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Standard completeness results for some neighbors of R-mingle*†

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【Abstract】In this paper we deal with new standard completeness proofs of some systems introduced by Metcalfe and Montagna in [10]. For this, this paper investigates several fuzzy-relevance logics, which can be regarded as neighbors of the **R** of Relevance with mingle (**RM**). First, the monoidal uninorm idempotence logic **MUIL**, which is intended to cope with the tautologies of left-continuous conjunctive idempotent uninorms and their residua, and some schematic extensions of it are introduced as neighbors of **RM**. The algebraic structures corresponding to them are defined, and standard completeness, completeness on the real unit interval [0, 1], results for them are provided.

【Key words】standard completeness, fuzzy-relevance logic, **RM**.

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1. Introduction

Hájek [5] introduced **BL** (the basic fuzzy logic), and showed that the well-known infinite-valued systems **Ł** (Łukasiewicz logic), **G** (Gödel-Dummett logic), and Π (Product logic) are its extensions. Esteva and Godo [2] introduced the monoidal t-norm logic **MTL** as a weakening of **BL** and a strengthening of Affine multiplicative additive intuitionistic linear logic **AMAILL** introduced by Höhle [7]. Metcalfe and Montagna [10] recently introduced the uninorm logic **UL** (calling this here Monoidal uninorm logic **MUL**) as a weakening of **MTL** and a strengthening of Multiplicative additive intuitionistic linear logic **MAILL**.

BL copes with the logic of *continuous* t-norms and their residua. The important result of **BL** is the (so called *standard*) completeness of **BL** with respect to (w.r.t.) the real unit interval structures defined by continuous t-norms. Initial related results on **BL** were obtained by Hajek, but the completeness of BL is proved by Cignoli et al. in [1], i.e., they finally proved that the theorems of **BL** are the formulas capturing tautologies for each continuous t-norm and its residuum. Esteva and Godo [2] conjectured that **MTL** would be standard complete w.r.t. evaluations on algebras on [0, 1] equipped by a *left-continuous* t-norm and its residuum. The conjecture was positively solved by Jenei and Montagna in [9]. **MUL** captures the tautologies of *left-continuous conjunctive* uninorms and their residua. Metcalfe and Montagna [10] gave standard completeness results for **MUL** and several schematic extensions of it, in particular, the uninorm mingle logic **UML** (calling this here Monoidal uninorm idempotence logic **MUIL**) and the involutive uninorm mingle logic **IUML**, which is the **RM** with **f** $\leftrightarrow$ **t** (calling this here Fixed-pointed **R**-mingle **FRM**).

The proofs of standard completeness of **MUL** and several extensions of it given by Metcalfe and Montagna in [10] are complicate in the sense that in each completeness proof they first add density rule to each system and then eliminate it by giving synthetic proof of it. But Jenei and Montagna-style proofs in [9] are simpler than theirs in the sense that these proofs do not require such a rule. Using this approach we provide standard completeness results for some neighbors of **RM**, more exactly, **MUIL**, min-maxed MUIL **MUMML**, and **FRM**.

For brevity, by *L (L-algebra resp)* we shall ambiguously express the systems ((corresponding) algebras resp) defined in section 2 (3 resp) all together, if we do not need distinguish them, but context should determine which system (algebra resp) is intended; and by **L**-algebra (i.e., boldface L-algebra), we mean L-algebra satisfying soundness. Also, for convenience, we shall adopt the notation and terminology similar to those in [1, 2, 3, 5], and assume being familiar with them (together with results found in them).

2. Syntax

Logical systems we shall define in this section are based on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **F**, **f**, **t**. Further definable connectives are:

df1. $\sim\phi := \phi \rightarrow \mathbf{f}$,

df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We moreover define **T** as $\sim\mathbf{F}$, and $\phi_{\mathbf{t}}$ as $\phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiom schemes and rules for Monoidal uninorm idempotence logic **MUIL**.

Definition 2.1 MUIL consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)

A2. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ ($\wedge$-elimination, $\wedge$-E)

A3. $((\phi\rightarrow\psi)\wedge(\phi\rightarrow\chi)) \rightarrow (\phi\rightarrow(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\phi \rightarrow (\phi \vee \psi)$, $\psi \rightarrow (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\phi\rightarrow\chi)\wedge(\psi\rightarrow\chi)) \rightarrow ((\phi\vee\psi)\rightarrow\chi)$ ($\vee$-elimination, $\vee$-E)

A6. $(\phi\wedge(\psi\vee\chi))\rightarrow((\phi\wedge\psi)\vee(\phi\wedge\chi))$ ($\wedge\vee$-distributivity, $\wedge\vee$-D)

A7. $\mathbf{F} \rightarrow \phi$ (ex falso quadlibet, EF)

A8. $(\phi \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\phi \,\&\, \psi) \,\&\, \chi)$ (&-associativity, AS)

A9. $(\phi \,\&\, \psi) \rightarrow (\psi \,\&\, \phi)$ (&-commutativity, &-C)

A10. $(\phi \,\&\, \mathbf{t}) \leftrightarrow \phi$ (push and pop, PP)

A11. $(\psi \to \chi) \to ((\phi \to \psi) \to (\phi \to \chi))$ (prefixing, PF)
A12. $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \,\&\, \psi) \to \chi)$ (residuation, RE)
A13. $(\phi \,\&\, \phi) \leftrightarrow \phi$ (idempotence, ID)
A14. $(\phi \to \psi)_{\mathbf{t}} \vee (\psi \to \phi)_{\mathbf{t}}$ (**t**-prelinearity, $\mathrm{PL}_{\mathbf{t}}$).
$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)
$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj)

Definition 2.2 A logic is a schematic extension of an arbitrary logic **L** if and only if (iff) it results from **L** by adding (finitely or infinitely many) axioms. In particular:

- Min-maxed MUIL **MUMML** is **MUIL** plus
(MM) $((\phi \,\&\, \psi) \to (\phi \wedge \psi)) \vee ((\phi \vee \psi) \to (\phi \,\&\, \psi))$.
- Fixed-pointed RM **FRM** is **MUIL** plus
(DNE) $\sim\sim\phi \to \phi$, and
(FP) $\mathbf{f} \leftrightarrow \mathbf{t}$.

For easy reference we let Ls be a set of logical systems defined previously.

Definition 2.3 Ls = {**MUIL**, **MUMML**, **FRM**}.

In L ($\in$ Ls), **f** can be defined as $\sim\mathbf{t}$ and vice versa. In **FRM**, $\wedge$ is defined using $\sim$ and $\vee$; & defined as (df3) $\phi \,\&\, \psi := \sim(\phi \to \sim\psi)$; and $\to$ instead as (df4) $\phi \to \psi := \sim(\phi \,\&\, \sim\psi)$.

A *theory* over L is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using the rules (mp) and (adj). T $\vdash \phi$, more exactly T $\vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. The relevant deduction theorem (RDT) for L is as follows:

Proposition 2.4 Let T be a theory, and ϕ, ψ formulas. T $\cup$ $\{\phi\}$ $\vdash_L \psi$ iff T $\vdash_L \phi_t \to \psi$.

Proof: It is just Enthymematic Deduction Theorem (see [8]). □

A theory T is *inconsistent* if T $\vdash$ **F**; otherwise it is *consistent*.

For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\to$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

3. Semantics

Suitable algebraic structures for Ls are obtained as varieties of idempotent commutative monoidal residuated lattices.

Definition 3.1 An *idempotent commutative monoidal residuated lattice* (icmr-lattice) is a structure **A** = (A, $\top$, $\bot$, $\top_t$, $\bot_f$, $\wedge$, $\vee$, $*$, $\to$) such that:

(I) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded distributive lattice with top element $\top$ and bottom element $\bot$.

(II) (A, $*$, $\top_t$) satisfies for all x, y, z $\in$ A,

(a) $x * y = y * x$ (commutativity)

(b) $\top_t * x = x$ (identity)

(c) $x \leq y$ implies $x * z \leq y * z$ (isotonicity)

(d) $x * (y * z) = (x * y) * z$ (associativity)

(e) $x * x = x$ (idempotence)

(III) $y \leq x \rightarrow z$ iff $x * y \leq z$, for all x,y,z $\in$ A (residuation)

(A, $*$, $\top_t$) satisfying (II-b, d) is a *monoid*. Thus (A, $*$, $\top_t$) satisfying (II-a, b, d) is a commutative monoid, and (A, $*$, $\top_t$) satisfying (II-a, b, d, e) an idempotent commutative monoid. (A, $*$, $\top_t$) satisfying (II-a, b, c, d) on [0, 1] is a *uninorm* and it is a *t-norm* in case $\top_t = \top$.

To define an icmr-lattice we may take in place of (II-c)

(IV) $x * (y \vee z) = (x * y) \vee (x * z)$.

Using $\rightarrow$ and $\perp_f$ we can define $\top_t$ as $\perp_f \rightarrow \perp_f$, and $\sim$ as in (df1). In an icmr-lattice, $\sim$ is a *weak* negation in the sense that for all x, $x \leq \sim\sim x$ holds in it. Then, an L-algebra corresponding to L is defined as follows.

Definition 3.2 (i) (MUIL-algebra) An *MUIL-algebra* is an icmr-lattice satisfying the condition: for all x, y,

(pl_t) $\top_t \leq (x \rightarrow y)_t \vee (y \rightarrow x)_t$.

(ii) (MUMML-algebra) An *MUMML-algebra* is an MUIL-algebra satisfying the condition: for all x, y, (mm) $\top_t \leq ((x * y) \rightarrow (x \wedge y)) \vee ((x \vee y) \rightarrow (x * y))$.

(iii) (FRM-algebra) A *FRM-algebra* is an MUIL-algebra satisfying the conditions: for all x,

(Inv) $x = \sim\sim x$, and

(FP) $\bot_f \leftrightarrow \top_t$.

For L ($\in$ Ls), L-algebra (defined in 3.2) is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 3.3 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function $v : FOR \rightarrow \mathcal{A}$ satisfying:

$v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$,
$v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$,
$v(\phi \vee \psi) = v(\phi) \vee v(\psi)$,
$v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$,
$v(\mathbf{F}) = \bot$,
$v(\mathbf{f}) = \bot_f$,

(and hence $v(\sim\phi) = \sim v(\phi)$, $v(\mathbf{T}) = \top$, and $v(\mathbf{t}) = \top_t$,).

Definition 3.4 Let $\mathcal{A}$ be an L-algebra, T a theory, ϕ a formula, and K a class of L-algebras.

(i) (Tautology) ϕ is a *$\top_t$-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq \top_t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq \top_t$ for each $\phi \in T$. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_{\mathsf{K}} \phi$, if Mod(T, $\mathcal{A}$) = Mod(T $\cup$ $\{\phi\}$, $\mathcal{A}$) for each $\mathcal{A} \in \mathsf{K}$.

In the next definition, we shall use the notational convention mentioned in the last paragraph of section 1.

Definition 3.5 (**L**-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is an ***L**-algebra* iff whenever ϕ is L-provable in T (i.e. $T \vdash_L \phi$), L an L, it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. $T \vDash_{\{A\}} \phi$), $\mathcal{A}$ a corresponding L-algebra. By *$MOD^{(l)}(L)$*, we denote the class of (linearly ordered) **L**-algebras. We write $T \vDash^{(l)}_L \phi$ in place of $T \vDash_{MOD^{(l)}_L} \phi$.

In [11, 12], Yang proved the following:

Proposition 3.6 MUIL (**MUMML** resp) is sound and complete w.r.t. the class of linearly ordered **MUIL**-algebras (**MUMML**-algebras resp).

4. L-uninorms and their residua

In this section, using *1, 0, and some 1_t, and 0_f* in the real unit interval [0, 1], we shall express $\top$, $\bot$, $\top_t$, and $\bot_f$, respectively. We also define standard L-algebras and L-uninorms on [0, 1].

Definition 4.1 An L-algebra is *standard* iff its lattice reduct is [0, 1].

Definition 4.2 (uninorm) A *uninorm* is a binary operation * on [0, 1] satisfying (II) (a) - (d) in Definition 3.1 for some $1_t \in [0, 1]$.

A uninorm satisfying unit element $1_t = 1$ is a *t-norm*. A uninorm is *decreasing* in case it satisfies (decreasing) $x * x \leq x$, for all $x \in [0, 1]$; *increasing* in case it satisfies (increasing) $x \leq x * x$, for all $x \in [0, 1]$; and *idempotent* in case it is both decreasing and increasing. Decreasing, increasing, and idempotent t-norms are defined analogously. Furthermore, an idempotent uninorm is *min-maxed* in case it satisfies that (mm) for all x, y $\in [0, 1]$, $x * y = \min\{x, y\}$ or $x * y = \max\{x, y\}$.

* is *residuated* iff there is $\rightarrow : [0, 1]^2 \rightarrow [0, 1]$ satisfying (residuation) on [0, 1]. A uninorm is called *conjunctive* if $0 * 1 = 0$, and *disjunctive* if $0 * 1 = 1$. For some $0_f \in [0, 1]$, a residuated uninorm has weak negation n defined as $n(x) := x \rightarrow 0_f$ because $x * (x \rightarrow 0_f) \leq 0_f$ holds in it and so by residuation $x * (x \rightarrow 0_f) \leq 0_f$ iff $x \leq (x \rightarrow 0_f) \rightarrow 0_f$. In case $n(n(x)) = x$, it has strong negation.

The most important property of a uninorm is that *left-continuity* holds in it. Given a uninorm *, *R-implication* $\rightarrow$ determined by * is defined as $x \to y := \sup\{z \in [0, 1]: x * z \leq y\}$ for all x, $y \in [0, 1]$.

Proposition 4.3 For any uninorm *, * and its R-implication → form a residuated pair iff * is conjunctive and left-continuous (in both arguments).

Proof: By the same proof as Proposition 5.4.2 in [4]. □

5. Standard completeness

Since we shall consider weak standard completeness of Ls, by an ***L****-algebra*, from now on we mean an algebra in which all the axioms of L are valid. We first show that finite or countable linearly ordered **MUIL**-algebras are embeddable into a standard algebra.

Proposition 5.1 For every finite or countable linearly ordered **MUIL**-algebra **A** = $(A, \leq_A, \top, \bot, \top_t, \bot_f, \wedge, \vee, *, \to)$, there is a countable ordered set X, a binary operation ○, and a map f from A into X such that the following conditions hold:

(I) X is densely ordered, and has a maximum Max, a minimum Min, and special elements e, ∂.

(II) $(X, \circ, \leq, e)$ is a linearly ordered isotonic commutative monoid, i.e., it satisfies for all $x, y, z \in X$,

(1) $x \circ y = y \circ x$ (commutativity)

(2) $e \bigcirc x = x$ (identity)

(3) $x \leq y$ implies $x \bigcirc z \leq y \bigcirc z$ (isotonicity)

(4) $x \bigcirc (y \bigcirc z) = (x \bigcirc y) \bigcirc z$ (associativity)

(III) $\bigcirc$ is conjunctive, idempotent, and left-continuous with respect to the order topology on $(X, \leq)$.

(IV) f is an embedding of the structure $(A, \leq_A, \top, \bot, \top_t, \bot_f, \wedge, \vee, *)$ into $(X, \leq, Max, Min, e, \partial, min, max, \bigcirc)$, and for all $m, n \in A$, $f(m \to n)$ is the residuum of $f(m)$ and $f(n)$ in $(X, \leq, Max, Min, e, \partial, max, min, \bigcirc)$.

Proof: For each $m \in A$, let m^+ denote the successor of m if it exists, otherwise take $m^+ = m$; and analogously, let m^- denote the predecessor of m if it exists, otherwise take $m^- = m$. Let

$X = \{(m, x)$: $m \in A$, $\exists m', m''$ such that $m'' >_A m''^- =_A m =_A m'^+ >_A m'$, and $x \in \mathbf{Q} \cap (0, 1)\} \setminus \{(\top_t, x): x \in \mathbf{Q} \cap (0, 1)\} \cup \{(\top, x): x \in \mathbf{Q} \cap (0, 1)\} \cup \{(\bot, x): x \in \mathbf{Q} \cap (0, 1)\} \cup \{(\top_t, 0)\} \cup \{(m, 1): \top_t <_A m \in A\} \cup \{(m, 0): \top_t >_A m \in A\}$.

For $(m, x), (n, y) \in X$, we define:

$$(m, x) \leq (n, y) \text{ iff either } m <_A n, \text{ or } m =_A n \text{ and } x \leq y.$$

It is clear that $\leq$ is a linear order with maximum $(\top, 1)$, minimum $(\bot, 0)$, and special elements $e = (\top_t, 0)$, $\partial = (\bot_f, 0)$ (noting that (decreasing) ensures $\bot_f \leq \top_t$ $(= \bot_f \to \bot_f)$ and so

$\partial \leq e$). Furthermore, $\leq$ is dense: let $(m, x) \prec (n, y)$. Then either $m <_A n$ or $m =_A n$ and $x < y$. If the first is the case (assuming that it is not the case that $m^+ = n$ and $y = 0$), then $(m, x) \prec (n, y/2) \prec (n, y)$. (In case $m^+ = n$ and $y = 0$, $(m, x) \prec (m, x+1/2) \prec (n, y)$.) If the second is the case, then $(m, x) \prec (n, x+y/2) \prec (n, y)$. This proves (I).

For convenience, we will from now on drop the index A of $\leq_A$ and $=_A$ if we do not need distinguish them. But context should make clear what we mean.

Define for $(m, x), (n, y) \in X$:

$(m,x) \circ (n,y) = \max\{(m,x), (n,y)\}$ if $m * n = m \vee n$, and $(m, x) \geq e$ or $(n, y) \geq e$;

$\min\{(m,x), (n,y)\}$ if $m * n = m \wedge z$, and $(m, x) \leq e$ or $(n, y) \leq e$;

$(m * n, 1)$ if $m * n \neq m \vee z$, $m \wedge z$, and $m * n > \top_t$;

$(m * n, 0)$ otherwise.

We verify that $\circ$ satisfies (II).

(1) It is obvious that $\circ$ is commutative.

(2) We prove that $(\top_t, 0)$ is the unit element, i.e., $(\top_t, 0) \circ (m, x) = (m, x)$. (i) Let $(\top_t, 0) \leq (m, x)$. Since $\top_t * m = m = \top_t \vee m$, $(\top_t, 0) \circ (m, x) = \max\{(\top_t, 0), (m, x)\} = (m, x)$. (ii) Let $(m, x) \prec (\top_t, 0)$. Since $\top_t * m = m = \top_t \wedge m$, $(\top_t, 0) \circ (m, x) = \min\{(\top_t, 0), (m, x)\} = (m, x)$.

(3) Since $\circ$ is commutative, it suffices to prove that if (l, x)

$\leq$ (m, y), then for all (n, z) $\in$ X, (l, x) $\circ$ (n, z) $\leq$ (m, y) $\circ$ (n, z). We distinguish several cases:

● Case (i). $l * n = l \vee n$ and $m * n = m \vee n$:

• Subcase (i-a). $(l, x) \geq e$ or $(n, z) \geq e$.

(a-1) $(m, y) \geq e$ or $(n, z) \geq e$. $(l, x) \circ (n, z) = \max\{(l, x), (n, z)\} \leq \max\{(m, y), (n, z)\} = (m, y) \circ (n, z)$.

(a-2) $(m, y), (n, z) \prec e$. It is not the case because $(m, y), (n, z) \prec e$ implies $m = n < \top_t$ and so $l \geq \top_t$, contrary to the supposition that $(l, x) \leq (m, y)$.

• Subcase (i-b). $(l, x), (n, z) \prec e$.

(b-1) $(m, y) \geq e$ or $(n, z) \geq e$. Then $l = n < \top_t$ and $(l, x) \circ (n, z) = \min\{(l, x), (n, z)\} \prec \max\{(m, y), (n, z)\} = (m, y) \circ (n, z)$.

(b-2) $(m, y), (n, z) \prec e$. This implies that $l = n = m < \top_t$. Thus $(l, x) \circ (n, z) = \min\{(l, x), (n, z)\} = \min\{(m, y), (n, z)\} = (m, y) \circ (n, z)$.

● Case (ii). $l * n = l \wedge n$ and $m * n = m \wedge n$. Its proof is analogous to that of Case (i).

● Case (iii). $l * n = l \vee n$ and $m * n \neq m \vee n$. We need to consider the subcases (a) $m * n = m \wedge n$, $m \neq n$, and (b) $m * n \neq m \wedge n$, $m \neq n$.

• Subcase (iii-a). $m * n = m \wedge n$, $m \neq n$. Since $m * n = m \wedge n$ and $m \neq n$, $l = n < m, \top_t$. Thus $(l, x) \circ (n, z) = \min\{(l, x), (n, z)\} \leq \min\{(m, y), (n, z)\} = (m, y) \circ (n, z)$.

- Subcase (iii-b). $m * n \neq m \wedge n$, $m \neq n$:

(b-1) $m * n > \top_t$. In case $l * n \geq \top_t$, $l * n < m * n$ and so $\max\{(l, x), (n, z)\} \prec (m * n, 1)$. In case $l * n < \top_t$, $l * n < \top_t < m * n$ and so $\min\{(l, x), (n, z)\} \prec (m * n, 1)$. Thus, this holds.

(b-2) $m * n \leq \top_t$. Since this implies that $l = n = l * n < m * n \leq \top_t$, this holds.

● Case (iv). $l * n \neq l \vee n$ and $m * n = m \vee n$. Its proof is analogous to that of Case (iii).

● Case (v). $l * n \neq l \vee n, l \wedge n$, and $m * n \neq m \vee n, m \wedge n$.

- Subcase (v-a). $l * n, m * n > \top_t$. $(l, x) \circ (n, z) = (l * n, 1) \preceq (m * n, 1) = (m, y) \circ (n, z)$.
- Subcase (v-b). $l * n \leq \top_t < m * n$. Since $l * n < m * n$, this holds.
- Subcase (v-c). $l * n > \top_t \geq m * n$. By the supposition, this is not the case.
- Subcase (v-d). Otherwise, i.e., $l * n, m * n \leq \top_t$. $(l, x) \circ (n, z) = (l * n, 0) \preceq (m * n, 0) = (m, y) \circ (n, z)$.

(4) For this, we need to prove that for all $(l, x), (m, y), (n, z) \in X$,

(AS) $(l, x) \circ ((m, y) \circ (n, z)) = ((l, x) \circ (m, y)) \circ (n, z)$.

We distinguish several cases:

● Case (i). $l * (m * n) = \vee(l, m, n)$. In case $l \geq \top_t$ or $m \geq \top_t$ or $n \geq \top_t$, both sides of AS are equal to $\max\{(l, x), (m, y), (n, z)\}$. Otherwise, i.e., in case $l = m = n < \top_t$, both sides of AS are equal to $\min\{(l, x), (m, y), (n, z)\}$.

● Case (ii). $l * (m * n) = \wedge(l, m, n)$. In case $l = m = n \geq \top_t$, both sides of AS are equal to $\max\{(l, x), (m, y), (n, z)\}$. In case $l < \top_t$ or $m < \top_t$ or $n < \top_t$, both sides of AS are equal to $\min\{(l, x), (m, y), (n, z)\}$.

● Case (iii). $l * (m * n) \neq \vee(l, m, n), \wedge(l, m, n)$, and $l * (m * n) \in \{l, m, n\}$.

• Subcase (iii-a). $l * (m * n) = l \vee (m * n)$. Since $l \vee (m * n) = l \vee (m \vee n)$ and $l \vee (m * n) = l \vee (m \wedge n) = m \wedge n$ ($= m = n$) are not the cases, $l \vee (m * n)$ is l. Then, it is not the case that $m * n = m \wedge n$ and $m \neq n$ because $*$ is isotone. Thus we consider the case $m * n \neq m \wedge n$, $m \neq n$. Let $m * n > \top_t$. Without loss of generality, we may assume that $m < \top_t < l < n$. Since $l * (m * n) = l$, $m * n \leq l$ by the supposition. But it is not the case that $m * n < l$ (otherwise, $l * n = n$ by $*$-decreasing and so $(l * n) * m = n * m < l$, contradicting $l * (m * n) = (l * m) * n = (l * n) * m$ by the associativity and commutativity of $*$). Thus, the left-hand side of AS is $\max\{(l, x), (m * n, 1)\} = (l\ (= m * n), 1)$. Then, since l

$*$ $(m * n) = (l * m) * n$, $l * m < \top_t$ and so the right-hand side of AS is equal to $(l, 1)$ as well. Let $m * n \leq \top_t$. Without loss of generality, we may assume that $m < l < \top_t < n$. Then, in an analogy to the above, $m * n = l$. Thus, the left-hand side of AS is $\min\{(l, x), (m * n, 0)\} = (l\ (= m * n), 0)$. Since $l * m = m$ and $(l * m) * n = l$, the right-hand side of AS is equal to $(l, 0)$ as well. Thus this holds.

- Subcase (iii-b). $l * (m * n) = l \wedge (m * n)$. Its proof is analogous to that of Subcase (iii-a).

- Subcase (iii-c). $l * (m * n) \neq l \vee (m * n), l \wedge (m * n)$.

(c-1). $m * n = m \vee n$, $m \neq n$. Without loss of generality, we may assume that $l < \top_t < m < n$ or $l < m < \top_t < n$. Let the first be the case. Since $(m, x) \circ (n, z) = (m * n\ (= n), z)$ and so $l * (m * n) = m$, the left-hand side of AS is equal to $(m, 1)$. $l * m > \top_t$ is not the case (otherwise, $(l * m) * n = n$ by $*$-decreasing and so the right-hand side of AS is (n, z)). Let $l * m \leq \top_t$. Since $m = l * (m * n) = (l * m) * n$, the right-hand side of AS is equal to $(m, 1)$ as well. Proof of the second is analogous to that of the first. Thus this holds.

(c-2). $m * n = m \wedge n$, $m \neq n$. Analogously to (c-1).

(c-3). $m * n \neq m \vee n, m \wedge n$, $m \neq n$. Analogously to (c-1).

● Case (iv). $l * (m * n) \notin \{l, m, n\}$ and either $l * (m * n)$

$= l \vee (m * n) = m * n$ or $l * (m * n) = l \wedge (m * n) = m * n$.

• Subcase (iv-a). $l * (m * n) = l \vee (m * n) = m * n$. Then $m * n \leq \top_t$ is not the case. Let $m * n > \top_t$ and $m \neq n$. Without loss of generality, we may assume that $l < m < \top_t < n$. Then, since $(m, y) \circ (n, z) = (m * n, 1)$, the left-hand side of AS is equal to $(m * n, 1)$. Since $l * m = l$ and $m * n = l * (m * n) = (l * m) * n$, the right-hand side of AS is equal to $(m * n, 1)$ as well. Thus this holds.

• Subcase (iv-b). $l * (m * n) = l \wedge (m * n) = m * n$. It suffices to consider the case that $m * n \leq \top_t$ and $m \neq n$. It follows from the supposition that the left-hand side of AS is equal to $(m * n, 0)$. Since $(l * m) * n = l * (m * n) = m * n$, the right-hand side of AS is equal to $(m * n, 0)$ as well. Thus this holds.

● Case (v). $l * (m * n) \not\in \{l, m, n\}$ and $l * (m * n) \neq l \vee (m * n), l \wedge (m * n)$. In case $l * (m * n) > \top_t$, both sides of AS are equal to $(l * (m * n), 1)$. In case $l * (m * n) \leq \top_t$, both sides of AS are equal to $(l * (m * n), 0)$.

We then prove (III). Since $\bot * \top = \bot$, it is immediate that $\circ$ is conjunctive, i.e., $(\bot, 0) \circ (\top, 1) = (\bot, 0)$. It is further idempotent, i.e., $(m, x) \circ (m, x) = (m, x)$ because $m * m = m$.
For left-continuity of $\circ$, we prove that if $<(m_i, x_i): i \in \mathbf{N}>$ is

any increasing sequence (w.r.t. $\leq$) of elements of X such that $\sup\{(m_i, x_i): i \in \mathbf{N}\} = (m, x)$, then for all $(n, y) \in X$, $\sup\{(m_i, x_i) \circ (n, y): i \in \mathbf{N}\} = (m, x) \circ (n, y)$. Note that for almost all i, $m_i = m$ (otherwise $(m, x/2) \prec (m, x)$ would be an upper bound of the sequence $<(m_i, x_i): i \in \mathbf{N}>$). By deleting a finite number of elements of the sequence $<(m_i, x_i): i \in \mathbf{N}>$, we can suppose that for all i, $m_i = m$ and that $x = \sup\{x_i: i \in \mathbf{N}\}$. Then we need to consider the following cases:

Case (i). $m * n = m \vee n$. In case $m \geq \top_t$ or $n \geq \top_t$, $(m, x) \circ (n, y) = \max\{(m, x), (n, y)\}$, $(m_i, x_i) \circ (n, y) = \max\{(m_i, x_i), (n, y)\}$, and left-continuity follows from left-continuity of max operation. Otherwise, i.e., in case $m = n < \top_t$, $(m, x) \circ (n, y) = (\min\{m, n\}, 0)$ and for all i, $(m_i, x_i) \circ (n, y) = (\min\{m_i, n\}, 0) = (\min\{m, n\}, 0)$. Thus $(m, x) \circ (n, y) = (m_i, x_i) \circ (n, y)$.

Case (ii). $m * n = m \wedge n$. Its proof is analogous to that of Case (i).

Case (iii). $m * n \neq m \vee n, m \wedge n$, and $m \neq n$. In case $m * n > \top_t$, $(m, x) \circ (n, y) = (m * n, 1)$ and for all i, $(m_i, x_i) \circ (n, y) = (m_i * n, 1) = (m * n, 1)$. Thus $(m, x) \circ (n, y) = (m_i, x_i) \circ (n, y)$. In case $m * n \leq \top_t$, $(m, x) \circ (n, y) = (m * n, 0)$ and for all i, $(m_i, x_i) \circ (n, y) = (m_i * n, 0) = (m * n, 0)$. Thus $(m, x) \circ (n, y) = (m_i, x_i) \circ (n, y)$.

This completes the proof of (III).

We finally prove (IV). First define for every $m \in A$,

$$f(m) = (m, 1) \qquad \text{if } \top_t < m,$$

$(m, 0)$ otherwise.

It is clear that f is increasing and so one-to-one. $f(\top)$, $f(\bot)$, $f(\top_t)$, and $f(\bot_f)$ are top, bottom, and special elements of $(X, \leq)$; and $f(\top_t)$ is the unit element of $\circ$. We then show that $f(m) \circ f(n) = f(m * n)$:

Case (i). $\top_t < m, n$. $f(m) \circ f(n) = (m, 1) \circ (n, 1) = (m * n, 1) = f(m * n)$.

Case (ii). $m \leq \top_t < n$.

Subcase (ii-a). $m * n = m \vee n$. $f(m) \circ f(n) = (m, 0) \circ (n, 1) = (m * n, 1) = (n, 1) = f(n) = f(m * n)$.

Subcase (ii-b). $m * n = m \wedge n$. $f(m) \circ f(n) = (m, 0) \circ (n, 1) = (m * n, 0) = (m, 0) = f(m) = f(m * n)$.

Subcase (ii-c). $m * n \neq m \vee n, m \wedge n$, and $m * n > \top_t$. $f(m) \circ f(n) = (m, 0) \circ (n, 1) = (m * n, 1) = f(m * n)$.

Subcase (ii-d). $m * n \neq m \vee n, m \wedge n$, and $m * n \leq \top_t$. $f(m) \circ f(n) = (m, 0) \circ (n, 1) = (m * n, 0) = f(m * n)$.

Case (iii). $n \leq \top_t < m$. Its proof is analogous to that of Case (ii).

Case (iv). $\top_t \geq m, n$. $f(m) \circ f(n) = (m, 0) \circ (n, 0) = (m * n, 0) = f(m * n)$.

Thus f is an embedding of partially ordered monoids. It remains to prove that for every $l, m, n \in A$, $f(l \to m)$ is the residuum of m and n w.r.t. $\circ$, i.e., (i) $f(l) \circ f(l \to m) \leq f(m)$, and (ii) if $f(l) \circ (n, z) \leq f(m)$, then $(n, z) \leq f(l \to m)$.

(i). Consider the case $\top_t < l \leq m$. $f(l) \circ f(l \to m) = (l, 1) \circ (l \to m, 1) = (l * (l \to m), 1) \leq (m, 1) = f(m)$. Proof of the other cases is analogous.

(ii). By contraposition, we prove this. Suppose that $f(l \to m) \prec (n, z)$, i.e., $(l \to m, 0 \text{ or } 1) \prec (n, z)$. Since $l \to m$ is the residuum of l and m in A, $m < l * n$. Thus $(m, 0 \text{ or } 1) \prec (l, 0 \text{ or } 1) \circ (n, z)$. This completes the proof. □

Remark 5.2 Every countable linearly ordered **MUIL**-algebra **A** in Proposition 5.1 has weak negation $\sim$ in the sense that for each $m \in A$, $m \leq \sim\sim m$, where $\sim m := m \to \bot_f$. Let A, X etc. be as in Proposition 5.1. Given a set X, weak negation $\urcorner$ can be defined as follows:

$$
\begin{aligned}
\urcorner(m, 1) &= (\sim m, 1) && \text{if } \top_t < m, \sim m; \\
&\ (\sim m, 0) && \text{if } \sim m \leq \top_t < m, \\
\urcorner(m, 0) &= (\sim m, 1) && \text{if } m \leq \top_t < \sim m; \\
&\ (\sim m, 0) && \text{if } m, \sim m \leq \top_t, \text{ and}
\end{aligned}
$$

for each $x \in \mathbf{Q} \cap (0, 1)$,

$$
\begin{aligned}
\urcorner(m, x) &= (\sim m, 1) && \text{if } \top_t < m, \sim m; \\
&\ (\sim m, 0) && \text{if } \sim m \leq \top_t < m; \\
&\ (\sim(m^+), 1) && \text{if } m < \top_t < \sim(m^+); \\
&\ (\sim(m^+), 0) && \text{if } m < \top_t \text{ and } \sim(m^+) \leq \top_t.
\end{aligned}
$$

An easy computation shows that the corresponding negation $\urcorner$ is weak. (Note that $\bot_f \leq \top_t$ and the definition of $\sim$ ensure that $\top_t < m, \sim m$ can not be the case. Thus we need not consider such cases in the definition of $\urcorner$.)

Proposition 5.3 Every countable linearly ordered **MUIL**-algebra can be embedded into a standard algebra.

Proof: In an analogy to the proof of Theorem 3.2 in [9], we prove this. Let X, **A**, etc. be as in Proposition 5.1. Since (X, $\leq$) is a countable, dense, linearly-ordered set with maximum and minimum, it is order isomorphic to (**Q** $\cap$ [0, 1], $\leq$). Let g be such an isomorphism. If (I), (II), (III), and (IV) hold, letting for α, $\beta \in [0, 1]$, $\alpha \circ' \beta = g(g^{-1}(\alpha) \circ g^{-1}(\beta))$, and, for all m $\in$ A, $f'(m) = g(f(m))$, we obtain that **Q** $\cap$ [0, 1], $\leq$, 1, 0, 1_t, 0_f, $\circ'$, f' satisfy the conditions (I) to (IV) of Proposition 5.1 whenever X, $\leq$, Max, Min, e, ∂, $\circ$, and f do. Thus we can without loss of generality assume that X = **Q** $\cap$ [0, 1] and $\leq$ = $\leq$.

Now we define for α, $\beta \in [0, 1]$,

$$\alpha \circ'' \beta = \sup_{x \in X: x \leq \alpha} \sup_{y \in X: y \leq \beta} x \circ y.$$

Commutativity of $\circ''$ follows from that of $\circ$. Its isotonicity and identity are easy consequences of the definition. Furthermore, it follows from the definition that $\circ''$ is conjunctive, i.e., $0 \circ'' 1 = 0$, and idempotent, i.e., $\alpha \circ'' \alpha = \alpha$.

We prove left-continuity. Suppose that $<\alpha_n: n \in \mathbf{N}>$, $<\beta_n: n \in \mathbf{N}>$ are increasing sequences of reals in [0, 1] such that $\sup\{\alpha_n: n \in \mathbf{N}\} = \alpha$ and $\sup\{\beta_n: n \in \mathbf{N}\} = \beta$. By the isotonicity of $\circ''$, $\sup\{\alpha_n \circ'' \beta_n\} = \alpha \circ'' \beta$. Since the restriction of $\circ''$ to **Q** $\cap$ [0, 1] is left-continuous, we obtain that

$$\alpha \circ'' \beta = \sup\{q \circ'' r: q, r \in \mathbf{Q} \cap [0, 1], q \leq \alpha, r \leq \beta\}$$
$$= \sup\{q \circ'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\}.$$

For each $q < \alpha$, $r < \beta$, there is n such that $q < \alpha_n$ and $r < \beta_n$. Thus,

$\sup\{\alpha_n \circ'' \beta_n: n \in \mathbf{N}\} \geq \sup\{q \circ'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\} = \alpha \circ'' \beta$.

Hence, $\circ''$ is a left-continuous uninorm on [0, 1].

It is an easy consequence of the definition that $\circ''$ extends $\circ$. By (I) to (IV), f is an embedding of $(A, \leq_A, \top, \bot, \top_t, \bot_f, \wedge, \vee, *)$ into $([0, 1], \leq, 1, 0, 1_t, 0_f, \min, \max, \circ'')$. Moreover, $\circ''$ has a residuum, calling it $\rightharpoonup$.

We finally prove that for $x, y \in A$, $f(x \to y) = f(x) \rightharpoonup f(y)$. By (IV), $f(x \to y)$ is the residuum of f(x) and f(y) in $(\mathbf{Q} \cap [0, 1], \circ, \leq, 1, 0, 1_t, 0_f, \min, \max, \circ'')$. Thus

$$f(x) \circ'' f(x \to y) = f(x) \circ f(x \to y) \leq f(y).$$

Suppose toward contradiction that there is $\alpha > f(x \to y)$ such that $\alpha \circ'' f(x) \leq f(y)$. Since $\mathbf{Q} \cap [0, 1]$ is dense in [0, 1], there is $q \in \mathbf{Q} \cap [0, 1]$ such that $f(x \to y) < q \leq \alpha$. Hence $q \circ'' f(x) = q \circ f(x) \leq f(y)$, contradicting (IV). □

Theorem 5.4 **MUIL** is complete w.r.t. left-continuous conjunctive idempotent uninorms and their residua. Namely, for each formula ϕ, if $\nvdash_{\mathbf{MUIL}}$ ϕ, then there is a left-continuous conjunctive idempotent uninorm $\circ$ and an evaluation v into ([0, 1], $\circ''$, $\rightharpoonup$, $\leq$, 1, 0, 1_t, 0_f), where $\rightharpoonup$ is the residuum of $\circ''$, such that $v(\phi) < 1_t$.

Proof: Let ϕ be a formula such that $\nvdash_{\mathbf{MUIL}}$ ϕ, **A** a linearly ordered **MUIL**-algebra, and v an evaluation in **A** such that $v(\phi) < \top_t$. Let f' be the embedding of **A** into the standard **MUIL**-algebra as in Proposition 5.3. Then $f' \circ v$ is an evaluation into the standard **MUIL**-algebra, and $f' \circ v(\phi) < 1_t$. □

In an analogy to the above, we can show that

Theorem 5.5 MUMML is complete w.r.t. left-continuous conjunctive, idempotent, and min-maxed uninorms and their residua.

Proof: By considering that (MM) $(m, x) \circ (n, y) = \max\{(m, x), (n, y)\}$ or $(m, x) \circ (n, y) = \min\{(m, x), (n, y)\}$ additionally, we can prove this. We first note that it is not the case that $m * n \neq m \vee n, m \wedge n$ because * satisfies (mm). Thus $m * n$ is $m \vee n$ or $m \wedge n$. Let $m * n = m \vee n$. In case $m \geq \top_t$ or $n \geq \top_t$, $(m, x) \circ (n, y) = \max\{(m, x), (n, y)\}$. In case $m = n < \top_t$, $m * n = m \wedge n$ and so $(m, x) \circ (n, y) = \min\{(m, x), (n, y)\}$. Let $m * n = m \wedge n$. In case $m < \top_t$ or $n < \top_t$, $(m, x) \circ (n, y) = \min\{(m, x), (n, y)\}$. In case $m = n \geq \top_t$, $m * n = m \vee n$ and so $(m, x) \circ (n, y) =$

max{(m, x), (n, y)}. Thus (MM) holds.

Proof of the remaining is almost the same as **MUIL**. □

Furthermore, we can show the standard completeness of **FRM** as follows:

Theorem 5.6 (See [10]) **FRM** is complete w.r.t. left-continuous conjunctive idempotent uninorms with identity element $e = n(e) = 1/2$ where $n(x) = 1 — x$, and their residua satisfying for all $x, y \in [0, 1]$,

$$x * y = \begin{cases} \min(x, y) & \text{if } x + y \leq 1; \\ \max(x, y) & \text{otherwise.} \end{cases}$$

References

[1] Cintula, P.(2006), "Weakly Implicative (Fuzzy) Logics I: Basic properties", *Archive for Mathematical Logic*.

[2] Esteva, F., and Godo, L.(2001), "Monoidal t-norm based logic: towards a logic for left-continous t-norms", *Fuzzy Sets and Systems*, 124, pp. 271-288.

[3] Esteva, F., Godo, L., Hájek, P., and Navara, M.(2000), "Residuated fuzzy logics with an involutive logics", *Archive for Mathematical Logic*, 39, pp. 103-124.

[4] Gottwald, S.(2001), *A Treatise on Many-valued Logics*, Baldock, Research studies press LTD.

[5] Hájek, P.(1998), *Metamathematics of Fuzzy Logic*, Amsterdam, Kluwer.

[6] Hájek, P.(1998), "Basic fuzzy logic and BL-algebras", *Soft computing*, 2, pp. 124-128.

[7] Höhle, U.(1995), "Commutative, residuated l-monoids", U. Höhle and E. P. Klement (eds.), *Non-Classical Logics and Their Applications to Fuzzy Subsets*, Amsterdam, Kluwer, pp. 53-106.

[8] Meyer, R. K., Dunn, J. M., and Leblanc, H.(1976), "Completeness of relevant quantification theories", *Notre Dame Journal of Formal Logic*, 15, pp. 97-121.

[9] Jenei, S and Montagna, F.(2002), "A Proof of Standard completeness for Esteva and Godo's Logic MTL", *Studia Logica*, 70, pp. 183-192.

[10] Metcalfe, G and Montagna, F.(2007), "Substructural Fuzzy Logics", *Journal of Symbolic Logic*, 72, pp. 834-864.
[11] Yang, E.(2007), "(weak) R-mingle: toward a fuzzy-relevance logic", *Korean Journal of Logic*, 10 No. 2, pp. 125-146.
[12] Yang, E.(2008), "Uninorm logic: toward a fuzzy-relevance logic(2)", *Korean Journal of Logic*, 11 No. 1, pp. 131-156.

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Standard completeness results for some neighbors of **R**-mingle

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이 논문에서 우리는 [10]에서 멧칼페와 몬테그나에 의해 소개된 몇 체계들에 대한 새로운 표준 완전성 증명을 다룬다. 이를 위해 이 논문은 연관 논리 R-mingle (**RM**)의 이웃들로 간주될 수 있는 몇몇 퍼지-연관 논리를 연구한다. 우선, 좌-연속 항등적 멱등 유니놈들과 그것들의 잔여(left-continuous conjunctive idempotent uninorms and their residua)의 동어반복을 다루도록 의도된 monoidal uninorm idempotence 논리 **MUIL**과 그것의 몇몇 확장이 **RM**의 이웃으로 소개된다. 그리고 그것들에 상응하는 대수적 구조가 정의된 후, 이 체계들을 위한 표준 완전성 즉 단위 실수 [0, 1] 위에서의 완전성이 제공된다.

[Key Words] 표준 완전성, 퍼지-연관 논리, RM.