Bull. Korean Math. Soc. **45** (2008), No. 2, pp. 253–261

ON GENERALIZED JORDAN LEFT DERIVATIONS IN RINGS

Mohammad Ashraf and Shakir Ali

Abstract. In this paper, we introduce the notion of generalized left derivation on a ring R and prove that every generalized Jordan left derivation on a 2-torsion free prime ring is a generalized left derivation on R. Some related results are also obtained.

1. Introduction

Throughout the present paper R will denote an associative ring with centre $Z(R)$. Recall that R is prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. As usual $[x, y]$ will denote the commutator $xy - yx$. An additive mapping $d : R \longrightarrow R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. An additive mapping $\delta : R \longrightarrow R$ is said to be a left derivation (resp. Jordan left derivation) if $\delta(xy) = x\delta(y) + y\delta(x)$ (resp. $\delta(x^2) = 2x\delta(x)$) holds for all $x, y \in R$. Clearly, every left derivation on a ring R is a Jordan left derivation but the converse need not be true in general; (see for example [18, Example 1.1]). First author together with Rehman [4] proved that a Jordan left derivation on a 2-torsion free prime ring is a left derivation. Further in [5], authors together with Rehman proved that if R is a 2-torsion free prime ring and $\delta : R \longrightarrow R$ is an additive mapping such that $\delta(u^2) = 2u\delta(u)$ for all u in a square closed Lie ideal U of R, then either $U \subseteq Z(R)$ or $\delta(U) = \{0\}$. During the last two decades, there has been ongoing interest concerning the relationship between the left derivation and Jordan left derivation on a prime ring (cf. [1, 4, 5, 7, 9, 14, 17, 18] and reference therein).

Following [12], an additive mapping $F : R \longrightarrow R$ is called a generalized derivation (resp. generalized Jordan derivation) if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ (resp. $F(x^2) = F(x)x + xd(x)$) holds for all $x, y \in R$. Clearly, every generalized derivation on a ring is a

Received January 3, 2007.

2000 *Mathematics Subject Classification.* 16W25, 16N60, 16U80.

Key words and phrases. prime rings, generalized left derivation, generalized Jordan left derivation.

generalized Jordan derivation. But the converse statement does not hold in general (see e.g., [6]). It is shown in [3] that if R is a ring with a commutator which is not a divisor of zero, then every generalized Jordan derivation on R is a generalized derivation. It should be mentioned that the result in [3] concerning generalized Jordan derivation has been improved in [2] and [6] by authors together with Rehman. More related results have also been obtained in [8], [13], and [15], where further references can be found.

Inspired by the definition of generalized derivation, we introduce the notion of generalized left derivation as follows: an additive mapping $G : R \longrightarrow R$ is called a *generalized left derivation* (*resp. generalized Jordan left derivation*) if there exists a Jordan left derivation $\delta : R \longrightarrow R$ such that $G(xy) = xG(y) + y\delta(x)$ (resp. $G(x^2) = xG(x) + x\delta(x)$) holds for all $x, y \in R$. It is obvious to see that every generalized left derivation on a ring R is a generalized Jordan left derivation. But the converse need not be true in general. The following example justifies this fact:

Example 1.1. Let S be a ring such that the square of each element in S is zero, but the product of some nonzero elements in S is nonzero. Next, let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \middle| a, b \in S \right\}.$$

Define a map $G : R \longrightarrow R$ such that

$$G \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we can find an associated Jordan left derivation $\delta : R \longrightarrow R$ such that

$$\delta \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to check that G is a generalized Jordan left derivation but not a generalized left derivation.

In the present paper, our aim is to establish set of conditions under which every generalized Jordan left derivation on a ring is a generalized left derivation. This lead to the discovery of some new results which can be regarded as a contribution to the theory of Jordan derivations in rings.

2. Preliminary results

To facilitate our discussion, we define a mapping $H : R^2 \longrightarrow R$ such that $H(x, y) = G(xy) - xG(y) - y\delta(x)$. Since G and δ both are additive, we have for any $x, y, z \in R$;

$$H(x, y + z) = H(x, y) + H(x, z) \text{ and } H(x + y, z) = H(x, z) + H(y, z).$$

Moreover, if H is zero then G is a generalized left derivation on R. We shall make use of commutator identities; $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$.

We begin with the following lemmas which are essential for developing the proof of our results.

Lemma 2.1 ([14, Proposition 2.2]). *Let R be a ring and X be a 2-torsion free left R-module. If $\delta : R \longrightarrow X$ is an additive mapping satisfying $\delta(x^2) = 2x\delta(x)$ for all $x \in R$, then*

(i) $\delta(x^2y) = x^2\delta(y) + (xy + yx)\delta(x) + x\delta(xy - yx)$ *for all* $x, y \in R$,
(ii) $\delta(yx^2) = x^2\delta(y) + (3yx - xy)\delta(x) - x\delta(xy - yx)$ *for all* $x, y \in R$,
(iii) $[x, y]\delta([x, y]) = 0$ *for all* $x, y \in R$,
(iv) $(x^2y - 2xyx + yx^2)\delta(y) = 0$ *for all* $x, y \in R$.

Lemma 2.2. *Let R be a 2-torsion free ring and $G : R \longrightarrow R$ be a generalized Jordan left derivation with associated Jordan left derivation $\delta : R \longrightarrow R$. Then*

(i) $G(xy + yx) = xG(y) + yG(x) + x\delta(y) + y\delta(x)$ *for all* $x, y \in R$,
(ii) $G(xyx) = xyG(x) + 2xy\delta(x) + x^2\delta(y) - yx\delta(x)$ *for all* $x, y \in R$,
(iii) $G(xyz + zyx) = xyG(z) + zyG(x) + 2xy\delta(z) + 2zy\delta(x) + xz\delta(y) + zx\delta(y) - yx\delta(z) - yz\delta(x)$ *for all* $x, y, z \in R$.

Proof. (i) We are given that G is a generalized Jordan left derivation of R such that

$$G(x^2) = xG(x) + x\delta(x) \quad \text{for all } x \in R. \tag{2.1}$$

Linearizing (2.1), we get

$$\begin{aligned} G((x + y)^2) &= (x + y)G(x + y) + (x + y)\delta(x + y) \\ &= xG(x) + xG(y) + yG(x) + yG(y) + x\delta(x) \\ &\quad + x\delta(y) + y\delta(x) + y\delta(y) \text{ for all } x, y \in R. \end{aligned} \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} &G((x + y)^2) \\ &= G(x^2 + xy + yx + y^2) \\ &= xG(x) + x\delta(x) + G(xy + yx) + yG(y) + y\delta(y) \text{ for all } x, y \in R. \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we get the required result.

(ii) Replacing y by $xy + yx$ in (i), we get

$$\begin{aligned} &G(x(xy + yx) + (xy + yx)x) \\ &= xG(xy + yx) + (xy + yx)G(x) + x\delta(xy + yx) \\ &\quad + (xy + yx)\delta(x) \text{ for all } x, y \in R. \end{aligned}$$

Since, $\delta : R \longrightarrow R$ is a Jordan left derivation, linearizing $\delta(x^2) = 2x\delta(x)$, we find that

$$\delta(xy + yx) = 2x\delta(y) + 2y\delta(x) \quad \text{for all } x, y \in R,$$

and hence

$$\begin{aligned} &G(x(xy+yx)+(xy+yx)x) \\ &= x^2G(y)+2xyG(x)+4xy\delta(x)+3x^2\delta(y) \\ &\quad + yx\delta(x)+yxG(x) \text{ for all } x,y\in R. \end{aligned} \tag{2.4}$$

Also,

$$\begin{aligned} &G(x(xy+yx)+(xy+yx)x) \\ &= G(x^2y)+2G(xyx)+G(yx^2) \\ &= x^2G(y)+yxG(x)+yx\delta(x)+x^2\delta(y) \\ &\quad + 2yx\delta(x)+2G(xyx) \text{ for all } x,y\in R. \end{aligned} \tag{2.5}$$

Comparing (2.4), (2.5) and using the fact that char$R\neq 2$, we obtain

$$G(xyx)=xyG(x)+2xy\delta(x)+x^2\delta(y)-yx\delta(x) \text{ for all } x,y\in R. \tag{2.6}$$

(iii) Replace x by $x+z$ in (2.6), to get

$$\begin{aligned} &G((x+z)y(x+z)) \\ &= xyG(x)+xyG(z)+zyG(x)+zyG(z)+2xy\delta(x) \\ &\quad + 2xy\delta(z)+2zy\delta(x)+2zy\delta(z)+x^2\delta(y)+xz\delta(y) \\ &\quad + zx\delta(y)+z^2\delta(y)-yx\delta(x)-yx\delta(z)-yz\delta(x)-yz\delta(z) \\ &\quad \text{for all } x,y,z\in R. \end{aligned} \tag{2.7}$$

On the other hand, we have

$$\begin{aligned} &G((x+z)y(x+z)) \\ &= G(xyx)+G(zyz)+G(xyz+zyz) \\ &= xyG(x)+2xy\delta(x)+x^2\delta(y)-yx\delta(x)+G(xyz+zyx) \\ &\quad + zyG(x)+2zy\delta(z)+z^2\delta(y)-yz\delta(z) \text{ for all } x,y,z\in R. \end{aligned} \tag{2.8}$$

Comparing (2.7) and (2.8), we get (iii). □

The following lemma play the key role in the proof of main theorem.

Lemma 2.3. *Let R be a 2-torsion free ring and $G:R\longrightarrow R$ be a generalized Jordan left derivation with associated Jordan left derivation $\delta:R\longrightarrow R$. Then*

$$[x,y]H(x,y)=0 \quad \textit{for all } x,y\in R. \tag{2.9}$$

Proof. Replace z by $xy-yx$ in Lemma 2.2(iii), to get

$$\begin{aligned} &G(xy(xy-yx)+(xy-yx)yx) \\ &= xyG(xy)-xyG(yx)+[x,y]yG(x)+[x,y]\delta([x,y])+xy\delta([x,y]) \\ &\quad + 2[x,y]y\delta(x)+x[x,y]\delta(y)+[x,y]x\delta(y)-y[x,y]\delta(x) \\ &\quad \text{for all } x,y,z\in R. \end{aligned} \tag{2.10}$$

Now, application of Lemma 2.1(iii) yields that

$$\begin{aligned}
&G(xy(xy-yx)+(xy-yx)yx)\\
&= xyG(xy) - xyG(yx) + [x,y]yG(x)\\
&\quad + 2[x,y]y\delta(x) + xy\delta([x,y]) + x[x,y]\delta(y)\\
&\quad + [x,y]x\delta(y) - y[x,y]\delta(x) \quad \text{for all } x,y \in R.
\end{aligned} \tag{2.11}$$

Also, we have

$$\begin{aligned}
&G(xy(xy-yx)+(xy-yx)yx)\\
&= G((xy)^2 - xy^2x + xy^2x - (yx)^2)\\
&= G((xy)^2) - G((yx)^2)\\
&= xyG(xy) + xy\delta(xy) - yxG(yx) - yx\delta(yx) \text{ for all } x,y \in R.
\end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12), we find that

$$\begin{aligned}
&yxG(yx) - xyG(yx) + [x,y]yG(x) + 2[x,y]y\delta(x)\\
&\quad + xy\delta([x,y]) + x[x,y]\delta(y) + [x,y]x\delta(y) - y[x,y]\delta(x)\\
&\quad + yx\delta(yx) - xy\delta(xy) = 0 \quad \text{for all } x,y \in R.
\end{aligned} \tag{2.13}$$

This implies that

$$\begin{aligned}
&[y,x]G(yx) + [x,y]yG(x) + [x,y]x\delta(y)) + 2[x,y]y\delta(x)\\
&\quad - 2y[x,y]\delta(x) + x[x,y]\delta(y) + y[x,y]\delta(x) + yx\delta(xy)\\
&\quad - xy\delta(xy) = 0 \quad \text{for all } x,y \in R.
\end{aligned} \tag{2.14}$$

By Lemma 2.1(iv), we have

$$\begin{aligned}
&x[x,y]\delta(y) + y[x,y]\delta(x) + yx\delta(xy) - xy\delta(xy)\\
&= (x^2y - 2xyx + yx^2)\delta(y) - (y^2x - 2yxy + xy^2)\delta(x)\\
&= 0 \quad \text{for all } x,y \in R
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&2[x,y]y\delta(x) - 2y[x,y]\delta(x)\\
&= 2(y^2x - 2yxy + xy^2)\delta(x) = 0 \text{ for all } x,y \in R.
\end{aligned} \tag{2.16}$$

Now, in view of (2.15) and (2.16), (2.14) reduces to

$$[y,x]G(yx) + [x,y]yG(x) + [x,y]x\delta(y) = 0 \quad \text{for all } x,y \in R.$$

This implies that

$$[x,y](G(xy) - xG(y) - y\delta(x)) = 0, \text{ i.e., } [x,y]H(x,y) = 0 \quad \text{for all } x,y \in R.$$

□

3. Main results

The main results of the present paper states as follows:

Theorem 3.1. *Let R be a 2-torsion free ring such that R has a commutator which is not a left zero divisor. Let $G : R \longrightarrow R$ be a generalized Jordan left derivation with associated Jordan left derivation $\delta : R \longrightarrow R$. Then every generalized Jordan left derivation on R is a generalized left derivation on R.*

Proof. By the assumption, for any fixed element $a, b \in R$ such that $[a,b]c = 0$ implies that $c = 0$. By Lemma 2.3, we have

$$H(a,b) = 0. \tag{3.1}$$

Replacing x by $x + a$ in (2.9) and using (2.9), we obtain

$$[x,y]H(a,y) + [a,y]H(x,y) = 0 \text{ for all } x,y \in R. \tag{3.2}$$

Linearizing (3.2) on y, we find that

$$\begin{aligned} &[x,b]H(a,y) + [a,y]H(x,b) + [a,b]H(x,y) + [a,b]H(x,b) \\ &= 0 \text{ for all } x,y \in R. \end{aligned} \tag{3.3}$$

Substituting a for x in (3.3) and using (3.1), we have $2[a,b]H(a,y) = 0$ for all $x,y \in R$. Since char$R \neq 2$, the last expression yields that $[a,b]H(a,y) = 0$ for all $x,y \in R$ and hence $H(a,y) = 0$ for all $y \in R$. Again, put b for y in (3.2), we find that $H(x,b) = 0$ for all $x \in R$. Therefore, equation (3.3) reduces to $[a,b]H(x,y) = 0$ for all $x,y \in R$ and hence $H(x,y) = 0$ for all $x,y \in R$, i.e., $G(xy) = xG(y) + y\delta(x)$ for all $x,y \in R$. This completes the proof of our theorem. □

Corollary 3.1. *Let R be a 2-torsion free ring such that R has a commutator which is not a left zero divisor. If $\delta : R \longrightarrow R$ is a Jordan left derivation, then δ is a left derivation on R.*

If the ring R is prime, then we have the following results:

Proposition 3.1. *Let R be a 2-torsion free prime ring. If R admits a generalized left derivation with associated Jordan left derivation δ, then either $\delta = 0$ or R is commutative.*

Proof. Let $G : R \longrightarrow R$ be a generalized left derivation with associated Jordan left derivation $\delta : R \longrightarrow R$. Then for any $x,y \in R$, we have

$$G(x^2y) = x^2G(y) + 2yx\delta(x) \text{ for all } x,y \in R. \tag{3.4}$$

On the other hand, we find that

$$G(x^2y) = G(x(xy)) = x^2G(y) + 2xy\delta(x) \text{ for all } x,y \in R. \tag{3.5}$$

Comparing (3.4) and (3.5), we obtain

$$2[x,y]\delta(x) = 0 \text{ for all } x,y \in R. \tag{3.6}$$

Since R is a 2-torsion free, the equation (3.6) implies that $[x,y]\delta(x)=0$ for all $x,y\in R$. Replacing y by yz in the last expression, we find that $[x,y]R\delta(x)=\{0\}$ for all $x,y\in R$. Thus for each $x\in R$, the primeness of R implies that either $[x,y]=0$ or $\delta(x)=0$ for all $y\in R$. Now, we put $A=\{x\in R\mid \delta(x)=0\}$, and $B=\{x\in R\mid [x,y]=\{0\}$ for all $y\in R\}$. Then, clearly A and B are additive subgroups of R whose union is R. But a group can not be written as a set theoretic union of two of its proper subgroups and hence we obtain that either $A=R$ or $B=R$. If $A=R$, then $\delta(x)=0$ for all $x\in R$. On the other hand, if $B=R$, then $[x,y]=0$ for all $x,y\in R$ and hence R is commutative. The proof of the proposition is complete. □

As a special case of above proposition, we have the following result:

Corollary 3.2. *Let R be a 2-torsion free prime ring. If R admits a nonzero Jordan left derivation δ, then R is commutative.*

Theorem 3.2. *Let R be a 2-torsion free prime ring. Let $G: R\longrightarrow R$ be a generalized Jordan left derivation with associated Jordan left derivation $\delta: R\longrightarrow R$. Then every generalized Jordan left derivation is a generalized left derivation on R.*

Proof. If the associated Jordan left derivation $\delta=0$, then G is a Jordan left multiplier on R. Therefore in view of Proposition 1.4 [19], G is a left multiplier (right centralizer). Hence for $\delta=0$, it is a generalized left derivation.

On the other hand suppose that the associated Jordan left derivation $\delta\neq 0$. Then, by Corollary 3.2, R is commutative. Notice that in view of main theorem of [4], every Jordan left derivation on a 2-torsion free prime ring is a left derivation. Hence by Lemma 2.2(i) and using the fact that R is 2-torsion free prime ring, we find that

$$\begin{aligned} G(xyz+zyx) &= G((xy)z+z(yx)) \\ &= xyG(z)+zG(yx)+xy\delta(z)+zy\delta(x)+zx\delta(y) \\ &\quad \text{for all } x,y,z\in R. \end{aligned} \tag{3.7}$$

Combining (3.7) with Lemma 2.2(iii), we find that

$$\begin{aligned} & zG(yx)+xy\delta(z)+zx\delta(y)+zy\delta(x) \\ = {} & zyG(x)++2xy\delta(z)+2zy\delta(x)+xz\delta(y)+zx\delta(y)-yz\delta(x)-yx\delta(z) \\ & \text{for all } x,y,z\in R. \end{aligned} \tag{3.8}$$

Since R is commutative, so equation (3.8) reduces to

$$z(G(yx)-yG(x)-x\delta(y))=0 \ \text{ for all } x,y,z\in R.$$

This implies that

$$(G(yx)-yG(x)-x\delta(y))R(G(yx)-yG(x)-x\delta(y))=\{0\} \ \text{ for all } x,y\in R.$$

Thus, the primeness of R yields that $G(yx)-yG(x)-x\delta(y)=0$ for all $x,y\in R$. That is, $G(xy)=xG(y)+y\delta(x)$ for all $x,y\in R$. Hence, G is a generalized left derivation on R. This completes the proof of our theorem. □

The following example demonstrates that R to be prime is essential in the hypotheses of the above theorem.

Example 3.1. Consider the rings S and R, as in Example 1.1, and define maps $G,\ \delta : R \longrightarrow R$ in similar manner. Then, it can be easily seen that $G(r^2)=rG(r)=rG(s)=r\delta(r)=s\delta(r)=0$ for all $r,s\in R$ but $G(rs)\neq 0$ for some nonzero elements $r,s\in R$.

In the end, it is to remark that the above result may be obtained for semiprime ring, but to our knowledge it has not yet been settled.

References

[1] M. Ashraf, *On left (θ,ϕ)-derivations of prime rings*, Arch. Math. (Brno) **41** (2005), no. 2, 157–166.

[2] M. Ashraf, A. Ali, and S. Ali, *On Lie ideals and generalized (θ,ϕ)-derivations in prime rings*, Comm. Algebra **32** (2004), no. 8, 2977–2985.

[3] M. Ashraf and N. Rehman, *On Jordan generalized derivations in rings*, Math. J. Okayama Univ. **42** (2000), 7–9.

[4] ———, *On Lie ideals and Jordan left derivations of prime rings*, Arch. Math. (Brno) **36** (2000), no. 3, 201–206.

[5] M. Ashraf, N. Rehman, and S. Ali, *On Jordan left derivations of Lie ideals in prime rings*, Southeast Asian Bull. Math. **25** (2001), no. 3, 379–382.

[6] ———, *On Lie ideals and Jordan generalized derivations of prime rings*, Indian J. Pure Appl. Math. **34** (2003), no. 2, 291–294.

[7] M. Brešar and J. Vukman, *On left derivations and related mappings*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 7–16.

[8] W. Cortes and C. Haetinger, *On Jordan generalized higher derivations in rings*, Turkish J. Math. **29** (2005), no. 1, 1–10.

[9] Q. Deng, *On Jordan left derivations*, Math. J. Okayama Univ. **34** (1992), 145–147.

[10] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.

[11] ———, *Topics in Ring Theory*, Univ. of Chicago Press, Chicago, 1969.

[12] B. Hvala, *Generalized derivations in rings*, Comm. Algebra **26** (1998), no. 4, 1147–1166.

[13] W. Jing and S. Lu, *Generalized Jordan derivations on prime rings and standard operator algebras*, Taiwanese J. Math. **7** (2003), no. 4, 605–613.

[14] K. W. Jun and B. D. Kim, *A note on Jordan left derivations*, Bull. Korean Math. Soc. **33** (1996), no. 2, 221–228.

[15] Y. S. Jung, *Generalized Jordan triple higher derivations on prime rings*, Indian J. Pure Appl. Math. **36** (2005), no. 9, 513–524.

[16] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[17] J. Vukman, *Jordan left derivations on semiprime rings*, Math. J. Okayama Univ. **39** (1997), 1–6.

[18] S. M. A. Zaidi, M. Ashraf, and S. Ali, *On Jordan ideals and left (θ,θ)-derivations in prime rings*, Int. J. Math. Math. Sci. **2004** (2004), no. 37-40, 1957–1964.

[19] B. Zalar, *On centralizers of semiprime rings*, Comment. Math. Univ. Carolin. **32** (1991), no. 4, 609–614.

Mohammad Ashraf
Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
E-mail address: mashraf80@hotmail.com

Shakir Ali
Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
E-mail address: shakir50@rediffmail.com