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OSCILLATION THEOREMS FOR PERTURBED DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. By means of a Riccati transform and averaging technique some oscillation criteria are established for perturbed nonlinear differential equations of second order

$$(P_1) \qquad (p(t)x'(t))' + q(t)|x(\phi(t))|^{\alpha+1}\operatorname{sgn} x(\phi(t)) + g(t,x(t)) = 0$$

(P_2) and (P_3) satisfying the condition (H). A comparison theorem and examples are given.

1. Introduction

The purpose of this paper is to study oscillatory properties of solutions of nonlinear delay differential equations

$$(P_1) \qquad \big[p(t)x'(t)\big]' + q(t)|x(\phi(t))|^{\alpha+1}\operatorname{sgn} x(\phi(t)) + g(t,x(t)) = 0,$$

$$(P_2) \qquad \big[p(t)x'(t)\big]' + q(t)|x(\phi(t))|^{\alpha+1}\operatorname{sgn} x(\phi(t)) + g(t,x(t),x'(t)) = 0,$$

and a nonlinear differential equation

$$(P_3) \qquad \big[p(t)x'(t)\big]' + q(t)|x(t)|^{\alpha+1}\operatorname{sgn} x(t) + g(t,x(t)) = 0,$$

where $p,\ q \in C([a,\infty),(0,\infty))$, $t \geq a > 0$, $\alpha > 0$. We assume that $\phi(t)$ is nondecreasing and

$$\phi(t) \leq t \quad \text{and} \quad \phi(t) \to \infty \text{ as } t \to \infty.$$

In this paper we always define a function $\varrho(t)$ as

$$\varrho(t) = \int_a^t 1/p(u)\,du \quad \text{for} \quad a \leq t,$$

and assume that

$$(H) \qquad \varrho(t) \to \infty \text{ as } t \to \infty.$$

By a solution of differential equations we mean a continuously differentiable function $x : [t_0,\infty) \to R$, for some t_0, such that $x(t)$ satisfies the differential

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equation for all $t \geq t_0$. A solution $x(t)$ is said to be oscillatory if it has unbounded zeros. Otherwise it is said to be nonoscillatory. An equation is said to be oscillatory if all solutions of the equation are oscillatory.

In the last two decades there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions for different classes of nonlinear differential equations of second order. We refer to the recent papers [1, 2, 3, 5, 6, 7, 12, 13, 15, 16] where further references can be founded therein.

For the Emden-Fowler differential equation $x''(t) + q(t)|x(t)|^{\alpha+1}\operatorname{sgn} x(t) = 0$ Atkinson and Belohorec proved that

$$(E_1) \qquad \int_a^\infty tq(t)\, dt = \infty$$

if and only if the differential equation is oscillatory. Applying the Leighton transformation to the above estimate (E_1), Cecci [4, Theorem 5, (ii)] obtained the more general result under the condition (H) and

$$(E_2) \qquad \int_0^\infty q(t)\varrho(t)\, dt = \infty$$

if and only if the generalized Emden-Fowler differential equation

$$(E_3) \qquad \big[p(t)x'(t)\big]' + q(t)|x(t)|^{\alpha+1}\operatorname{sgn} x(t) = 0$$

is oscillatory. We note that the estimate (E_2) is valid if $\int_a^\infty q(t)\, dt = \infty$.

We consider a delay differential equation of the form

$$(E_4) \quad \big[tx'(t)\big]' + 2(\sin t)t^{-\alpha}\,|x(t-c)|^{\alpha+1}\operatorname{sgn} x(t-c) - (1+2t\sin t)\operatorname{sgn} x(t-c) = 0,$$

where $t > 0$. Even if this equation contains a perturbed term $-(1 + 2t\sin t)$ oscillatory, it has a nonoscillatory solution $x(t) = t + c$. In this paper we seek the sufficient conditions for the equations (P_i), $1 \leq i \leq 3$, to be oscillatory.

To prove the oscillatory properties of differential equations we make use of $H(t,s)$ as a weight function. While the function $(t-s)^n$, $n \geq 1$, of Kamenev type [9, 10, 11, 14] or more general classes of weight functions [15] are very popular in various applications. Let $H(t,s)$ be defined on $D = \{(t,s) : t \geq s \geq a\}$. We shall assume that $H(t,s)$ is sufficiently smooth in both variables t and s so that the following conditions are satisfied

(H_1) $H(t,t) \equiv 0$ for $t \geq a$, $\quad H(t,s) > 0$ for $t > s \geq a$

(H_2) $\frac{\partial H(t,s)}{\partial s} = -h(t,s)H(t,s)^{1/2}$,

where $h \in C(D, [0, \infty))$.

2. Main results

In this paper we assume that

(A_1) $1/p(t)$ is nonincreasing, differentiable and locally integrable.

Lemma 1 ([8]). *Let X, Y be positive constants and $\sigma \geq 0$. Then the inequality*

$$(\sigma+1)XY^{\sigma} \leq X^{\sigma+1} + \sigma Y^{\sigma+1}$$

is valid where the equality holds if and only if $X = Y$.

Lemma 2. *Let $K > 0$, $a > 0$, $b \geq 0$ and c be constants. Then*

$$\limsup_{t\to\infty} \frac{1}{(t-b)^2} \int_b^t \left[K(t-s)^2(s-c)s^{\gamma}|\sin 2\pi s|^{\delta} - (2s-t)^2\right] ds = \infty$$

is valid where $\delta > 0$, $\gamma > 0$.

Proof. By means of a Taylor polynomial with remainder there exists a constant θ, $0 < \theta < 1$, such that

$$\begin{aligned} &\int_b^t \left[K(t-s)^2(s-c)s^{\gamma}|\sin 2\pi s|^{\delta} - (2s-t)^2\right] ds \\ &= \frac{K(t-b)^3}{3}\big(\theta(t-b)+b-c\big)\big(\theta(t-b)+b\big)^{\gamma}\left|\sin 2\pi\big(\theta(t-b)+b\big)\right|^{\delta} \\ &\quad - \frac{2(t-b)^3}{3} - b(t-b)^2 - b^2(t-b). \end{aligned}$$

Then $\frac{1}{(t-b)^2}\int_b^t \left[K(t-s)^2(s-c)s^{\gamma}|\sin 2\pi s|^{\delta} - (2s-t)^2\right] ds$ is a function of t with degree $2+\gamma$. Choose an increasing sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} t_n = \infty, \quad b + \theta(t_n - b) = n + \frac{1}{4},$$

where n is a positive integer. So our lemma follows because $\gamma > 0$ and coefficient of $(t_n - b)^{2+\gamma}$ is $K\theta^{1+\gamma}/3 > 0$. □

Theorem 3. *Let the conditions (H), (A_1), and*

$$(A_2) \qquad g(t) = f(t)\cdot \operatorname{sgn} x(t) \text{ with } f(t) \geq 0 \text{ for all } t > a$$

be satisfied. Assume that there exists a positive function $\rho(t) \in C^1([a, \infty))$ such that for some $\lambda \in (0,1)$

$$(1) \qquad \begin{aligned} \limsup_{t\to\infty} \frac{1}{H(t,a)} \int_a^t \Bigg[&\lambda \left\{\frac{(\alpha+1)^{\alpha+1}}{\alpha^{\alpha}}\right\}^{\frac{1}{\alpha+1}} H(t,s)\rho(s) \\ &\times q(s)^{\frac{1}{\alpha+1}} \frac{\phi(s)}{s} f(s)^{\frac{\alpha}{\alpha+1}} - V(t,s)^2\Bigg] ds = \infty \end{aligned}$$

is valid where

$$(2) \qquad V(t,s) = \frac{1}{2}\sqrt{\rho(s)p(s)}\left[h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\right].$$

Then the equation (P_1) is oscillatory.

Proof. Assume that $x(t)$ is a nonoscillatory solution of the equation (P_1) and that there exists $T_0 \geq a$ such that $x(t) > 0$, $x(\phi(t)) \geq 0$ for all $t \geq T_0$. So $p(t)x'(t)$ is decreasing for $t \geq T_0$. It is not difficult to show there exists a T_1 with $T_1 \geq T_0$ such that $x'(t) > 0$ for $t \geq T_1$. Consider the function $\Psi(t)$ defined by

$$\Psi(t) = \int_{T_1}^{t} p(\tau)x'(\tau)\, d\tau. \tag{3}$$

By mean value theorem it follows that

$$\Psi(t) = \Psi'(\xi)(t - T_1) \geq \Psi'(t)(t - T_1) \quad \text{for} \quad t > T_1, \tag{4}$$

from which we can derive $\dfrac{\Psi'(t)}{\Psi(t)} \leq \dfrac{1}{t - T_1}$. It is clear from (3) and (A_1) that $\Psi(t) \leq p(t)(x(t) - x(T_1))$. It follows that

$$\frac{x'(t)}{x(t) - x(T_1)} \leq \frac{1}{t - T_1}. \tag{5}$$

Therefore for $\lambda \in (0, 1)$ there exists $T_2 > T_1$ such that

$$\frac{\lambda\phi(t)}{t} \leq \frac{x(\phi(t))}{x(t)} \quad \text{for} \quad T_2 \leq t. \tag{6}$$

Now we consider a Riccati transform

$$W(t) = \rho(t)\frac{p(t)x'(t)}{x(t)}. \tag{7}$$

It is obvious that

$$W'(t) = \frac{\rho'(t)}{\rho(t)}\, W(t) - \rho(t)\left\{q(t)\frac{x(\phi(t))^{\alpha+1}}{x(t)} + \frac{f(t)}{x(t)}\right\} - \frac{1}{\rho(t)p(t)}W(t)^2. \tag{8}$$

By Lemma 1 with $\sigma = \alpha$, $X = q(t)^{1/(\alpha+1)}\frac{\lambda\phi(t)}{t}\, x(t)$, and $Y = (f(t)/\alpha)^{1/(\alpha+1)}$ we can derive the inequality

$$C_\alpha\, q(t)^{\frac{1}{\alpha+1}}\, \frac{\lambda\phi(t)}{t} f(t)^{\frac{\alpha}{\alpha+1}}\, x(t) \leq q(t)\left(\lambda\frac{\phi(t)}{t}\right)^{\alpha+1} x(t)^{\alpha+1} + f(t) \tag{9}$$

from which $q(t)\frac{x(\phi(t))^{\alpha+1}}{x(t)} + \dfrac{f(t)}{x(t)}$ is bounded below by

$$C_\alpha\, q(t)^{1/(\alpha+1)}\frac{\lambda\phi(t)}{t} f(t)^{\alpha/(\alpha+1)},$$

where

$$C_\alpha = \left\{\frac{(\alpha+1)^{\alpha+1}}{\alpha^\alpha}\right\}^{\frac{1}{\alpha+1}}. \tag{10}$$

Thus (8) is reduced to

(11)

$$W'(t) \le \frac{\rho'(t)}{\rho(t)} W(t) - \lambda C_\alpha \rho(t) q(t)^{1/(\alpha+1)} \frac{\phi(t)}{t} f(t)^{\alpha/(\alpha+1)} - \frac{1}{\rho(t)p(t)} W(t)^2.$$

Integrating for s from $T_2 \ge a$ to t after multiplying (11) by $H(t,s)$ we obtain

$$\begin{aligned} &\lambda C_\alpha \int_{T_2}^t H(t,s)\rho(s)q(s)^{1/(\alpha+1)} \frac{\phi(s)}{s} f(s)^{\alpha/(\alpha+1)}\,ds \\ &\le -\int_{T_2}^t H(t,s)W'(s)\,ds - \int_{T_2}^t \frac{1}{\rho(s)p(s)} H(t,s)W(s)^2\,ds \\ &\quad + \int_{T_2}^t \frac{\rho'(s)}{\rho(s)} H(t,s)W(s)\,ds. \end{aligned} \tag{12}$$

In view of (H_1), (H_2) it follows that

$$\int_{T_2}^t H(t,s)W'(s)\,ds = -H(t,T_2)W(T_2) + \int_{T_2}^t h(t,s)\sqrt{H(t,s)}W(s)\,ds. \tag{13}$$

Thus the below terms of the inequality (12) are transformed into

$$\begin{aligned} H(t,T_2)W(T_2) - \int_{T_2}^t \Bigg[&\left\{h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\right\}\sqrt{H(t,s)}W(s) \\ &+ \frac{1}{\rho(s)p(s)} H(t,s)W(s)^2\Bigg]\,ds. \end{aligned} \tag{14}$$

Using (14) we obtain

$$\begin{aligned} &\lambda C_\alpha \int_{T_2}^t H(t,s)\rho(s)q(s)^{1/(\alpha+1)} \frac{\phi(s)}{s} f(s)^{\alpha/(\alpha+1)}\,ds \\ &\le H(t,T_2)W(T_2) - \int_{T_2}^t \left[\sqrt{\frac{H(t,s)}{\rho(s)p(s)}}W(s) + V(t,s)\right]^2 ds + \int_{T_2}^t V(t,s)^2\,ds, \end{aligned}$$

where $V(t,s)$ is given by (2). From the latter inequality and (H_2) it follows that

$$\begin{aligned} &\int_{T_2}^t \left[\lambda C_\alpha H(t,s)\rho(s)q(s)^{1/(\alpha+1)} \frac{\phi(s)}{s} f(s)^{\alpha/(\alpha+1)} - V(t,s)^2\right] ds \\ &\le H(t,T_2)W(T_2) \le H(t,a)W(T_2). \end{aligned}$$

Since this inequality is valid for all $t \geq T_2$, by (H_2) we have

$$\begin{aligned}
&\int_a^t \left[\lambda C_\alpha\, H(t,s)\rho(s)q(s)^{1/(\alpha+1)}\,\frac{\phi(s)}{s}f(s)^{\alpha/(\alpha+1)} - V(t,s)^2\right] ds\\
(15)\qquad &\leq \int_a^{T_2} \left[\lambda C_\alpha\, H(t,s)\rho(s)\; q(s)^{1/(\alpha+1)}\,\frac{\phi(s)}{s}f(s)^{\alpha/(\alpha+1)} - V(t,s)^2\right] ds\\
&+ H(t,a)W(T_2).
\end{aligned}$$

On the other hand, since $H(t,s)$ is nonincreasing for the second variable by (H_2), we get

$$\begin{aligned}
&\int_a^{T_2} \left[\lambda C_\alpha\, H(t,s)\rho(s)q(s)^{1/(\alpha+1)}\,\frac{\phi(s)}{s}f(s)^{\alpha/(\alpha+1)} - V(t,s)^2\right] ds\\
&\leq H(t,a)\int_a^{T_2} \lambda C_\alpha\, \rho(s)q(s)^{1/(\alpha+1)}\,\frac{\phi(s)}{s}f(s)^{\alpha/(\alpha+1)}\, ds.
\end{aligned}$$

Thus (15) is reduced to

$$\begin{aligned}
&\int_a^t \left[\lambda C_\alpha\, H(t,s)\rho(s)q(s)^{1/(\alpha+1)}\,\frac{\phi(s)}{s}f(s)^{\alpha/(\alpha+1)} - V(t,s)^2\right] ds\\
&\leq H(t,a)\left[\int_a^{T_2} \lambda C_\alpha\, \rho(s)q(s)^{1/(\alpha+1)}\,\frac{\phi(s)}{s}f(s)^{\alpha/(\alpha+1)}\, ds + W(T_2)\right],
\end{aligned}$$

which contradicts the equality (1). Thus (P_1) is oscillatory. In the case of $x(t) < 0$ for $t \geq T_0$, we put $y(t) = -x(t)$ and take (A_2) into account. Then $y(t)$ is a positive solution of $\left(p(t)y'(t)\right)' + q(t)\left|y(\phi(t))\right|^{\alpha+1} \operatorname{sgn} y(\phi(t)) + f(t)\operatorname{sgn} y(t) = 0$. By means of the similar argument we reach the same conclusion. □

Corollary 4. *Let the conditions in Theorem 3 be satisfied. Assume that there exists a positive function $\rho(t) \in C^1([a,\ \infty))$ such that*

$$(16)\qquad \limsup_{t\to\infty} \frac{1}{H(t,a)}\int_a^t H(t,s)\rho(s)q(s)^{\frac{1}{\alpha+1}}\,\frac{\phi(s)}{s}f(s)^{\frac{\alpha}{\alpha+1}}\, ds = \infty$$

and

$$(17)\qquad \limsup_{t\to\infty} \frac{1}{H(t,a)}\int_a^t \rho(s)p(s)\left[h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds < \infty$$

are valid. Then the equation (P_1) is oscillatory.

If we choose several appropriate functions $H(t,s)$ and $h(t,s)$ we can obtain the various results from Theorem 3. Consider, for example, $H(t,s) = (t-s)^n$, $n \geq 2$, $(t,s) \in D$. Then $h(t,s) = n(t-s)^{(n-2)/2}$ and $\sqrt{H(t,s)} = (t-s)^{n/2}$.

Corollary 5. *Under the conditions in Theorem 3, if there exists a positive function $\rho(t) \in C^1([a, \infty))$ such that for a constant λ, $0 < \lambda < 1$,*

$$\limsup_{t\to\infty} t^{-n} \int_a^t \left[\lambda C_\alpha (t-s)^n \rho(s) q(s)^{\frac{1}{\alpha+1}} \frac{\phi(s)}{s} f(s)^{\frac{\alpha}{\alpha+1}} - V(t,s)^2\right] ds = \infty$$

is valid where C_α is a constant given by (10) and

$$V(t,s) = \frac{1}{2}\sqrt{\rho(s)p(s)}(t-s)^{(n-2)/2}\left[n - \frac{\rho'(s)}{\rho(s)}(t-s)\right].$$

Then the equation (P_1) is oscillatory.

Example 1. Consider a differential equation

$$\begin{aligned} &x''(t) + t^{-1}|x(t-1)|^{\alpha+1}\ \operatorname{sgn} x(t-1) \\ &+ \left(4\pi^2|\sin 2\pi t| - t^{-1}|\sin 2\pi t|^{\alpha+1}\right) \operatorname{sgn} x(t) = 0, \quad t \geq 1. \end{aligned} \tag{18}$$

Put $f(t) = \left(4\pi^2|\sin 2\pi t| - t^{-1}|\sin 2\pi t|^{\alpha+1}\right)$. Observe that $\phi(t) = t - 1$ and that $p(t) = 1$ satisfies (H). If we take $\rho(t) = t^2$ and $H(t-s) = (t-s)^2$ it follows that $V(t,s) = 2s - t$. Moreover $f(t) \geq (4\pi^2 - 1)|\sin 2\pi t|^{\alpha+1}$ is valid. Thus the integrand of (1) for the equation (18) is not less than

$$K(t-s)^2(s-1)s^{\alpha/(\alpha+1)}|\sin 2\pi s|^\alpha - (2s-t)^2,$$

where $K = (4\pi^2-1)^{\alpha/(\alpha+1)}\lambda C_\alpha$, C_α is the constant given by (10). Therefore by means of Lemma 2 and Theorem 3 the differential equation (18) is oscillatory. We note that $x(t) = \sin 2\pi t$ is an oscillatory solution.

Theorem 6. *Let the conditions (H), (A_1), and*

$$\begin{aligned} &g(t,u) = \operatorname{sgn} u \cdot f(t,u) \text{ such that for some constant } M > 0, \\ &f(t,u) = f_1(t)f_2(u),\ f_1(t) \geq 0, f_2(u) \geq M \text{ for all } t > a \text{ and } u \neq 0 \end{aligned} \tag{A_3}$$

be satisfied. Assume that there exists a positive function $\rho(t) \in C^1([a, \infty))$ such that for $\lambda \in (0,1)$

(19)
$$\limsup_{t\to\infty} \frac{1}{H(t,a)} \int_a^t \left[\lambda C_{\alpha,M} H(t,s)\rho(s)q(s)^{\frac{1}{\alpha+1}} \times \frac{\phi(s)}{s} f_1(s)^{\frac{\alpha}{\alpha+1}} - V(t,s)^2\right] ds = \infty$$

is valid where $V(t,s)$ is given by (2) and $C_{\alpha,M} = M^{\frac{\alpha}{\alpha+1}}\left\{\frac{(\alpha+1)^{\alpha+1}}{\alpha^\alpha}\right\}^{\frac{1}{\alpha+1}}$. Then the equation (P_1) is oscillatory.

Proof. We consider a Riccati transform (7). Then

$$q(t)\frac{x(\phi(t))^{\alpha+1}}{x(t)} + \frac{f(t,x(t))}{x(t)} \geq q(t)\left(\lambda\frac{\phi(t)}{t}\right)^{\alpha+1} x(t)^\alpha + \frac{Mf_1(t)}{x(t)}$$

is valid. The rest part of proof is same as that of Theorem 3. □

Example 2. Consider a delay differential equation

$$(20)\qquad \begin{aligned} x'' + \frac{1}{1+|\sin 2t|^{\alpha}}\big|x(t-\pi)\big|^{\alpha+1}\operatorname{sgn} x(t-\pi) \\ +|\sin 2t|\left(4-\frac{|x(t)|^{\alpha}}{1+|x(t)|^{\alpha}}\right)\operatorname{sgn} x(t) = 0, \end{aligned}$$

where $t \geq 2\pi$. We observe that

$$p(t)=1, \quad q(t)\geq 1/2, \quad \phi(t)=t-\pi, \quad f(t,u)=f_1(t)f_2(u),$$
$$f_1(t)=|\sin 2t|\geq 0, \quad f_2(u)=4-\frac{|u|^{\alpha}}{1+|u|^{\alpha}}\geq 3 = M.$$

Choose $\rho(t)=t^2$ and $H(t-s)=(t-s)^2$. By means of Theorem 6 we have

$$\begin{aligned} &\int_a^t \left[\lambda C_{\alpha,M} H(t,s)\rho(s)q(s)^{\frac{1}{\alpha+1}}\frac{\phi(s)}{s}f_1(s)^{\frac{\alpha}{\alpha+1}} - V(t,s)^2\right] ds \\ \geq &\int_a^t \left[C\,(t-s)^2 s(s-\pi)|\sin 2s|^{\frac{\alpha}{\alpha+1}} - (2s-t)^2\right] ds, \end{aligned}$$

where $\lambda \in (0,1)$ and $C = 2^{\frac{-1}{\alpha+1}}3^{\frac{\alpha}{\alpha+1}}\lambda\left[\frac{(\alpha+1)^{\alpha+1}}{\alpha^{\alpha}}\right]^{\frac{1}{\alpha+1}}$. Modifying the minor part of the proof of Lemma 2 we can easily show that the differential equation (20) is oscillatory. We note that $x(t)=\sin 2t$ is an oscillatory solution.

The following theorem is obvious from the proof of Theorem 6.

Theorem 7. *Let the conditions* (H), (A_1), *and*

$$(A_4)\qquad \begin{aligned} &g(t,u,v)=\operatorname{sgn} u\cdot f(t,u,v) \text{ such that for some positive constants } L,\ M \\ &f(t,u,v)\geq f_1(t)f_2(u)f_3(v),\ f_1(t)\geq 0,\ f_2(u)\geq L,\ f_3(v)\geq M \\ &\qquad \text{for all } t>a, u\neq 0 \text{ and for all } v \end{aligned}$$

be satisfied. Assume that there exists a positive function $\rho(t)\in C^1([a,\ \infty))$ *such that for* $\lambda\in(0,1)$

$$(21)\qquad \begin{aligned} \limsup_{t\to\infty}\frac{1}{H(t,a)}\int_a^t \Big[&\lambda C_{\alpha,L,M}H(t,s)\rho(s)q(s)^{\frac{1}{\alpha+1}} \\ &\times\frac{\phi(s)}{s}f_1(s)^{\frac{\alpha}{\alpha+1}} - V(t,s)^2\Big]\,ds = \infty \end{aligned}$$

is valid where $V(t,s)$ *is given by* (2) *and* $C_{\alpha,L,M}=\lambda(LM)^{\frac{\alpha}{\alpha+1}}\left\{\frac{(\alpha+1)^{\alpha+1}}{\alpha^{\alpha}}\right\}^{\frac{1}{\alpha+1}}$. *Then the equation* (P_2) *is oscillatory.*

Proof. We consider a Riccati transform (7). Then for some $\lambda \in (0,1)$

$$q(t)\frac{x(\phi(t))^{\alpha+1}}{x(t)} + \frac{f(t,x(t),x'(t))}{x(t)} \geq q(t)\left(\lambda\frac{\phi(t)}{t}\right)^{\alpha+1} x(t)^{\alpha} + \frac{LMf_1(t)}{x(t)}$$

is valid. The rest part of proof is same as that of Theorem 3. □

Example 3. Consider a delay differential equation

$$\begin{aligned} x'' &+ \left(\frac{1+2\sin^2\pi t}{2+\sin^2\pi t}\right)\left|x(t-2)\right|^{\alpha+1}\operatorname{sgn} x(t-2) \\ &+ \operatorname{sgn} x(t)\left(\pi^2 - \frac{1+2x(t)^2}{2+x(t)^2}|\sin\pi t|^{\alpha}\right)|\sin\pi t| \\ &\cdot\frac{20+\pi^2\cos^2\pi t}{10+\pi^2\cos^2\pi t}\cdot\frac{10+(x'(t))^2}{20+(x'(t))^2} \\ &= 0, \end{aligned} \tag{22}$$

where $t > b \geq 2$. We observe that $p(t) = 1$, $q(t) = \frac{1+2\sin^2\pi t}{2+\sin^2\pi t} \geq \frac{1}{2}$, $\phi(t) = t-2$ and $f_1(t) = |\sin\pi t|\,\frac{20+\pi^2\cos^2\pi t}{10+\pi^2\cos^2\pi t} \geq |\sin\pi t|$, $f_2(u) = \pi^2 - \frac{1+2u^2}{2+u^2}|\sin\pi t|^{\alpha} \geq \pi^2-2$, $f_3(v) = \frac{10+v^2}{20+v^2} \geq \frac{1}{2}$. Thus if we choose $\rho(t) = t^2$, $H(t,s) = (t-s)^2$ the integrand of (21) is not less than

$$K(t-s)^2 s(s-2)|\sin\pi s|^{\alpha/(\alpha+1)} - (2s-t)^2,$$

where $K = (\pi^2-2)^{\alpha/(\alpha+1)}C_{\alpha}/2$, C_{α} is the constant given by (10). By Lemma 2 and Theorem 3 the differential equation (22) is oscillatory. We note that $x(t) = \sin\pi t$ is an oscillatory solution.

Put $\phi(t) = t$. The differential equation (P_1) is then reduced to (P_3). By means of Theorem 3 we obtain the following:

Theorem 8. *Let the conditions in Theorem 3 be satisfied. Assume that there exists a positive function $\rho(t) \in C^1([a,\ \infty))$ such that for $\lambda \in (0,1)$*

$$\limsup_{t\to\infty}\frac{1}{H(t,a)}\int_a^t\left[C_{\alpha}\,H(t,s)\rho(s)q(s)^{\frac{1}{\alpha+1}}f(s)^{\frac{\alpha}{\alpha+1}} - V(t,s)^2\right]ds = \infty$$

is valid where $V(t,s)$ and C_{α} is given by (2) and (10), *respectively. Then the differential equation (P_3) is oscillatory.*

Corollary 9. *Let the conditions in Theorem 3 be satisfied. If there exists a positive function $\rho(t) \in C^1([a,\ \infty))$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,a)}\int_a^t H(t,s)\rho(s)q(s)^{\frac{1}{\alpha+1}}f(s)^{\frac{\alpha}{\alpha+1}}\,ds = \infty \tag{23}$$

and (17) *are valid, then the equation (P_3) is oscillatory.*

The next theorem is a comparison theorem with Theorem 3.

Theorem 10. *Let the conditions* (A_1), (A_2), *and* (H) *be satisfied. Consider an equation*

$$(P_1^*) \qquad \left[P(t)x'(t)\right]' + Q(t)|x(\phi(t))|^{\alpha+1}\operatorname{sgn} x(\phi(t)) + F(t)\operatorname{sgn} x(t) = 0$$

satisfying that for all $t > a$

$$p(t) \geq P(t) > 0, \quad q(t) \leq Q(t), \quad f(t) \leq F(t).$$

Assume that there exists a positive function $\rho(t) \in C^1([a,\ \infty))$ *such that* (1) *and* (2) *are valid. Then the equation* (P_1^*) *is oscillatory.*

Proof. By (H) we have $\int_a^\infty \frac{1}{P(u)}\, du = \infty$. From (1) and (2) it follows. □

Remark. In Theorem 10 let the conditions (A_1), (A_3), and (H) be satisfied. we reach then the same conclusion provided $f(t)$, $F(t)$ are replaced with $f(t,u)$, $F(t,u)$, respectively, such that $F(t,u) = F_1(t)F_2(u)$, $F_1(t) \geq 0$, $F_2(u) \geq L$ for some $L > 0$ and $f(t,u) \leq F(t,u)$ for all t and u.

Example 4. Consider a differential equation

$$(24) \qquad \left[t^{1/3}\, x'(t)\right]' + t^{-2}|x(t)|^{\alpha+1}\, \operatorname{sgn} x(t) + te^{t^2}\, \operatorname{sgn} x(t) = 0, \quad t \geq 1.$$

Put $f(t) = te^{t^2}$. Observe that $p(t) = t^{1/3}$, $q(t) = t^{-2}$ and $f(t) \geq 1$ for $t \geq 1$. It is obvious that $p(t)$ satisfies (H). On the other hand, we have

$$\int_a^\infty q(s)\varrho(s)\, ds = \frac{3}{2}\int_a^\infty \left[s^{-4/3} - a^{2/3}\, s^{-2}\right] ds < \infty.$$

Therefore the differential equation $\left[t^{1/3}\, x'(t)\right]' + t^{-2}|x(t)|^{\alpha+1}\, \operatorname{sgn} x(t) = 0$ is nonoscillatory. Choose $H(t,s) = (t-s)^2$ and $\rho(s) = s^\delta$ so that $\delta \geq \frac{1-\alpha}{\alpha+1}$. Here the value of δ is determined later. Since then

$$\int_a^\infty \rho(s)q(s)^{\frac{1}{\alpha+1}}\, ds = \infty$$

it follows by Lemma [15] that

$$\limsup_{t\to\infty} \frac{1}{(t-a)^2}\int_a^t (t-s)^2\rho(s)q(s)^{\frac{1}{\alpha+1}}\, ds = \infty$$

and so (23) is valid. Now we calculate the integrand of (17). We note that the integrand of (17) is both positive and equal to

$$\left\{(2+\delta)^2\, s^2 - 2\delta(2+\delta)ts + \delta^2 t^2\right\} s^{\delta-5/3}.$$

Thus the left side of (17) equals

$$(25) \qquad \lim_{t\to\infty}\left[\left\{\frac{(2+\delta)^2}{\delta+4/3} - \frac{2\delta(2+\delta)}{\delta+1/3} + \frac{\delta^2}{\delta-2/3}\right\}\frac{t^{\delta+4/3}}{(t-a)^2} - \left\{\frac{(2+\delta)^2 a^{\delta+4/3}}{\delta+4/3} - \frac{2\delta(2+\delta)a^{\delta+1/3}}{\delta+1/3}t + \frac{\delta^2 a^{\delta-2/3}}{\delta-2/3}t^2\right\}\frac{1}{(t-a)^2}\right].$$

According to α, if we take δ with $\frac{1-\alpha}{\alpha+1} \le \delta \le 2/3$, for example $\delta = 1/3$ when $\alpha = 2$, (25) is finite. The differential equation (24) is therefore oscillatory. So the oscillatory property of (25) under (H) is determined by a suitable choice of $\rho(t)$.

The following is an extension of Theorem 6. Consider the differential equation (P_1) and assume that the function $f(t,u)$ satisfies

$$(A_3') \qquad \begin{aligned} &g(t,u) = \operatorname{sgn} u \cdot f(t,u) \text{ such that for some positive constant } M, \\ &f(t,u) = f_1(t) f_2(u), \ f_1(t) \ge 0, \text{ and } f_2(u) \ge M|u|^\beta \text{ with } \beta < 1. \end{aligned}$$

Theorem 11. *Let the conditions* (H), (A_1), *and* (A_3') *be satisfied. Assume that there exists a positive function* $\rho(t) \in C^1([a,\ \infty))$ *such that for* $\lambda \in (0,1)$

$$(26) \qquad \limsup_{t\to\infty} \frac{1}{H(t,a)} \int_a^t \Bigg[C_{\alpha,\beta,\lambda} H(t,s)\rho(s) q(s)^{\frac{1-\beta}{\alpha-\beta+1}} \times \left(\frac{\phi(s)}{s}\right)^{\frac{(\alpha+1)(1-\beta)}{\alpha-\beta+1}} f_1(s)^{\frac{\alpha}{\alpha-\beta+1}} - V(t,s)^2 \Bigg] ds = \infty$$

is valid where $C_{\alpha,\beta,\lambda} = \frac{\alpha-\beta+1}{1-\beta}\left(\frac{1-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta+1}} M^{\frac{\alpha}{\alpha-\beta+1}} \lambda^{\frac{(\alpha+1)(1-\beta)}{\alpha-\beta+1}}$, *and* $V(t,s)$ *is given by* (2). *Then the equation* (P_1) *is oscillatory.*

Proof. We may assume that $x(t) > 0$ for $t \ge T_0$ for some T_0. Now we consider a Riccati transform given by (7). Using Lemma 1 with

$$X = \left[q(t)\left(\frac{\lambda\phi(t)}{t}\right)^{\alpha+1} x(t)^\alpha\right]^{\frac{1-\beta}{\alpha-\beta+1}}, \quad Y = \left[\frac{(1-\beta)x(t)^{\beta-1} f(t,x(t))}{\alpha}\right]^{\frac{\alpha}{\alpha-\beta+1}}$$

and taking (A_3') into account we can derive then the inequality

$$\begin{aligned} &q(t)\left(\lambda\frac{\phi(t)}{t}\right)^{\alpha+1} x(t)^\alpha + x(t)^{\beta-1} f(t,x(t)) \\ &\ge C_{\alpha,\beta,\lambda} q(t)^{\frac{1-\beta}{\alpha-\beta+1}} \left(\frac{\phi(t)}{t}\right)^{\frac{(\alpha+1)(1-\beta)}{\alpha-\beta+1}} f_1(t)^{\frac{\alpha}{\alpha-\beta+1}}. \end{aligned}$$

The rest part of proof is same as that of Theorem 3. □

References

[1] B. Ayanlar and A. Tiryaki, *Oscillation theorems for nonlinear second order differential equations with damping*, Acta Math. Hungar. **89** (2000), no. 1-2, 1–13.

[2] J. W. Baker, *Oscillation theorems for a second order damped nonlinear differential equation*, SIAM J. Appl. Math. **25** (1973), 37–40.

[3] L. E. Bobisud, *Oscillation of solutions of damped nonlinear equations*, SIAM J. Appl. Math. **19** (1970), 601–606.

[4] M. Cecchi, M. Marini, and G. Villari, *Comparison results for oscillation of nonlinear differential equations*, NoDEA Nonlinear Differential Equations Appl. **6** (1999), no. 2, 173–190.

[5] A. Elbert and T. Kusano, *Oscillation and nonoscillation theorems for a class of second order quasilinear differential equations*, Acta Math. Hungar. **56** (1990), no. 3-4, 325–336.
[6] H. E. Gollwitzer, *Nonoscillation theorems for a nonlinear differential equation*, Proc. Amer. Math. Soc. **26** (1970), 78–84.
[7] S. R. Grace, *Oscillation theorems for second order nonlinear differential equations with damping*, Math. Nachr. **141** (1989), 117–127.
[8] G. H. Hardy, J. E. Litlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1988.
[9] M. K. Kong and J. S. W. Wong, *Nonoscillation theorems for a second order sublinear ordinary differential equation*, Proc. Amer. Math. Soc. **87** (1983), no. 3, 467–474.
[10] A. H. Nasr, *Sufficient conditions for the oscillation of forced super-linear second order differential equations with oscillatory potential*, Proc. Amer. Math. Soc. **126** (1998), no. 1, 123–125.
[11] Ch. G. Philos, *Oscillation theorems for linear differential equations of second order*, Arch. Math. (Basel) **53** (1989), no. 5, 482-492.
[12] Y. G. Sun, *A Note on Nasr and Wong papers*, J. Math. Anal. Appl. **286** (2003), no. 1, 363–367.
[13] ———, *New Kamenev-type oscillation criteria for second-order nonlinear differential equations with damping*, J. Math. Anal. Appl. **291** (2004), no. 1, 341–351.
[14] J. S. W. Wong, *Oscillation criteria for second order nonlinear differential equations involving general means*, J. Math. Anal. Appl. **247** (2000), no. 2, 489–505.
[15] ———, *On Kamenev-type oscillation theorems for second-order differential equations with damping*, J. Math. Anal. Appl. **258** (2001), no. 1, 244–257.
[16] C. C. Yeh, *Oscillation theorems for nonlinear second order differential equations with damped term*, Proc. Amer. Math. Soc. **84** (1982), no. 3, 397–402.

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