R}^d)$ and k, m be integers with $0 \leq k < m$. For any $a > 0$,*

$$\|u\|_{k,t} \lesssim a\|u\|_{m,t} + \left(a^{k/(k-m)} + at^m + t^k\right)\|u\|_{0,t}. \tag{2.3}$$

Proposition 2.4 ([3, Proposition 4.3]). *Let $u, v \in C_0^\infty(\mathbb{R}^d)$. For any non-negative integers m, k with $0 < k < m$, and for each $a > 0$,*

$$\|u\|_{k,t}|v|_{m-k} \lesssim a\|u\|_{m,t}|v|_0 + a^{-\frac{k}{m-k}}\|u\|_{0,t}|v|_m + at^m\|u\|_{0,t}|v|_0. \tag{2.4}$$

We note that the total order of $t^{s_1}|\cdot|_{s_2}\|\cdot\|_{s_3}$ satisfies $s_1 + s_2 + s_3 = m$. Following standard estimates of Hörmander [10] or Catlin [1] or Kohn [11], and using the estimates (2.1)–(2.4), one obtains the following precise estimates. One may refer the proof in ([3], Theorem 4.9).

Proposition 2.5. *There is a sufficiently small interval $I \subset \mathbb{R}^1$ such that for each non-negative integer $m \geq 0$, there exist constants $C_m, T_m > 0$, independent of $\tau \in I$, such that for all $t \geq T_m$, the following estimate holds:*

$$\begin{aligned} &t\|f^\tau\|^2_{m,t} + \|\overline{\partial}_\tau f^\tau\|^2_{m,t} + \|\overline{\partial}^*_\tau f^\tau\|^2_{m,t} \\ &\leq C_m\|\Box^\tau_t f^\tau\|^2_{m,t} + C_m(|\overline{\partial}|^2_{m+1}\|\Box^\tau_t f^\tau\|^2_{0,t} + t^{2m}\|\Box^\tau_t f^\tau\|^2_{0,t}) \end{aligned} \tag{2.5}$$

for $f^\tau \in Dom(\overline{\partial}^*_\tau) \cap C^\infty(\overline{M})$, where $\Box^\tau_t$ denotes the complex Laplacian with respect to the complex structure $\mathcal{L}^\tau$ with weight $e^{-\chi_{s_0,t}}$.

By the method of elliptic regularization it follows that if $\Box^\tau_t u_\tau \in H_m(M)$, then $u_\tau \in H_m(M)$ and the estimate (2.5) holds independent of $\tau \in I$ and this proves the stability of the estimate for $\overline{\partial}$ in Sobolev spaces.

Now if $\alpha_\tau \in \Lambda^{p,q}_\tau$ and $\overline{\partial}_\tau\alpha_\tau = 0$, $\alpha_\tau \perp \mathcal{H}^{p,q}_\tau$, then $u_\tau = \overline{\partial}^*_{t,\tau}N^t_\tau\alpha_\tau$ is the unique solution of $\overline{\partial}_{t,\tau}u = \alpha_\tau$ orthogonal to $ker(\overline{\partial}_{t,\tau})$, where $\overline{\partial}_{t,\tau}$ and N^t_τ denote the $\overline{\partial}$ and the Neumann operator with respect to $\mathcal{L}^\tau$ with weight $e^{-\chi_{s_0,t}}$. Therefore if $U_\tau = N^t_\tau\alpha_\tau$ is the Neumann solution of $\Box^t_\tau U_\tau = \alpha_\tau$, then it follows from (2.5) that U_τ and the canonical solution $u_\tau = \overline{\partial}^*_{t,\tau}U_\tau$ satisfy the following estimates :

$$\|U_\tau\|_{m,t} \leq C_m\left(\|\alpha_\tau\|_{m,t} + |\overline{\partial}|_{m+1}\|\alpha_\tau\|_{0,t} + t^{m-1}\|\alpha_\tau\|_{0,t}\right), \tag{2.6}$$

$$\|u_\tau\|_{m,t} \leq C_m\left(\|\alpha_\tau\|_{m,t} + |\overline{\partial}|_{m+1}\|\alpha_\tau\|_{0,t} + t^{m}\|\alpha_\tau\|_{0,t}\right), \tag{2.7}$$

where C_m does not depend on $\tau \in I$.

When $\overline{M}$ is strongly pseudoconvex, we use the estimates in Lemma 2.4 with $t = 0$ and follow standard estimates as in [11]. Then the solutions $U_\tau = N_\tau\alpha_\tau$

and $u_\tau = \overline{\partial}_\tau^* N_\tau \alpha_\tau$ satisfy the following estimates:

$$\|U_\tau\|_{k+\frac{1}{2}} + \|u_\tau\|_k \leq C_k \left(\|\alpha_\tau\|_{k-\frac{1}{2}} + |\overline{\partial}|_{k+1}\|\alpha_\tau\|_0\right) \tag{2.8}$$

for each $k = 1, 2, \ldots$, and

$$\|U_\tau\|_{\frac{1}{2}} + \|u_\tau\|_0 \leq C_0\|\alpha_\tau\|_{-\frac{1}{2}}, \tag{2.9}$$

where C_k does not depend on $\tau \in I$. By the definition of the space H_{-s} we have

$$|(h, g)| \leq \|h\|_s\|g\|_{-s} \tag{2.10}$$

for any $s \leq \frac{1}{2}$ where $h \in H_s$ and $g \in H_{-s}$. Let $[s]$ be the smallest integer bigger than or equal to s. Then it follows from (2.8)–(2.10), and the interpolation inequality in Sobolev spaces, that

$$\|U_\tau\|_{s+1} + \|u_\tau\|_{s+\frac{1}{2}} \leq C_s \left(\|\alpha_\tau\|_s + |\overline{\partial}|_{[s]+1}\|\alpha_\tau\|_{-\frac{1}{2}}\right) \tag{2.11}$$

for any $s \geq -\frac{1}{2}$ because $|\overline{\partial}|_{1,4} \approx 1$.

3. Sobolev estimates on parameters on weakly pseudoconvex domains

In this section we estimate the solutions of $\overline{\partial}_\tau$ on parameter variables. In the sequel, we let $A \lesssim B$ mean that there is an independent constant C_m (depending only on m and independent of t and parameter τ) such that $A \leq C_m B$.

Let $(\overline{M}, \mathcal{L}^\tau)_{\tau\in I}$ and $M_{\mu_0} \Subset M_{\mu_1} \Subset M_{\mu_2} \Subset M$ be as in Section 1. Let $0 \leq h \leq 1$ be a smooth functions such that $h = 1$ on $\overline{M} - M_2^0$, and $h = 0$ on $\overline{M}_1$. We may assume that $\overline{M} \subset M'$ and $M = \{z \in M' : r(z) < 0\}$ where $r \in C^\infty(M')$ is a real valued function such that $|dr| = 1$ on bM. Set $L_n^\tau = \frac{\partial}{\partial r} - iJ_\tau(\frac{\partial}{\partial r})$ and $\mathcal{L}_{tan}^\tau = \{L^\tau \in \mathcal{L}^\tau : L^\tau r = 0\}$, where J_τ denotes the complex structure on $\mathcal{L}^\tau$, $\tau \in I$.

Lemma 3.1 ([6, Lemma 3.1]). *There exists a bundle isomorphism $B_\tau : \overline{\mathcal{L}}^\tau \to \overline{\mathcal{L}}^0$ such that $B_\tau(\overline{\mathcal{L}}^\tau|_p) = \overline{\mathcal{L}}^0|_p$ for every $p \in \overline{M}$, B_τ depends smoothly on τ, $B_\tau(\overline{L}_n^\tau) = \overline{L}_n^0$ and $B_\tau(\overline{\mathcal{L}}_{tan}^\tau) = \overline{\mathcal{L}}_{tan}^0$ on bM.*

Let $\Lambda_\tau^{p,q}$ denote the space of all smooth (p, q) forms over the complex space $(\overline{M}, \mathcal{L}^\tau)_{\tau\in I}$. For a convenience, we will consider only for the case that $p = 0$ and $q = 1$. For any $U_\tau \in \Lambda_\tau^{0,1}$, define $P_\tau : \Lambda_\tau^{0,1} \longrightarrow \Lambda_0^{0,1}$ by

$$(P_\tau U_\tau)(\overline{X}^0) = U_\tau(B_\tau^{-1}\overline{X}^0), \quad \overline{X}^0 \in \overline{\mathcal{L}}^0. \tag{3.1}$$

Then P_τ is a bundle isomorphism which depends smoothly on τ and it follows, from (2.1) and (3.1), that

$$\|P_\tau G\|_{s,t} \leq C_s \left(\|G\|_{s,t} + |\overline{\partial}|_s\|G\|_{0,t} + t^s\|G\|_{0,t}\right), \tag{3.2}$$

$$\|P_\tau^{-1} F\|_{s,t} \leq C_s \left(\|F\|_{s,t} + |\overline{\partial}|_s\|F\|_{0,t} + t^s\|F\|_{0,t}\right), \tag{3.3}$$

where C_s does not depend on $\tau \in I$.

We use the family $\{P_\tau\}_{\tau\in I}$ to change the $\overline{\partial}$-equation on $\Lambda^{0,1}_\tau$ forms into the family of equations over the same space $\Lambda^{0,1}_0$. Consider the family of equations

$$\Box_\tau U_\tau = \alpha_\tau, \quad \tau \in I, \tag{3.4}$$

where $\Box_\tau = \overline{\partial}_\tau\overline{\partial}^*_\tau + \overline{\partial}^*_\tau\overline{\partial}_\tau$, and $U_\tau, \alpha_\tau \in \Lambda^{0,1}_\tau$. Set $\Delta_\tau = P_\tau\Box_\tau P_\tau^{-1}$, $V_\tau = P_\tau U_\tau$ and $\beta_\tau = P_\tau\alpha_\tau$. Then each equation in (3.4) is equivalent to the equation:

$$\Delta_\tau V_\tau = \beta_\tau, \quad \tau \in I,$$

where $V_\tau, \beta_\tau \in \Lambda^{0,1}_0$, i.e., on the same space $\Lambda^{0,1}_0$. By virtue of the estimates in (2.6), (3.2) and (3.3), one obtains that

$$\begin{aligned} \|V_\tau\|_{m,t} &\lesssim \|U_\tau\|_{m,t} + |\overline{\partial}|_m\|U_\tau\|_{0,t} + t^m\|U_\tau\|_{0,t} \\ &\lesssim \|\alpha_\tau\|_{m,t} + |\overline{\partial}|_{m+1}\|\alpha_\tau\|_{0,t} + t^m\|\alpha_\tau\|_{0,t} \\ &\approx \|\beta_\tau\|_{m,t} + |\overline{\partial}|_{m+1}\|\beta_\tau\|_{0,t} + t^m\|\beta_\tau\|_{0,t}. \end{aligned} \tag{3.5}$$

We first prove Theorem 1.3. Let $\nu \in I$ be fixed for a moment. To show the continuity of the Neumann (or canonical) solution of the $\overline{\partial}$ equation on parameter, we try to use the solution of the equation

$$\Box_\tau(U_\tau - U_\nu) = \alpha_\tau - \alpha_\nu - (\Box_\tau - \Box_\nu)U_\nu. \tag{3.6}$$

Unfortunately $U_\nu \notin Dom(\Box_\tau)$ in general and hence (3.6) should be corrected. Instead of U_τ, we perturb U_τ by $U_\tau + E^\nu_\tau$, where $E^\nu_\tau \in \Lambda^{0,1}_0$ is defined by

$$E^\nu_\tau(\overline{L}) = \overline{L}^\tau_n(V_\nu(\overline{L})) + (P_\tau^{-1}V_\nu)[B_\tau^{-1}\overline{L}, \overline{L}^\tau_n], \quad \overline{L} \in \overline{\mathcal{L}}^0, \quad \tau \in I. \tag{3.7}$$

Lemma 3.2 ([6, Lemma 3.2]). *Let $V_\tau \in Dom(\Delta_\tau)$ and $V_\nu \in Dom(\Delta_\nu)$. Then $V_\nu - rhE^\nu_\tau \in Dom(\Delta_\tau)$ and $rhE^\nu_\nu \in Dom(\Delta_\tau)$.*

From the definition of E^ν_τ in (3.7) and by (2.1)–(2.4) (with $a = t^{-m}$), (3.3) and (3.5), one obtains that

$$\begin{aligned} \|E^\nu_\tau\|_{m,t} &\lesssim \|V_\nu\|_{m+1,t} + |\overline{\partial}|_m\|V_\nu\|_{1,t} + t^m\|V_\nu\|_{1,t} \\ &\lesssim \|\alpha_\nu\|_{m+1,t} + \left(|\overline{\partial}|_{m+2} + t^{m+1}\right)\|\alpha_\nu\|_{0,t}. \end{aligned} \tag{3.8}$$

Similarly, applying (2.1) and (2.4) again, we obtain that

$$\begin{aligned} &\|E^\nu_\tau - E^\nu_\nu\|_{m,t} \\ &\lesssim |\overline{\partial}_\tau - \overline{\partial}_\nu|_0\left(\|V_\nu\|_{m+1,t} + t^m\|V_\nu\|_{1,t}\right) + |\overline{\partial}_\tau - \overline{\partial}_\nu|_m\|V_\nu\|_{1,t} \\ &\lesssim |\overline{\partial}_\tau - \overline{\partial}_\nu|_0\left(\|\alpha_\nu\|_{m+1,t} + |\overline{\partial}|_{m+2}\|\alpha_\nu\|_{0,t} + t^{m+1}\|\alpha_\nu\|_{0,t}\right) \\ &\quad + |\overline{\partial}_\tau - \overline{\partial}_\nu|_{m+1}\|\alpha_\nu\|_{0,t}. \end{aligned} \tag{3.9}$$

Remark 3.3. We note that the total order of $t^{s_1}|\cdot|_{s_2}\|\cdot\|_{s_3,t}$ is $s_1+s_2+s_3 = m+1$ in (3.8) and (3.9) even though we have $|\overline{\partial}|_{m+2}$ term which is not a major term in our estimates. These interpolation results occur frequently in the rest of the estimates in this article.

Set $w_\tau^\nu = V_\tau + rhE_\tau^\nu \in \Lambda_0^{0,1}$. Since $U_\nu = P_\nu^{-1}V_\nu \in Dom(\overline{\partial}_\nu^*)$, it follows that

$$P_\tau^{-1}V_\nu(\overline{L}_n^\tau) = V_\nu(B_\tau \overline{L}_n^\tau) = V_\nu(\overline{L}_n^0) = P_\nu^{-1}V_\nu(B_\nu^{-1}\overline{L}_n^0) = P_\nu^{-1}V_\nu(\overline{L}_n^\nu) = 0$$

on bM, and hence $P_\tau^{-1}V_\nu \in Dom(\overline{\partial}_\tau^*)$. Therefore we can write

$$\begin{aligned} u_\tau - u_\nu = \overline{\partial}_\tau^* P_\tau^{-1} V_\tau - \overline{\partial}_\nu^* P_\nu^{-1} V_\nu &= \overline{\partial}_\tau^* P_\tau^{-1}(w_\tau^\nu - w_\nu^\nu) \\ &- (\overline{\partial}_\nu^* P_\nu^{-1} - \overline{\partial}_\tau^* P_\tau^{-1})V_\nu - \overline{\partial}_\tau^* P_\tau^{-1} rh(E_\tau^\nu - E_\nu^\nu). \end{aligned} \tag{3.10}$$

By applying (2.3), (2.4), (3.3) and (3.9), we obtain that

$$\begin{aligned} &\|\overline{\partial}_\tau^* P_\tau^{-1} rh(E_\tau^\nu - E_\nu^\nu)\|_{m,t} \\ &\lesssim \|E_\tau^\nu - E_\nu^\nu\|_{m+1,t} + |\overline{\partial}|_{m+1}\|E_\tau^\nu - E_\nu^\nu\|_{0,t} + t^{m+1}\|E_\tau^\nu - E_\nu^\nu\|_{0,t} \\ &\lesssim |\overline{\partial}_\tau - \overline{\partial}_\nu|_0 \left(\|\alpha_\nu\|_{m+2,t} + |\overline{\partial}|_{m+3}\|\alpha_\nu\|_{0,t} + t^{m+2}\|\alpha_\nu\|_{0,t}\right) \\ &\quad + |\overline{\partial}_\tau - \overline{\partial}_\nu|_{m+2}\|\alpha_\nu\|_{0,t}, \end{aligned} \tag{3.11}$$

and similarly, $\|(\overline{\partial}_\nu^* P_\nu^{-1} - \overline{\partial}_\tau^* P_\tau^{-1})V_\nu\|_{m,t}$ is bounded by the right hand side of (3.9).

In view of (3.10) and (3.11), it is enough to estimate $\|\overline{\partial}_\tau^* P_\tau^{-1}(w_\tau^\nu - w_\nu^\nu)\|_{m,t}$ to obtain the estimate of $\|u_\tau - u_\nu\|_{m,t}$. Let $\tilde{\Delta}_\tau$ be the same differential operator as Δ_τ without boundary conditions, i.e., $\tilde{\Delta}_\tau = P_\tau(\overline{\partial}_\tau\Theta_\tau + \Theta_\tau\overline{\partial}_\tau)P_\tau^{-1}$, where Θ_τ is the formal adjoint of $\overline{\partial}_\tau$. Set

$$f_\tau^\nu = \beta_\tau + \tilde{\Delta}_\tau rhE_\tau^\nu.$$

Then it follows that

$$\tilde{\Delta}_\tau w_\tau^\nu = \tilde{\Delta}_\tau(V_\tau + rhE_\tau^\nu) = f_\tau^\nu.$$

By Lemma 3.2 we have $w_\tau^\nu - w_\nu^\nu \in Dom(\Delta_\tau)$, and hence

$$\begin{aligned} \Delta_\tau(w_\tau^\nu - w_\nu^\nu) = \tilde{\Delta}_\tau(w_\tau^\nu - w_\nu^\nu) &= f_\tau^\nu - f_\nu^\nu - (\tilde{\Delta}_\tau - \tilde{\Delta}_\nu)w_\nu^\nu \\ &= \beta_\tau - \beta_\nu + (\tilde{\Delta}_\tau - \tilde{\Delta}_\nu)rhE_\tau^\nu \\ &\quad + \tilde{\Delta}_\nu rh(E_\tau^\nu - E_\nu^\nu) - (\tilde{\Delta}_\tau - \tilde{\Delta}_\nu)w_\nu^\nu. \end{aligned} \tag{3.12}$$

Let R_τ^ν be the right hand side of (3.12). By virtue of the estimates in (2.7) and (3.3), for each $m \geq 0$, there is a constant $T_m > 0$ such that if $t \geq T_m$ then

$$\begin{aligned} \|\overline{\partial}_\tau^* P_\tau^{-1}(w_\tau^\nu - w_\nu^\nu)\|_{m,t} &\lesssim \|P_\tau^{-1}R_\tau^\nu\|_{m,t} + \left(|\overline{\partial}|_m + t^m\right)\|P_\tau^{-1}R_\tau^\nu\|_{0,t} \\ &\lesssim \|R_\tau^\nu\|_{m,t} + |\overline{\partial}|_m\|R_\tau^\nu\|_{0,t} + t^m\|R_\tau^\nu\|_{0,t}. \end{aligned} \tag{3.13}$$

Let us estimate $\|R_\tau^\nu\|_{m,t}$. We first recall an interpolation formula in C^k norm. Let B be a bounded convex domain in $\mathbb{R}^d$. For any $a > 0$ and for any integers $0 \leq k < m$, there exists a constant $C = C(k, m, d)$ such that

$$|f|_k \leq C|f|_m^{\frac{k}{m}}|f|_0^{\frac{m-k}{m}} \leq C\left(a|f|_m + a^{-\frac{k}{m-k}}|f|_0\right) \tag{3.14}$$

for $f \in C^m(B)$. If we apply the interpolation formulas (2.1)–(2.4), (3.14) and the estimate (3.8), we obtain the following estimates:

$$\begin{aligned} &\|(\tilde{\Delta}_\tau - \tilde{\Delta}_\nu) rhE^\nu_\tau\|_{m,t} \\ &\lesssim |\overline{\partial}_\tau - \overline{\partial}_\nu|_0 \left(\|E^\nu_\tau\|_{m+2,t} + t^m \|E^\nu_\tau\|_{2,t}\right) + |\overline{\partial}_\tau - \overline{\partial}_\nu|_m \|E^\nu_\tau\|_{2,t}, \\ &\lesssim |\overline{\partial}_\tau - \overline{\partial}_\nu|_0 \left(\|E^\nu_\tau\|_{m+2,t} + t^{m+2} \|E^\nu_\tau\|_{0,t}\right) + |\overline{\partial}_\tau - \overline{\partial}_\nu|_{m+2} \|E^\nu_\tau\|_{0,t} \\ &\lesssim |\overline{\partial}_\tau - \overline{\partial}_\nu|_0 \left(\|\alpha_\nu\|_{m+3,t} + |\overline{\partial}|_{m+4} \|\alpha_\nu\|_{0,t} + t^{m+3} \|\alpha_\nu\|_{0,t}\right) \\ &\quad + |\overline{\partial}_\tau - \overline{\partial}_\nu|_{m+3} \|\alpha_\nu\|_{0,t}. \end{aligned} \tag{3.15}$$

Similarly, applying (3.9), we have

$$\|\tilde{\Delta}_\nu rh(E^\nu_\tau - E^\nu_\nu)\|_{m,t} + \|(\tilde{\Delta}_\tau - \tilde{\Delta}_\nu) w^\nu_\nu\|_{m,t} \lesssim \text{r.h.s. of (3.15)}. \tag{3.16}$$

Since we can write

$$\beta_\tau - \beta_\nu = P_\tau(\alpha_\tau - \alpha_\nu) + (P_\tau - P_\nu)\alpha_\nu,$$

it follows, from (2.1) and (3.2), that

$$\begin{aligned} &\|\beta_\tau - \beta_\nu\|_{m,t} \\ &\lesssim \|\alpha_\tau - \alpha_\nu\|_{m,t} + |\overline{\partial}|_m \|\alpha_\tau - \alpha_\nu\|_{0,t} + |\overline{\partial}_\tau - \overline{\partial}_\nu|_0 \|\alpha_\nu\|_{m,t} \\ &\quad + |\overline{\partial}_\tau - \overline{\partial}_\nu|_m \|\alpha_\nu\|_{0,t} + t^m \left(\|\alpha_\tau - \alpha_\nu\|_{0,t} + |\overline{\partial}_\tau - \overline{\partial}_\nu|_0 \|\alpha_\nu\|_{0,t}\right). \end{aligned} \tag{3.17}$$

Combining (3.15) – (3.17), one obtains that

$$\begin{aligned} \|R^\nu_\tau\|_{m,t} \lesssim \|\alpha_\tau - \alpha_\nu\|_{m,t} + |\overline{\partial}|_m \|\alpha_\tau - \alpha_\nu\|_{0,t} + t^m \|\alpha_\tau - \alpha_\nu\|_{0,t} \\ + \text{r.h.s. of (3.15)}. \end{aligned} \tag{3.18}$$

Therefore it follows from (3.13) and (3.18) that

$$\|\overline{\partial}^*_\tau P^{-1}_\tau (w^\nu_\tau - w^\nu_\nu)\|_{m,t} \lesssim \text{r.h.s. of (3.18)}. \tag{3.19}$$

If we combine (3.10), (3.11) and (3.19), we see that

$$\begin{aligned} &\|u_\tau - u_\nu\|_{m,t} \\ &\lesssim \|\alpha_\tau - \alpha_\nu\|_{m,t} + |\overline{\partial}|_m \|\alpha_\tau - \alpha_\nu\|_{0,t} + t^m \|\alpha_\tau - \alpha_\nu\|_{0,t} \\ &\quad + |\overline{\partial}_\tau - \overline{\partial}_\nu|_0 \left(\|\alpha_\nu\|_{m+3,t} + |\overline{\partial}|_{m+4} \|\alpha_\nu\|_{0,t} + t^{m+3} \|\alpha_\nu\|_{0,t}\right) \\ &\quad + |\overline{\partial}_\tau - \overline{\partial}_\nu|_{m+3} \|\alpha_\nu\|_{0,t}. \end{aligned} \tag{3.20}$$

This proves Theorem 1.3.

Next, let us prove Theorem 1.5. That is, we estimate the derivatives of V_τ (or U_τ, u_τ) at $\tau = \nu$. Let D^q_τ denote the q-th order derivative operator in τ-variable and we denote a typical element of $D^q_\tau g_\tau$ by $g^{(q)}_\tau$. We recall that $f^\nu_\tau = \beta_\tau + \tilde{\Delta}_\tau rhE^\nu_\tau$ and $w^\nu_\tau = V_\tau + rhE^\nu_\tau$. Let S_τ be the solution of

$$\Delta_\tau S_\tau = -\tilde{\Delta}'_\tau w^\nu_\tau + (f^\nu_\tau)' = \beta'_\tau + \tilde{\Delta}_\tau (rhE^\nu_\tau)' - \tilde{\Delta}'_\tau V_\tau := R^\nu_{1,\tau}, \quad \tau \in I, \tag{3.21}$$

where $\tilde{\Delta}'_\tau$ and $(f^\nu_\tau)'$ are the derivatives of $\tilde{\Delta}_\tau$ and f^ν_τ with respect to $\tau \in I$, in the Sobolev space $H_m(M)$. To estimate $\|R^\nu_{1,\nu}\|_{m,t}$, we first estimate $\tilde{\Delta}_\nu(rhE^\nu_\nu)'$. Note that

$$\tilde{\Delta}_\nu(rh\overline{L}^\nu_n)'V_\nu = rh(\overline{L}^\nu_n)'\tilde{\Delta}_\nu V_\nu + Y_\nu V_\nu, \tag{3.22}$$

where $Y_\nu = \left[\tilde{\Delta}_\nu, rh(\overline{L}^\nu_n)'\right]$. Therefore it follows from the interpolation formulas (2.2)–(2.4), and from the estimate (3.5), that

$$\begin{aligned}\|Y_\nu V_\nu\|_{m,t} &\lesssim \|V_\nu\|_{m+2,t} + |\overline{\partial}|_{1,m}\|V_\nu\|_{2,t} + t^m\|V_\nu\|_{2,t}\\ &\lesssim \|V_\nu\|_{m+2,t} + |\overline{\partial}|_{1,m+2}\|V_\nu\|_{0,t} + t^{m+2}\|V_\nu\|_{0,t}\\ &\lesssim \|\beta_\nu\|_{m+2,t} + \left(|\overline{\partial}|_{1,m+3} + t^{m+2}\right)\|\beta_\nu\|_{0,t},\end{aligned} \tag{3.23}$$

because $|\overline{\partial}|_{1,4} \approx 1$.

Combining (3.22) and (3.23), we obtain that

$$\begin{aligned}&\|\tilde{\Delta}_\nu(rhE^\nu_\nu)'\|_{m,t}\\ &\lesssim \|\beta_\nu\|_{m+1,t} + |\overline{\partial}|_{1,m}\|\beta_\nu\|_{1,t} + t^m\|\beta_\nu\|_{1,t} + \|Y_\nu V_\nu\|_{m,t}\\ &\lesssim \|\beta_\nu\|_{m+2,t} + \left(|\overline{\partial}|_{1,m+3} + t^{m+2}\right)\|\beta_\nu\|_{0,t}.\end{aligned} \tag{3.24}$$

Similarly, $\|\tilde{\Delta}'_\tau V_\tau\|_{m,t}$ is bounded by the right hand side of (3.23). Hence it follows from (3.3), (3.21) and (3.24) that

$$\|P^{-1}_\nu R^\nu_{1,\nu}\|_{m,t} \lesssim \|\beta'_\nu\|_{m,t} + \|\beta_\nu\|_{m+2,t} + \left(|\overline{\partial}|_{1,m+3} + t^{m+2}\right)\|\beta_\nu\|_{0,t}. \tag{3.25}$$

Therefore, one obtains, from (2.3), (2.4), (2.7), (3.5) and (3.25), that

$$\begin{aligned}&\|\overline{\partial}^*_\nu P^{-1}_\nu S_\nu\|_{m,t}\\ &\lesssim \|P^{-1}_\nu R^\nu_{1,\nu}\|_{m,t} + |\overline{\partial}|_{m+1}\|P^{-1}_\nu R^\nu_{1,\nu}\|_{0,t} + t^m\|P^{-1}_\nu R^\nu_{1,\nu}\|_{0,t}\\ &\lesssim \|\alpha'_\nu\|_{m,t} + \left(|\overline{\partial}|_m + t^m\right)\|\alpha'_\nu\|_{0,t} + \|\alpha_\nu\|_{m+2,t}\\ &\quad + \left(|\overline{\partial}|_{1,m+3} + t^{m+2}\right)\|a_\nu\|_{0,t}.\end{aligned} \tag{3.26}$$

Since $w^\nu_\tau - w^\nu_\nu \in Dom(\Delta_\tau)$, we have

$$\Delta_\tau(w^\nu_\tau - w^\nu_\nu) = \tilde{\Delta}_\tau(w^\nu_\tau - w^\nu_\nu) = f^\nu_\tau - f^\nu_\nu - (\tilde{\Delta}_\tau - \tilde{\Delta}_\nu)w^\nu_\nu.$$

Hence it follows that

$$\Delta_\tau\left(\frac{w^\nu_\tau - w^\nu_\nu}{\tau - \nu} - S_\tau\right) = \frac{f^\nu_\tau - f^\nu_\nu}{\tau - \nu} - (f^\nu_\tau)' - \left(\frac{\tilde{\Delta}_\tau - \tilde{\Delta}_\nu}{\tau - \nu}w^\nu_\nu - \tilde{\Delta}'_\tau w^\nu_\tau\right) := \tilde{R}^\nu_\tau.$$

Therefore one obtains, from (2.6), (3.2) and (3.3), that

$$\begin{aligned}&\left\|\frac{\overline{\partial}^*_\tau P^{-1}_\tau w^\nu_\tau - \overline{\partial}^*_\nu P^{-1} w^\nu_\nu}{\tau - \nu} - \overline{\partial}^*_\nu P^{-1}_\nu S_\nu\right\|_{m,t}\\ &\lesssim \|\tilde{R}^\nu_\tau\|_{m,t} + |\overline{\partial}|_{m+1}\|\tilde{R}^\nu_\tau\|_{0,t} + t^m\|\tilde{R}^\nu_\tau\|_{0,t}\\ &\quad + \|\overline{\partial}^*_\tau P^{-1}_\tau S_\tau - \overline{\partial}^*_\nu P^{-1}_\nu S_\nu\|_{m,t} + \left\|\frac{(\overline{\partial}^*_\tau P^{-1}_\tau - \overline{\partial}^*_\nu P^{-1}_\nu)w^\nu_\nu}{\tau - \nu}\right\|_{m,t}\end{aligned} \tag{3.27}$$

for a sufficiently large t, independent of $\tau \in I$. From (3.21) and Theorem 1.3, we have $\|\overline{\partial}_\tau^* P_\tau^{-1} S_\tau - \overline{\partial}_\nu^* P_\nu^{-1} S_\nu\|_{m,t} \longrightarrow 0$ as $\tau \to \nu$, and if we write

$$\tilde{R}_\tau^\nu = \left(\frac{f_\tau^\nu - f_\nu^\nu}{\tau - \nu} - (f_\nu^\nu)'\right) + (-(f_\tau^\nu)' + (f_\nu^\nu)') - \left(\frac{\tilde{\Delta}_\tau - \tilde{\Delta}_\nu}{\tau - \nu} w_\nu^\nu - \tilde{\Delta}_\nu' w_\nu^\nu\right) - \tilde{\Delta}_\nu'(w_\nu^\nu - w_\tau^\nu) + (\tilde{\Delta}_\tau' - \tilde{\Delta}_\nu')w_\tau^\nu,$$

it follows from Theorem 1.3 that $\|\tilde{R}_\tau^\nu\|_{m,t} \to 0$ as $\tau \to \nu$ and hence one obtains from (3.27) that

$$(\overline{\partial}_\nu^* P_\nu^{-1} w_\nu^\nu)' = u_\nu' + (\overline{\partial}_\nu^* P_\nu^{-1} rhE_\nu^\nu)' = \overline{\partial}_\nu^* P_\nu^{-1} S_\nu + (\overline{\partial}_\nu^* P_\nu^{-1})' w_\nu^\nu \tag{3.28}$$

in $H_m(M)$ where $u_\nu = \overline{\partial}_\nu^* P_\nu^{-1} V_\nu$. Combining (2.1), (3.8), (3.26) and (3.28), one then obtains, for sufficiently large $t \geq 0$ and for each $\nu \in I$, that

$$\begin{aligned} \|u_\nu'\|_{m,t} &\lesssim \|\overline{\partial}_\nu^* P_\nu^{-1} S_\nu\|_{m,t} + \|(\overline{\partial}_\nu^* P_\nu^{-1})' w_\nu^\nu\|_{m,t} + \|(\overline{\partial}_\nu^* P_\nu^{-1} rhE_\nu^\nu)'\|_{m,t} \\ &\lesssim \|\alpha_\nu'\|_{m,t} + \|\alpha_\nu\|_{m+2,t} + \left(|\overline{\partial}|_{m+1} + t^m\right) \|\alpha_\nu'\|_{0,t} \\ &\quad + \left(|\overline{\partial}|_{1,m+3} + t^{m+2}\right) \|a_\nu\|_{0,t}, \end{aligned} \tag{3.29}$$

provided $\alpha_\nu' \in H_m$ and $\alpha_\nu \in H_{m+2}(M)$. Similarly, following the same method, one can show that the solution of $\Box_\nu U_\nu = \alpha_\nu$ satisfies the same estimates as in (3.29) with u replaced by U, and hence we obtain that

$$\|u_\nu'\|_{m,t} + \|u_\nu\|_{m+2,t} + \|U_\nu'\|_{m,t} + \|U_\nu\|_{m+2,t} \lesssim \text{ r.h.s. of (3.29)}, \tag{3.30}$$

independent of ν.

For a simplicity, we let $\|f\|_{l,m,t}$ denote the norm $\|f\|_{l,m,t\lambda}$ defined in Section 1. We recall that $u = u(\tau, x) = u_\tau(x)$ and $\alpha = \alpha(\tau, x) = \alpha_\tau(x)$. In the following, we let $\Delta V = \beta$, and $\overline{\partial} u = \alpha$ mean that $\Delta_\nu V_\nu = \beta_\nu$, and $\overline{\partial}_\nu u_\nu = \alpha_\nu$, respectively, for each $\nu \in I$. Since the estimate in (3.30) is independent of $\nu \in I$ we integrate both sides of (3.30) with respect to $\nu \in I$ variable. Then we can summarize the above results as follows:

Proposition 3.4. *Under the notations as above, for each nonnegative integer m there are constants C_m and $T_{1,m}$, independent of $\nu \in I$, such that if $\alpha \in H_{1,m}(I \times M) \cap H_{0,m+2}(I \times M)$ and if $t \geq T_{1,m}$ then the canonical solution u of $\overline{\partial} u = \alpha$ and the Neumann solution U of $\Box U = \alpha$ with weight $e^{-t\lambda}$ satisfy the estimate*

$$\begin{aligned} &\|u\|_{1,m,t} + \|u\|_{0,m+2,t} + \|U\|_{1,m,t} + \|U\|_{0,m+2,t} \\ &\lesssim \|\alpha\|_{1,m,t} + \|\alpha\|_{0,m+2,t} + \left(|\overline{\partial}|_{m+1} + t^m\right) \|\alpha\|_{1,0,t} \\ &\quad + \left(|\overline{\partial}|_{1,m+3} + t^{m+2}\right) \|\alpha\|_{0,0,t}. \end{aligned} \tag{3.31}$$

For higher order derivative estimates, we use inductive step. For each non-negative integers q and m, and for each $\tau \in I$, we set

$$G_\tau^t(\alpha, q, m) = \sum_{r=0}^{q} |\overline{\partial}|_{q-r,0} \left(\|\alpha_\tau^{(r)}\|_{m+2q-2r,t} + \left(t^{m+2q-2r} + |\overline{\partial}|_{q,m+1+2q-2r}\right) \|\alpha_\tau^{(r)}\|_{0,t} \right).$$

Let u, U and V be the canonical, Neumann solutions of $\overline{\partial} u = \alpha$ respectively. In [6], the first author proved that the solutions u, U and V are smooth on parameter variables provided α is smooth on parameter variable. Assume $\alpha \in H_{k,m}(I \times M) \cap H_{0,m+2k}(I \times M) \cap C^\infty(I \times M)$ and assume that there exists a constant $T_{k,m} > 0$, independent of $\tau \in I$, such that if $t \geq T_{k,m}$, then for each $q \leq k$ we have

$$\|u_\tau^{(q)}\|_{m,t} + \|u_\tau^{(q-1)}\|_{m+2,t} + \|V_\tau^{(q)}\|_{m,t} + \|V_\tau^{(q-1)}\|_{m+2,t} \lesssim G_\tau^t(\alpha, q, m). \tag{3.32}$$

From (3.30), it follows that (3.32) holds for $k = 1$. Let us prove (3.32) for $k+1$.

Assume that $\Delta V = \beta$ where $\beta \in H_{k+1,m}(I \times M) \cap H_{k,m+2}(I \times M) \cap C^\infty(I \times M)$. Fix $\nu \in I$ for a moment. For each $\tau \in I$, define $E_{k,\tau}^\nu \in \Lambda_0^{0,1}$ by

$$E_{k,\tau}^\nu(\overline{L}) = \overline{L}_n^\tau(V_\nu^{(k)}(\overline{L})) + (P_\tau^{-1} V_\nu^{(k)})[B_\tau^{-1}\overline{L}, \overline{L}_n^\tau], \quad \overline{L} \in \overline{\mathcal{L}}^0,$$

and set

$$\begin{aligned} w_{k,\tau}^\nu &= V_\tau^{(k)} + rhE_{k,\tau}^\nu, \\ f_{k,\tau}^\nu &= \tilde{\Delta}_\tau V_\tau^{(k)} + \tilde{\Delta}_\tau rhE_{k,\tau}^\nu, \\ R_{k+1,\tau}^\nu &= -\tilde{\Delta}'_\tau w_{k,\tau}^\nu + (f_{k,\tau}^\nu)'. \end{aligned}$$

Note that

$$\tilde{\Delta}_\tau V_\tau^{(k+1)} = \left(\tilde{\Delta}_\tau V_\tau\right)^{(k+1)} + X_\tau^{(k+1)} V_\tau, \tag{3.33}$$

where

$$X_\tau^{(k+1)} = \left[\tilde{\Delta}_\tau, D_\tau^{k+1}\right].$$

From the interpolation formulas (2.1)–(2.4), it follows that

$$\|X_\tau^{(k+1)} V_\tau\|_{m,t} \lesssim \sum_{\substack{s+l=k+1 \\ s \leq k}} |\overline{\partial}|_{l,0} \|V_\tau^{(s)}\|_{m+2,t} + |\overline{\partial}|_{l,m+2} \|V_\tau^{(s)}\|_{0,t} + t^{m+2} |\overline{\partial}|_{l,0} \|V_\tau^{(s)}\|_{0,t} \tag{3.34}$$

independent of $\tau \in I$. Note that if we use the interpolation formula (3.14) in τ variable, and space variable separately, we obtain that

$$|\overline{\partial}|_{l,0} |\overline{\partial}|_{s-r,0} \lesssim |\overline{\partial}|_{l+s-r,0} \quad \text{and} \quad |\overline{\partial}|_{p,k_1} |\overline{\partial}|_{p,k_2} \lesssim |\overline{\partial}|_{p,k_1+k_2} |\overline{\partial}|_{p,0}. \tag{3.35}$$

Combining (3.33)–(3.35), and using the induction hypothesis (3.32), we see that

$$\|\tilde{\Delta}_\tau V_\tau^{(k+1)}\|_{m,t} \lesssim \|\beta_\tau^{(k+1)}\|_{m,t} + G_\tau^t(\alpha, k+1, m), \tag{3.36}$$

independent of τ. Also, as in (3.24) (with V_ν replaced by $V_\nu^{(k)}$) and using (3.36), it follows that

$$\|\tilde{\Delta}_\nu (rhE^\nu_{k,\nu})'\|_{m,t} \lesssim G^t_\nu(\alpha, k+1, m), \tag{3.37}$$

and hence, one obtains, from (3.2), (3.3), (3.36) and (3.37), that

$$\|P_\nu^{-1} R^\nu_{k+1,\nu}\|_{m,t} \lesssim \|\alpha_\tau^{(k+1)}\|_{m,t} + G^t_\nu(\alpha, k+1, m), \tag{3.38}$$

independent of ν. Using (3.38) and the interpolation formulas (2.3) and (2.4), one can show that

$$|\overline{\partial}|_{m+1} \|P_\nu^{-1} R^\nu_{k+1,\nu}\|_{0,t} + t^m \|P_\nu^{-1} R^\nu_{k+1,\nu}\|_{0,t} \lesssim G^t_\nu(\alpha, k+1, m). \tag{3.39}$$

Let $S_{k+1,\nu}$ be the solution of $\Delta_\nu S_{k+1,\nu} = R^\nu_{k+1,\nu}$. It follows from (2.7), (3.3), (3.38) and (3.39) that

$$\begin{aligned} &\|\overline{\partial}^*_\nu P_\nu^{-1} S_{k+1,\nu}\|_{m,t} \\ &\lesssim \|R^\nu_{k+1,\nu}\|_{m,t} + |\overline{\partial}|_{m+1} \|R^\nu_{k+1,\nu}\|_{0,t} + t^m \|R^\nu_{k+1,\nu}\|_{0,t} \\ &\lesssim \|\alpha_\tau^{(k+1)}\|_{m,t} + G^t_\nu(\alpha, k+1, m) \\ &\lesssim G^t_\nu(\alpha, k+1, m), \end{aligned} \tag{3.40}$$

independent of $\nu \in I$ provided $t \geq 0$ is sufficiently large. If we follow the same method leading to (3.28), we see that $(w^\nu_{k,\nu})'$ exists and

$$w^\nu_{k+1,\nu} := S_{k+1,\nu} = \frac{d}{d\tau}(w^\nu_{k,\tau})(\nu) = V_\nu^{(k+1)} + rh(E^\nu_{k,\nu})' \tag{3.41}$$

in $H_m(M)$, and hence it follows from (3.41) that

$$\begin{aligned} \overline{\partial}^*_\nu P_\nu^{-1} S_{k+1,\nu} &= \overline{\partial}^*_\nu P_\nu^{-1} V_\nu^{(k+1)} + \overline{\partial}^*_\nu P_\nu^{-1} rh(E^\nu_{k,\nu})' \\ &= \left(\overline{\partial}^*_\nu P_\nu^{-1} V_\nu\right)^{(k+1)} + Z_\nu^{(k+1)} V_\nu + \overline{\partial}^*_\nu P_\nu^{-1} rh(E^\nu_{k,\nu})', \end{aligned} \tag{3.42}$$

where

$$Z_\tau^{(k+1)} = \left[\overline{\partial}^*_\tau P_\tau^{-1}, D_\tau^{k+1}\right].$$

As in (3.34)–(3.37), we have

$$\|Z_\nu^{(k+1)} V_\nu\|_{m,t} + \|\overline{\partial}^*_\nu P_\nu^{-1} rh(E^\nu_{k,\nu})'\|_{m,t} \lesssim G^t_\nu(\alpha, k+1, m), \tag{3.43}$$

and it follows from (3.40)–(3.43) that

$$\|u_\nu^{(k+1)}\|_{m,t} \lesssim G^t_\nu(\alpha, k, m+2) \leq G^t_\nu(\alpha, k+1, m),$$

because $u_\nu = \overline{\partial}^*_\nu P_\nu^{-1} V_\nu$. Similarly, $\|V_\nu^{(k+1)}\|_{m,t}$ is bounded by $G^t_\nu(\alpha, k+1, m)$. This proves the inductive step of (3.32).

Since (3.32) holds independent of ν, we integrate both sides with respect to $\nu \in I$. Then for each integers k and m, there is $T_m > 0$ such that for each $t \geq T_m$, we have

$$\|U\|_{k,m,t} + \|U\|_{k-1,m+2,t} + \|u\|_{k,m,t} + \|u\|_{k-1,m+2,t} \lesssim G^t(\alpha, k, m),$$

where $G^t(\alpha, k, m)$ is defined in (1.7). Obviously, this estimates holds for $\alpha \in H_{k,m}(I \times M) \cap H_{0,m+2k}(I \times M)$. This proves Theorem 1.5.

4. Sobolev estimates on parameters on strongly pseudoconvex domains

When we study several complex variables, we sometimes fill a domain by a smooth parameter family of strongly pseudoconvex domains and use the estimates of $\bar\partial$-equation on space and parameter variables. Therefore we need to study a precise estimate both of space and parameter variables when $\overline{M}$ is strongly pseudoconvex.

Let $\{\overline{M}, \mathcal{L}^\tau\}_{\tau\in I}$, be a smooth family of diffeomorphic complex structures on $\overline{M}$ and assume that $\overline{M}$ is strongly pseudoconvex with respect to $\mathcal{L}^0$. In this case, we gain one derivative for Neumann operator and $\frac{1}{2}$-derivative for the canonical solution. In the following we assume that $s \geq -\frac{1}{2}$, and $|\cdot|_s$ should be understood that $|\cdot|_{[s]}$ where $[s]$ denotes the smallest nonnegative integer bigger than or equal to s. If we use the estimate (2.11) instead of (2.6) and (2.7), then (3.8) and (3.9) become

$$\|E^\nu_\tau\|_s \lesssim \|\alpha_\nu\|_s + |\bar\partial|_{s+2}\|\alpha_\nu\|_{-\frac{1}{2}}, \tag{4.1}$$

and

$$\|E^\nu_\tau - E^\nu_\nu\|_s \lesssim |\bar\partial_\tau - \bar\partial_\nu|_0\left(\|\alpha_\nu\|_s + |\bar\partial|_{s+1}\|\alpha_\nu\|_{-\frac{1}{2}}\right) + |\bar\partial_\tau - \bar\partial_\nu|_s\|\alpha_\nu\|_0. \tag{4.2}$$

Therefore the estimates analogous to (3.13) become

$$\begin{aligned} &\|P_\tau^{-1}(w^\nu_\tau - w^\nu_\nu)\|_{s+1} + \|\bar\partial^*_\tau P_\tau^{-1}(w^\nu_\tau - w^\nu_\nu)\|_{s+\frac{1}{2}} \\ &\lesssim \|P_\tau^{-1}R^\nu_\tau\|_s + |\bar\partial|_{s+2}\|P_\tau^{-1}R^\nu_\tau\|_{-\frac{1}{2}} \end{aligned} \tag{4.3}$$

where R^ν_τ is the right hand side of (3.12).

If we use the estimates (2.4) (with $t = 0$), (4.1) and (4.2) we obtain that

$$\begin{aligned} \|P_\tau^{-1}R^\nu_\tau\|_s \lesssim &\|\alpha_\tau - \alpha_\nu\|_s + |\bar\partial|_s\|\alpha_\tau - \alpha_\nu\|_0 + |\bar\partial_\tau - \bar\partial_\nu|_{s+2}\|\alpha_\nu\|_0 \\ &+ |\bar\partial_\tau - \bar\partial_\nu|_0\left(\|\alpha_\nu\|_{s+2} + |\bar\partial|_{s+4}\|\alpha_\nu\|_0\right). \end{aligned} \tag{4.4}$$

As in (3.18)–(3.20), it follows, from (4.3) and (4.4), that

$$\|U_\tau - U_\nu\|_{s+1} + \|u_\tau - u_\nu\|_{s+\frac{1}{2}} \lesssim \text{r.h.s. of (4.4)}, \tag{4.5}$$

provided that $\alpha_\nu \in H_{s+2}(M)$ for some $\nu \in I$ and $\alpha_\tau \in H_s(M)$ for each $\tau \in I$ where $s \geq -1/2$. Here we have used the fact that $|\bar\partial|_{1,4} \approx 1$. This proves Theorem 1.6.

Next we estimate the derivatives in parameter variable, i.e., Theorem 1.7. For each $q \geq 1$ and $s \geq -\frac{1}{2}$, and for each $\alpha \in H_{r,m+2k-2r}(I \times M) \cap C^\infty(I \times M)$,

$0 \le r \le k$, and for each $\tau \in I$, we set

$$\begin{aligned} &H_\tau(\alpha, q, s) \\ &= \sum_{r=0}^{q} |\overline{\partial}|_{q-r,0} \|\alpha_\tau^{(r)}\|_{s+q-r} \\ &+ \left(|\overline{\partial}|_{q-r,0}|\overline{\partial}|_{0,s+2q+1} + |\overline{\partial}|_{q-r,s+2q+1}\right) \|\alpha_\tau^{(r)}\|_{q-r-\frac{1}{2}}. \end{aligned}$$

Recall that $R_{1,\tau}^\nu = \beta_\tau' + \tilde{\Delta}_\tau(rhE_\tau^\nu)' - \tilde{\Delta}_\tau' V_\tau$. To estimate $\tilde{\Delta}_\tau(rhE_\tau^\nu)'$, we use the expression (3.22). Since $\overline{M}$ is strongly pseudoconvex, we note that the Neumann solution U_ν gets one derivative and hence the analogue of (3.23) becomes

$$\|Y_\nu V_\nu\|_s \lesssim \|V_\nu\|_{s+2} + |\overline{\partial}|_{1,s+3}\|V_\nu\|_0 \lesssim \|\alpha_\nu\|_{s+1} + |\overline{\partial}|_{1,s+3}\|\alpha_\nu\|_{-\frac{1}{2}}. \tag{4.6}$$

Hence it follows from (3.22) and (4.6) that

$$\begin{aligned} \|\tilde{\Delta}_\nu(rhE_\nu^\nu)'\|_s &\lesssim \|\beta_\nu\|_{s+1} + |\overline{\partial}|_{1,s}\|\beta_\nu\|_{\frac{1}{2}} + \|Y_\nu V_\nu\|_s \\ &\lesssim \|\alpha_\nu\|_{s+1} + |\overline{\partial}|_{1,s+3}\|\alpha_\nu\|_{-\frac{1}{2}}. \end{aligned} \tag{4.7}$$

Similarly, $\|\tilde{\Delta}_\tau' V_\tau\|_s$ is bounded by the right hand side of (4.7), and hence one obtains that

$$\|P_\nu^{-1} R_{1,\nu}^\nu\|_s \lesssim \|\alpha_\nu'\|_s + \|\alpha_\nu\|_{s+1} + |\overline{\partial}|_{1,s+3}\|\alpha_\nu\|_{-\frac{1}{2}}. \tag{4.8}$$

From (4.6)–(4.8), it follows that the estimates analogous to (3.26) become

$$\begin{aligned} \|P_\nu^{-1} S_\nu\|_{s+1} &\lesssim \|P_\nu^{-1} R_{1,\nu}^\nu\|_s + |\overline{\partial}|_{s+2}\|P_\nu^{-1} R_{1,\nu}^\nu\|_{-\frac{1}{2}} \\ &\lesssim \|\alpha_\nu'\|_s + \|\alpha_\nu\|_{s+1} + |\overline{\partial}|_{1,s+3}\|\alpha_\nu\|_{-\frac{1}{2}}. \end{aligned} \tag{4.9}$$

If we use (4.9) and follow the same method leading to (3.29), we obtain that

$$\|U_\nu'\|_{s+1} + \|u_\nu'\|_{s+\frac{1}{2}} \lesssim \|\alpha_\nu'\|_s + \|\alpha_\nu\|_{s+1} + |\overline{\partial}|_{1,s+3}\|\alpha_\nu\|_{-\frac{1}{2}}, \tag{4.10}$$

where $u_\nu = \overline{\partial}_\nu^* P_\nu^{-1} V_\nu$.

Assume, by induction (on k), that

$$\|U_\nu^{(q)}\|_{s+1} + \|V_\nu^{(q)}\|_{s+1} + \|u_\nu^{(q)}\|_{s+\frac{1}{2}} \le C_{q,s} H_\nu(\alpha, q, s), \tag{4.11}$$

for $s \ge -\frac{1}{2}$ and $q \le k$, where $C_{q,s}$ does not depend on $\nu \in I$. From (4.10), it follows that (4.11) is true for $k = 1$. Recall that

$$R_{k+1,\tau}^\nu = \tilde{\Delta}_\tau V_\tau^{(k+1)} + \tilde{\Delta}_\tau(rhE_{k,\tau}^\nu)' = \beta_\tau^{(k+1)} + X_\tau^{k+1} V_\tau + \tilde{\Delta}_\tau(rhE_{k,\tau}^\nu)', \tag{4.12}$$

where

$$X_\tau^{k+1} = \left[\tilde{\Delta}_\tau, \frac{d^q}{d\tau^q}\right].$$

If we use the interpolation formulas (2.1)–(2.2), and the estimate (3.34), and the induction hypothesis (4.11), we obtain that

$$\begin{aligned} &\|X_\tau^{(k+1)} V_\tau\|_s \\ (4.13)\qquad &\lesssim \sum_{\substack{p+q=k+1 \\ q \le k}} |\bar\partial|_{p,0} \|V_\tau^{(q)}\|_{s+2} + |\bar\partial|_{p,s+2} \|V_\tau^{(q)}\|_0 \lesssim H_\tau(\alpha, k+1, s) \end{aligned}$$

because $|\bar\partial|_{1,4} \approx 1$ and $H_\tau(\alpha, q, s+1) \lesssim H_\tau(\alpha, q+1, s) \lesssim H_\tau(\alpha, p+q, s)$. Therefore it follows, from (4.13), that

$$(4.14)\qquad \|\tilde\Delta_\tau V_\tau^{(k+1)}\|_s \lesssim \|\beta_\tau^{(k+1)}\|_s + \|X_\tau^{(k+1)} V_\tau\|_s \lesssim H_\tau(\alpha, k+1, s).$$

Also, as in (4.7), one can show that

$$(4.15)\qquad \|\tilde\Delta_\tau (rhE_{k,\tau}^\nu)'\|_s \lesssim H_\tau(k, s+1) \lesssim H_\tau(\alpha, k+1, s).$$

Therefore, it follows from (4.12), (4.14) and (4.15) that

$$(4.16)\qquad \|R_{k+1,\tau}^\nu\|_s \lesssim H_\tau(\alpha, k+1, s).$$

As in (3.41), we see that $(w_{k,\nu}^\nu)'$ exists and

$$(4.17)\qquad S_{k+1,\nu} = \frac{d}{d\tau}(w_{k,\tau}^\nu)(\nu) = V_\nu^{(k+1)} + rh(E_{k,\nu}^\nu)'$$

in $H_s(M)$ where $S_{k+1,\nu}$ is the solution of $\Delta_\nu S_{k+1,\nu} = R_{k+1,\nu}^\nu$. From (4.16), we have

$$\begin{aligned} (4.18)\qquad \|P_\nu^{-1} S_{k+1,\nu}\|_{s+1} &\lesssim \|P_\nu^{-1} R_{k+1,\tau}^\nu\|_s + |\bar\partial|_{s+2} \|P_\nu^{-1} R_{k+1,\nu}^\nu\|_{-\frac{1}{2}} \\ &\lesssim H_\nu(\alpha, k+1, s), \end{aligned}$$

and hence it follows from (3.3) and (4.17) that

$$(4.19)\qquad \|U_\nu^{(k+1)}\|_{s+1} + \|V_\nu^{(k+1)}\|_{s+1} \lesssim H_\nu(\alpha, k+1, s).$$

Similarly, $\|u_\nu^{(k+1)}\|_{s+\frac{1}{2}} \lesssim H_\nu(\alpha, k+1, s)$. This proves the inductive step.

Since (4.19) is independent of $\nu \in I$, we integrate both sides with respect to $\nu \in I$. Then for each integers k and $s \ge -1/2$, there is $C_{k,s} > 0$ such that

$$(4.20)\qquad \|U\|_{k,s+1} + \|V\|_{k,s+1} + \|u\|_{k,s+\frac{1}{2}} \le C_{k,s} H(\alpha, k, s).$$

This proves Theorem 1.7.

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