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NEAR-RINGS WITH LEFT BAER LIKE CONDITIONS

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ABSTRACT. Kaplansky introduced the Baer rings as rings in which every left (or right) annihilator of each subset is generated by an idempotent. On the other hand, Hattori introduced the left (resp. right) P.P. rings as rings in which every principal left (resp. right) ideal is projective. The purpose of this paper is to introduce the near-rings with Baer like condition and near-rings with P.P. like condition which are somewhat different from ring case, and to extend the results of Armendariz and Jøndrup.

1. Introduction

All rings and near-rings are assumed to be with identity. In [6], Kaplansky introduced the Baer rings as rings in which every left (right) annihilator is generated by an idempotent. On the other hand, Hattori [4] introduced the left P.P. rings as rings in which any principal left ideal is projective. Also, Berberian [3] called this concept as *left Rickart ring.*

In this paper, we introduce left Baer near-rings and left P.P. near-rings and give some examples of them and study some of their properties.

Let G be an additively written (but not necessarily abelian) group with zero element 0 and

$$M_0(G) = \{f : G \to G \mid f(0) = 0\}$$

the near-ring of all zero respecting mappings on G. We shall show that $M_0(G)$ is a left Baer near-ring. Also, as a corollary, we shall show that every zero-symmetric near-ring can be embedded into a left Baer near-ring.

Let R be a commutative ring with identity. When R is reduced, it is well known that R is a Baer (resp. P.P.) ring if and only if the polynomial ring $R[x]$ is a Baer (resp. P.P.) ring (see e.g., Armendariz [1] and Jøndrup [5]). Corresponding to this result, we will prove that the zero-symmetric part of $R[x]$ is a left Baer (resp. left P.P.) near-ring if and only if R is a Baer (resp. left P.P.) ring.

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Finally we shall study some properties of a zero-symmetric reduced near-ring with identity and the structure of a zero-symmetric reduced P.P. near-ring with identity.

2. Baer like near-rings and P.P. like near-rings

A (right) near-ring is a set N with two binary operations $+$ and $\cdot$ such that $(N,+)$ is a (not necessarily abelian) group with zero 0, $(N,\cdot)$ is a semigroup and $(x+y)z = xz+yz$ for all $x,y,z \in N$.

Some notations, basic definitions and concepts in near-ring theory can be found in Meldrum [7] and Pilz [8].

For any nonempty subset S of a near-ring N, the set $\{a \in N \mid aS = 0\}$ is called the *left annihilator of* S in N which is denoted by $l_N(S)$, simply, $l(S)$.

Note that $l(S)$ is a left ideal of N, if S is an N-subset of N, then $l(S)$ is an ideal of N and $l(S) = \bigcap_{x\in S} l(x)$. Similarly, the set $\{a \in N \mid Sa = 0\}$ is called the *right annihilator of* S in N which is denoted by $r_N(S)$, simply, $r(S)$.

A near-ring N is called a *left Baer near-ring* if, for any subset S of N, $l(S) = l(e)$ for some idempotent $e \in N$. The right Baer near-rings are defined similarly.

The following remark is obtained obviously:

Let N_i $(i \in I)$ be a family of near-rings. Then the direct product $\prod_{i\in I} N_i$ is a left Baer near-ring if and only if N_i is a left Baer near-ring for each $i \in I$.

A near-ring N is said to be *integral* if N has no nonzero divisors of zero ([8, 1.14, p.11]).

Example 1. (1) Every constant near-ring is a left Baer near-ring.

(2) A direct product of integral near-rings with identity is a left Baer near-ring.

Following Beidleman [2], we call that a near-ring N is von Neumann regular if, for any $x \in N$, there exists $y \in N$ such that $xyx = x$. Beidleman [2, Theorem 1] proved that $M_0(G)$ is a regular near-ring. Now, we can show that $M_0(G)$ is left Baer. But, in general, $M(G)$ is not left Baer.

Theorem 1. *The near-ring $M_0(G)$ is a left Baer near-ring.*

Proof. Let S be a nonempty subset of $M_0(G)$ and let $H = \{s(g) \mid s \in S,\ g \in G\}$. Let e be a mapping on G such that if $x \in H$, then $e(x) = x$ and $e(y) = 0$ for any $y \in G - H$. Then e is an idempotent of $M_0(G)$ and clearly, we see that $l(S) = l(e)$. This implies that $M_0(G)$ is a left Baer near-ring. □

Corollary 1. *Every zero-symmetric near-ring can be embedded into a left Baer near-ring.*

Proof. By [8, 1.102], every zero-symmetric near-ring can be embedded into a zero-symmetric near-ring with identity. Let N be a zero-symmetric near-ring with identity. By Theorem 1, $M_0(N)$ is a left Bear near-ring. For any $r \in N$, the mapping $f_r : t \in N \to rt \in N$ is an element of $M_0(N)$. Since

N contains an identity, the mapping $f : N \to M_0(N); r \mapsto f_r$ is a near-ring monomorphism. □

An associative ring R called a left P.P. ring if every principal left ideal of R is projective. This is equivalent to the condition that, for any $a \in R$, $l(a) = l(e)$ for some idempotent $e \in R$. A right P.P. ring is defined in a symmetric way. A right and left P.P. ring is called a P.P. ring.

Now we call a near-ring N a *left P.P. near ring* if, for any $a \in N$, $l(a) = l(e)$ for some idempotent $e \in N$. Also, a right P.P. near-ring is defined in a symmetric way. A right and left P.P. near-ring is called a *P.P. near-ring.* Clearly a left Baer near-ring is a left P.P. near-ring.

Example 2. Every von Neumann regular near-ring is a left P.P. near ring.

In fact, for any $x \in N$, there exists $y \in N$ such that $xyx = x$. Then xy is an idempotent and obviously, we have $l(x) = l(xy)$.

Let R be a commutative ring with identity and let $R[x]$ denote the set of all polynomials in one indeterminate x over R. Under usual addition $+$ and substitution $\circ$ of polynomials, $(R[x], +, \circ)$ becomes a near-ring.

A zero symmetric near-ring is a near-ring N with $a0 = 0$ for all $a \in R$.

Following Pilz [8, 7.78, p.221], $R_0[x]$ denotes the zero symmetric part of $R[x]$, that is,

$$R_0[x] = \left\{ \sum_{i=1}^{n} a_i x^i \mid a_i \in R, n \geq 1 \right\}.$$

The following is a near-ring theoretic modification of Jøndrup [5, Theorem 1.2]. Recall that a ring R with no nonzero nilpotent elements is called *reduced*, equivalently, $a^2 = 0$ in R implies $a = 0$.

Theorem 2. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) *$R_0[x]$ is a left P.P. near-ring;*
2) *R is a reduced and P.P. ring.*

Proof. 1) $\Rightarrow$ 2). First we claim that R is reduced. Suppose that $a \in R$ with $a^2 = 0$. By hypothesis, there exists an idempotent $f \in R_0[x]$ such that $l(ax) = l(f)$. Let $f = a_1 x + a_2 x^2 + \cdots + a_n x^n$ with $a_i \in R$. Since f is an idempotent, we have $a_1^2 = a_1$. Since $ax \in l(ax)$, $ax \circ f = af = 0$. In particular, $aa_1 = 0$. Also, since f is an idempotent, $x - f \in l(f)$, and we have

$$0 = (x - f) \circ ax = ax - f(ax).$$

Hence $ax^2 = a_1 ax = 0$, that is $a = 0$. This proves that R is reduced. Since R is reduced, the set of idempotents of $R_0[x]$ is just $\{ex \mid e^2 = e \in R\}$. Now let r be an arbitrary element of R. By hypothesis, there exists an idempotent $e \in R$ such that $l(rx) = l(ex)$. Clearly this implies that

$$l(r) = \{s \in R \mid sr = 0\} = R(1 - e) = l(e).$$

Hence R is a P.P. ring.

2) $\Rightarrow$ 1). Let $f = a_1x + \cdots + a_nx^n \in R_0[x]$ and $g = b_1x + \cdots + b_mx^m \in R_0[x]$. First we claim that $f \circ g = 0$ if and only if $a_ib_j = 0$ for all i, j. It suffices to prove the 'only if' part. Let P be an arbitrary prime ideal of R and let $\bar{f}$ and $\bar{g}$ denote the image of f and g in $(R/P)[x]$ respectively. Since R/P is an integral domain and since $\bar{f} \circ \bar{g} = \bar{0}$, we can easily see that either $\bar{f} = \bar{0}$ or $\bar{g} = \bar{0}$ holds. Hence $a_ib_j \in P$ for all i, j. Since P is an arbitrary prime ideal, this implies that $a_ib_j \in \mathrm{Rad}(R)$, where $\mathrm{Rad}(R)$ denote the prime radical of R. Since R is reduced, $\mathrm{Rad}(R) = 0$. This proves our claim. Therefore $a_1, \ldots, a_n \in l_R(b_1, \ldots, b_m)$.

Since R is a P.P. ring, for each i, there exists an idempotent $e_i \in R$ such that $l(b_i) = l(e_i)$. If $n = 2$, then $f = e_1 + e_2 - e_1e_2$ is an idempotent and $l_R(b_1, b_2) = l(f)$. Using induction on n, we can find an idempotent e of R such that $l_R(b_1, \ldots, b_m) = l(e)$. Then ex is an idempotent of $R_0[x]$ and $l(g) = l(ex)$. Therefore $R_0[x]$ is a left P.P. near-ring. □

Corollary 2. *Let R be a commutative reduced ring with identity. Then the following conditions are equivalent:*

1) *$R_0[x]$ is a left P.P. near-ring;*
2) *R is a P.P. ring;*
3) *$R[x]$ is a P.P. ring.*

The next theorem provides more examples of left Baer near-rings.

Theorem 3. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) *$R_0[x]$ is a left Baer near-ring;*
2) *R is a reduced and Baer ring.*

Proof. 1) $\Rightarrow$ 2). Let T be a nonempty subset of R and consider the subset $S = \{tx \mid t \in T\}$ of $R_0[x]$. As saw in the proof of 1) $\Rightarrow$ 2) of Theorem 2, the set of idempotents of $R_0[x]$ is just $\{ex \mid e^2 = e \in R\}$. Since $R_0[x]$ is left Baer, $l(S) = l(ex)$ for some idempotent $e \in R$. Then we can easily see that $l_R(T) = l_R(e)$. Hence R is a Baer ring.

2) $\Rightarrow$ 1). Let S be a subset of $R_0[x]$ and consider the set T of all coefficients of $g(x) \in S$. Let $f = a_1x + \cdots + a_nx^n \in l(S)$. As saw in the proof of 2) $\Rightarrow$ 1) of Theorem 2, $a_i \in l_R(T)$ for all i. Since R is a Baer ring, there exists an idempotent e such that $l_R(T) = l_R(e)$. Now we can easily see that $l(S) = l(ex)$. This proves that $R_0[x]$ is a left Baer near-ring. □

Corollary 3. *Let R be a commutative reduced ring with identity. Then the following conditions are equivalent:*

1) *$R_0[x]$ is a left Baer near-ring;*
2) *R is a Baer ring;*
3) *$R[x]$ is a Baer ring.*

Corollary 4. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) *R is a von Neumann regular ring;*

2) *$(R/I)_0[x]$ is a left P.P. near-ring for all proper ideals I of R.*

Proof. 1) $\Rightarrow$ 2). If R is von Neumann regular, then R/I is von Neumann regular for every proper ideal I of R, so that R/I is a P.P. ring. Hence this follows from Theorem 2.

2) $\Rightarrow$ 1). As saw in the proof of 1) $\Rightarrow$ 2) of Theorem 2, R/I is reduced for every proper ideal I of R. Let $a \in R$ and consider the ideal Ra^2 of R. Since R/Ra^2 is reduced and since $a + Ra^2 \in R/Ra^2$ is nilpotent, we have $a \in Ra^2$. This implies that R is von Neumann regular. □

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