Journal of Korean
Data & Information Science Society
2008, Vol. 19, No. 3, pp. 995 ~ 1006

Bayesian One-Sided Hypothesis Testing for Shape Parameter in Inverse Gaussian Distribution

Sang Gil Kang[1)] · Dal Ho Kim[2)] · Woo Dong Lee[3)]

Abstract

This article deals with the one-sided hypothesis testing problem in inverse Gaussian distribution. We propose Bayesian hypothesis testing procedures for the one-sided hypotheses of the shape parameter under the noninformative prior. The noninformative prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. So we propose the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor, the median intrinsic Bayes factor and the encompassing intrinsic Bayes factor under the reference prior. Simulation study and a real data example are provided.

Keywords: Fractional Bayes Factor; Intrinsic Bayes Factor; Inverse Gaussian Distribution; One-sided Hypothesis Testing; Reference Prior; Shape Parameter.

1. Introduction

The inverse Gaussian distribution has potentially useful applications in a wide variety of fields such as biology, economics, reliability theory, life testing and social sciences as discussed in Chhikara and Folks (1989), Folks and Chhikara (1978), Mudholkar and Natarajan (2002), Seshadri (1999) and Whitmore (1979).

The inverse Gaussian distribution has its origin in Wiener process as a first

1) Associate Professor, Department of Computer & Data Information, Sangji University, Wonju, 220-712, Korea
E-mail : sangkg@mail.sangji.ac.kr

2) Corresponding Author : Professor, Department of Statistics, Kyungpook National University, Taegu, 702-701, Korea
E-mail : dalkim@knu.ac.kr

3) Professor, Department of Asset Management, Daegu Haany University, Kyungsan, 712-240, Korea
E-mail : wdlee@dhu.ac.kr

passage time distribution. It is also an approximation to the sample size distribution in a sequential probability ratio test. For these reasons, the inverse Gaussian distribution is also known as the first passage time or the Wald distribution.

Tweedie (1957a, 1957b) established many important statistical properties of the inverse Gaussian distribution and discussed the similarity between statistical methods based on the inverse Gaussian distribution and those based on the normal theory. A comprehensive discussion on the inverse Gaussian distributions can be found in books of Chhikara and Folks (1989) and Seshadri (1999).

The probability density function of the inverse Gaussian distribution $IG(\mu,\lambda)$ is given as

$$f(x)=\sqrt{\frac{\lambda}{2\pi}}\,x^{-\frac{3}{2}}\exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2x}\right\},\ x>0, \tag{1}$$

where $\mu(>0)$ is the mean parameter and $\lambda(>0)$ is the scale parameter. Let $\theta=\lambda/\mu$. Then the shape of the inverse Gaussian distribution depends on θ, and hence θ is the shape parameter.

For reliability point of view, Chhikara and Folks (1977) showed that if the lifetime of a machine has the inverse Gaussian distribution, then given that the machine has survived up to time t_0, the mean residual time will eventually exceed the original mean lifetime if the shape parameter is less than 2. In practice, however, the shape parameter is typically unknown. So one may want to estimate it or perform a statistical test to decide whether the shape parameter is less than 2. This motivates testing the null hypothesis $H_0:\theta<\theta_0$ versus $H_1:\theta\geq\theta_0$, where θ_0 is prespecified. Chhikara (1972) has discussed these problems based on approximate procedures. Hsieh (1990) derived the likelihood ratio test and proposed the method for constructing the confidence bounds.

However, there is a little work in this problem from the viewpoint of Bayesian framework. It is well known that the role of the objective priors such as the probability matching prior or the reference prior in the presence of nuisance parameters is very important in Bayesian inference. Kang, Kim and Lee (2004) developed the noninformative priors for the shape parameter in the inverse Gaussian distribution. They showed that the one-at-a-time reference prior satisfying the first order matching criterion matches the target coverage probability much more accurately than Jeffreys' prior.

In Bayesian model selection or testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the device of imaginary training samples in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so, there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang, Kim and Lee, 2005, 2006). An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001).

In this paper, we propose the objective Bayesian one-sided hypothesis testing procedures based on the Bayes factors for the shape parameter in the inverse Gaussian distribution. The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factor. In Section 3, using the reference prior, we provide the Bayesian one-sided hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factor. In Section 4, simulation study and a real data example are given.

2. Intrinsic and Fractional Bayes Factors

Suppose that hypotheses $H_1, H_2, \cdots, H_q$ are under consideration, with the data $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)$ under hypothesis H_i. The parameter vectors $\boldsymbol{\theta}_i$ are unknown. Let $\pi_i(\boldsymbol{\theta}_i)$ be the prior distributions of hypothesis H_i, and let p_i be the prior probabilities of hypothesis H_i, $i = 1, 2, \cdots, q$. Then the posterior probability that the hypothesis H_i is true is

$$P(H_i \mid \boldsymbol{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{2}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\boldsymbol{x} \mid \boldsymbol{\theta}_j) \pi_j(\boldsymbol{\theta}_j) d\boldsymbol{\theta}_j}{\int f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i) \pi_i(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i} = \frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})}. \tag{3}$$

The B_{ji} interpreted as the comparative support of the data for H_j versus H_i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\boldsymbol{\theta}_i)$ and $\pi_j(\boldsymbol{\theta}_j)$. Often in Bayesian analysis, one can use noninformative priors π_i^N. Common choices are the uniform prior, Jeffreys' prior and the reference prior. The noninformative prior π_i^N is typically improper. Hence the use of noninformative prior $\pi_i^N(\cdot)$ in (3) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\boldsymbol{x}(l)$ denote the part of the data to be so used and let $\boldsymbol{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\boldsymbol{x}(l)) < \infty, i = 1, \cdots, q. \tag{4}$$

In view (4), the posteriors $\pi_i^N(\boldsymbol{\theta}_i \mid \boldsymbol{x}(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, with the remainder of the data $\boldsymbol{x}(-l)$, using $\pi_i^N(\boldsymbol{\theta}_i \mid \boldsymbol{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int f(\boldsymbol{x}(-l) \mid \boldsymbol{\theta}_j, \boldsymbol{x}(l)) \pi_j^N(\boldsymbol{\theta}_j \mid \boldsymbol{x}(l)) d\boldsymbol{\theta}_j}{\int f(\boldsymbol{x}(-l) \mid \boldsymbol{\theta}_i, \boldsymbol{x}(l)) \pi_i^N(\boldsymbol{\theta}_i \mid \boldsymbol{x}(l)) d\boldsymbol{\theta}_i} = B_{ji}^N \cdot B_{ij}^N(\boldsymbol{x}(l)) \tag{5}$$

where

$$B_{ji}^N = B_{ji}^N(\boldsymbol{x}) = \frac{m_j^N(\boldsymbol{x})}{m_i^N(\boldsymbol{x})} \quad \text{and} \quad B_{ij}^N(\boldsymbol{x}(l)) = \frac{m_i^N(\boldsymbol{x}(l))}{m_j^N(\boldsymbol{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\boldsymbol{x}$ and training samples $\boldsymbol{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\boldsymbol{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ij}^N(\boldsymbol{x}(l)). \tag{6}$$

where L is the number of all possible minimal training samples. However the AIBF are often not suitable for nonnested situations, especially when testing one-sided hypotheses (See Dmochowski, 1996). An attractive alternative, given by Berger and Pericchi (1996) is to embed the competing hypotheses in a larger encompassing hypothesis H_0 so that all of the H_i are nested within H_0. The encompassing arithmetic intrinsic Bayes factor (EIBF) is then defined as

$$B_{ji}^{EI} = B_{ji}^N \cdot \frac{\sum_{l=1}^{L} B_{i0}^N(\boldsymbol{x}(l))}{\sum_{l=1}^{L} B_{j0}^N(\boldsymbol{x}(l))}, \tag{7}$$

where $B_{i0}^N(\boldsymbol{x}(l)) = m_i^N(\boldsymbol{x}(l)) / m_0^N(\boldsymbol{x}(l))$. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \cdot ME[B_{ij}^N(\boldsymbol{x}(l))], \tag{8}$$

where ME indicates the median for all the training sample Bayes factors. Therefore we can also calculate the posterior probability of H_i using (2), where B_{ji} is replaced by B_{ji}^{EI} and B_{ji}^{MI} from (7) and (8), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b, of each likelihood function, $L(\boldsymbol{\theta}_i) = f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int L^b(\boldsymbol{x} \mid \boldsymbol{\theta}_i)\pi_i^N(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i}{\int L^b(\boldsymbol{x} \mid \boldsymbol{\theta}_j)\pi_j^N(\boldsymbol{\theta}_j)d\boldsymbol{\theta}_j} = B_{ji}^N \cdot \frac{m_i^b(\boldsymbol{x})}{m_j^b(\boldsymbol{x})}. \tag{9}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. (See O'Hagan (1995, 1997) and the discussion by Berger and Mortera in O'Hagan (1995)).

3. Bayesian One-Sided Hypothesis Testing Procedures

Consider that $X_1, \cdots, X_n$ are independent and identically distributed random variables from the inverse Gaussian distribution $IG(\mu, \lambda)$. Then the joint probability density function is

$$f(\boldsymbol{x} \mid \mu, \lambda) = (2\pi)^{-n/2}\left[\prod_{i=1}^{n} x_i^{-3/2}\right]\lambda^{-n/2} \exp\left\{-\sum_{i=1}^{n}\frac{\lambda(x_i-\mu)^2}{2\mu^2 x_i}\right\}, \tag{10}$$

where $\boldsymbol{x} = (x_1, \cdots, x_n)$. The shape parameter $\theta(=\lambda/\mu)$ determines the shape of the distribution and the density function is highly skewed for moderate values of θ. As θ increases the inverse Gaussian converges to the normal. And also the coefficient of variation, the skewness and the kurtosis measure of the inverse Gaussian distribution are closely related with θ, and these values are $\theta^{-1/2}$, $3\theta^{-1/2}$ and $15\theta^{-1}$, respectively.

Let $\theta_1 = \lambda/\mu$ and $\theta_2 = 2\mu^{-1} + \lambda^{-1}$. With this parameterization, Kang, Kim and Lee (2004) derived the noninformative priors for θ_1 and θ_2, and showed that the posterior distribution under the general priors including the reference prior is proper. The reference prior is

$$\pi^N(\theta_1, \theta_2) \propto \theta_1^{-1/2}(2\theta_1 + 1)^{-1/2}\theta_2^{-1}.$$

We are interest to testing the hypotheses $H_1 : \theta_1 < \theta_{10}$ versus $H_2 : \theta_1 \geq \theta_{10}$ based on the fractional Bayes factor and the intrinsic Bayes factor.

3.1 Bayesian Hypothesis Testing based on the Fractional Bayes Factor

Under the hypothesis H_1, the reference prior for θ_1 and θ_2 is

$$\pi_1^N(\theta_1,\theta_2) \propto \theta_1^{-1/2}(2\theta_1+1)^{-1/2}\theta_2^{-1}, \tag{11}$$

where $\theta_1 < \theta_{10}$. The likelihood function under the hypothesis H_1 is

$$L(\theta_1,\theta_2 \mid \boldsymbol{x}) = (2\pi)^{-\frac{n}{2}}\left[\prod_{i=1}^n x_i^{-\frac{3}{2}}\right]\theta_2^{-\frac{n}{2}}(1+2\theta_1)^{\frac{n}{2}}\exp\left\{-\sum_{i=1}^n \frac{[x_i-\theta_1^{-1}\theta_2^{-1}(1+2\theta_1)]^2}{2\theta_1^{-2}\theta_2^{-1}(1+2\theta_1)x_i}\right\}, \tag{12}$$

where $\theta_1 < \theta_{10}$. Then from the likelihood (12) and the reference prior (11), the element of the FBF under H_1 is given by

$$\begin{aligned} m_1^b(\boldsymbol{x}) &= \int_0^{\theta_{10}}\int_0^\infty L^b(\theta_1,\theta_2 \mid \boldsymbol{x})\pi_1^N(\theta_1,\theta_2)d\theta_2 d\theta_1 \\ &= 2(2\pi)^{-\frac{nb}{2}}\left[\prod_{i=1}^n x_i^{-\frac{3b}{2}}\right]\left(\sum_{i=1}^n x_i / \sum_{i=1}^n x_i^{-1}\right)^{\frac{bn}{4}} \\ &\times \int_0^{\theta_{10}} \frac{\theta_1^{(nb-1)/2}}{(1+2\theta_1)^{1/2}}\exp\{nb\theta_1\}\mathrm{BesselK}\left[\frac{bn}{2}, b\theta_1\sqrt{\sum_{i=1}^n x_i}\sqrt{\sum_{i=1}^n x_i^{-1}}\right]d\theta_1, \end{aligned}$$

where $\mathrm{BesselK}[\nu, az] = 1/2a^\nu \int_0^\infty t^{-\nu-1}\exp\{-1/2z(t+a^2t^{-1})\}dt$ is the modified Bessel function of the second order. For the hypothesis H_2, the reference prior for θ_1 and θ_2 is

$$\pi_2^N(\theta_1,\theta_2) \propto \theta_1^{-1/2}(2\theta_1+1)^{-1/2}\theta_2^{-1}, \tag{13}$$

where $\theta_1 \geq \theta_{10}$. The likelihood function under the hypothesis H_2 is

$$L(\theta_1,\theta_2 \mid \boldsymbol{x}) = (2\pi)^{-\frac{n}{2}}\left[\prod_{i=1}^n x_i^{-\frac{3}{2}}\right]\theta_2^{-\frac{n}{2}}(1+2\theta_1)^{\frac{n}{2}}\exp\left\{-\sum_{i=1}^n \frac{[x_i-\theta_1^{-1}\theta_2^{-1}(1+2\theta_1)]^2}{2\theta_1^{-2}\theta_2^{-1}(1+2\theta_1)x_i}\right\}, \tag{14}$$

where $\theta_1 \geq \theta_{10}$. Thus from the likelihood (14) and the reference prior (13) the element of FBF under H_2 gives as follows.

$$\begin{aligned} m_2^b(\boldsymbol{x}) &= \int_{\theta_{10}}^\infty\int_0^\infty L^b(\theta_1,\theta_2 \mid \boldsymbol{x})\pi_2^N(\theta_1,\theta_2)d\theta_2 d\theta_1 \\ &= 2(2\pi)^{-\frac{nb}{2}}\left[\prod_{i=1}^n x_i^{-\frac{3b}{2}}\right]\left(\sum_{i=1}^n x_i / \sum_{i=1}^n x_i^{-1}\right)^{\frac{bn}{4}} \\ &\times \int_{\theta_{10}}^\infty \frac{\theta_1^{(nb-1)/2}}{(1+2\theta_1)^{1/2}}\exp\{nb\theta_1\}\mathrm{BesselK}\left[\frac{bn}{2}, b\theta_1\sqrt{\sum_{i=1}^n x_i}\sqrt{\sum_{i=1}^n x_i^{-1}}\right]d\theta_1. \end{aligned}$$

Therefore the element B_{21}^N of the FBF is given by

$$B_{21}^{N} = \frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})}, \tag{15}$$

where

$$S_1(\boldsymbol{x}) = \int_0^{\theta_{10}} \frac{\theta_1^{(n-1)/2}}{(1+2\theta_1)^{1/2}} \exp\{n\theta_1\} \mathrm{BesselK}\left[\frac{n}{2}, \theta_1 \sqrt{\sum_{i=1}^{n} x_i} \sqrt{\sum_{i=1}^{n} x_i^{-1}}\right] d\theta_1$$

and

$$S_2(\boldsymbol{x}) = \int_{\theta_{10}}^{\infty} \frac{\theta_1^{(n-1)/2}}{(1+2\theta_1)^{1/2}} \exp\{n\theta_1\} \mathrm{BesselK}\left[\frac{n}{2}, \theta_1 \sqrt{\sum_{i=1}^{n} x_i} \sqrt{\sum_{i=1}^{n} x_i^{-1}}\right] d\theta_1.$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\boldsymbol{x})}{m_2^b(\boldsymbol{x})} = \frac{S_1(\boldsymbol{x};b)}{S_2(\boldsymbol{x};b)},$$

where

$$S_1(\boldsymbol{x};b) = \int_0^{\theta_{10}} \frac{\theta_1^{(nb-1)/2}}{(1+2\theta_1)^{1/2}} \exp\{nb\theta_1\} \mathrm{BesselK}\left[\frac{bn}{2}, b\theta_1 \sqrt{\sum_{i=1}^{n} x_i} \sqrt{\sum_{i=1}^{n} x_i^{-1}}\right] d\theta_1$$

and

$$S_2(\boldsymbol{x};b) = \int_{\theta_{10}}^{\infty} \frac{\theta_1^{(nb-1)/2}}{(1+2\theta_1)^{1/2}} \exp\{nb\theta_1\} \mathrm{BesselK}\left[\frac{bn}{2}, b\theta_1 \sqrt{\sum_{i=1}^{n} x_i} \sqrt{\sum_{i=1}^{n} x_i^{-1}}\right] d\theta_1.$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^{F} = \frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})} \cdot \frac{S_1(\boldsymbol{x};b)}{S_2(\boldsymbol{x};b)}. \tag{16}$$

Note that the calculation of the FBF of H_2 versus H_1 requires an one dimensional numerical integration.

3.2 Bayesian Hypothesis Testing based on the Intrinsic Bayes Factor

The element B_{21}^{N} of the encompassing arithmetic intrinsic Bayes factor is computed in the fractional Bayes factor. So under minimal training sample, we only calculate the marginal densities for the hypotheses H_0, H_1 and H_2, respectively. The marginal density of (X_j, X_l) is finite for all $1 \le j < l \le n$ under each hypothesis (Kang, Kim and Lee, 2004, 2008). Thus we conclude that any training sample of size 2 is a minimal training sample.

The marginal densities $m_0^N(x_j, x_l)$ under $H_0 (= H_1 \cup H_2): \theta_1 > 0$ is given by

$$\begin{aligned} m_0(x_j, x_l) &= \int_0^{\infty}\int_0^{\infty} f(x_j, x_l \mid \theta_1, \theta_2)\, \pi^N(\theta_1, \theta_2)\, d\theta_2\, d\theta_1 \\ &= 2(2\pi)^{-1}(x_j x_l)^{-\frac{3}{2}} \left[(x_j + x_l)/(x_j^{-1} + x_l^{-1})\right]^{\frac{1}{2}} \end{aligned}$$

$$\times \int_0^{\infty} \frac{\theta_1^{1/2}}{(1+2\theta_1)^{1/2}} \exp\{2\theta_1\} \text{BesselK}\left[1, \theta_1 \sqrt{x_j + x_l} \sqrt{x_j^{-1} + x_l^{-1}}\right] d\theta_1,$$

where $1 \le j < l \le n$. The marginal density $m_1^N(x_j, x_l)$ under H_1 is given by

$$\begin{aligned} m_1(x_j, x_l) &= \int_0^{\theta_{10}} \int_0^{\infty} f(x_j, x_l \mid \theta_1, \theta_2) \pi_1^N(\theta_1, \theta_2) d\theta_2 d\theta_1 \\ &= 2(2\pi)^{-1} (x_j x_l)^{-\frac{3}{2}} \left[(x_j + x_l)/(x_j^{-1} + x_l^{-1})\right]^{\frac{1}{2}} \\ &\times \int_0^{\theta_{10}} \frac{\theta_1^{1/2}}{(1+2\theta_1)^{1/2}} \exp\{2\theta_1\} \text{BesselK}\left[1, \theta_1 \sqrt{x_j + x_l} \sqrt{x_j^{-1} + x_l^{-1}}\right] d\theta_1. \end{aligned}$$

And the marginal density $m_2^N(x_j, x_l)$ under H_2 is given by

$$\begin{aligned} m_2(x_j, x_l) &= \int_{\theta_{10}}^{\infty} \int_0^{\infty} f(x_j, x_l \mid \theta_1, \theta_2) \pi_2^N(\theta_1, \theta_2) d\theta_2 d\theta_1 \\ &= 2(2\pi)^{-1} (x_j x_l)^{-\frac{3}{2}} \left[(x_j + x_l)/(x_j^{-1} + x_l^{-1})\right]^{\frac{1}{2}} \\ &\times \int_{\theta_{10}}^{\infty} \frac{\theta_1^{1/2}}{(1+2\theta_1)^{1/2}} \exp\{2\theta_1\} \text{BesselK}\left[1, \theta_1 \sqrt{x_j + x_l} \sqrt{x_j^{-1} + x_l^{-1}}\right] d\theta_1. \end{aligned}$$

Therefore the EIBF of H_2 versus H_1 is given by

$$B_{21}^{EI} = \frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})} \cdot \left(\frac{\sum_{j,l} T_1(x_j, x_l)/T_0(x_j, x_l)}{\sum_{j,l} T_2(x_j, x_l)/T_0(x_j, x_l)} \right), \tag{17}$$

where

$$T_0(x_j, x_l) = \int_0^{\infty} \frac{\theta_1^{1/2}}{(1+2\theta_1)^{1/2}} \exp\{2\theta_1\} \text{BesselK}\left[1, \theta_1 \sqrt{x_j + x_l} \sqrt{x_j^{-1} + x_l^{-1}}\right] d\theta_1,$$

$$T_1(x_j, x_l) = \int_0^{\theta_{10}} \frac{\theta_1^{1/2}}{(1+2\theta_1)^{1/2}} \exp\{2\theta_1\} \text{BesselK}\left[1, \theta_1 \sqrt{x_j + x_l} \sqrt{x_j^{-1} + x_l^{-1}}\right] d\theta_1$$

and

$$T_2(x_j, x_l) = \int_{\theta_{10}}^{\infty} \frac{\theta_1^{1/2}}{(1+2\theta_1)^{1/2}} \exp\{2\theta_1\} \text{BesselK}\left[1, \theta_1 \sqrt{x_j + x_l} \sqrt{x_j^{-1} + x_l^{-1}}\right] d\theta_1.$$

And also the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})} \cdot ME\left[\frac{T_1(x_j, x_l)}{T_2(x_j, x_l)}\right]. \tag{18}$$

Note that the calculations of the EIBF and the MIBF of H_2 versus H_1 require only one dimensional integration.

4. Numerical Studies

In order to assess the Bayesian hypothesis testing procedures, we evaluate the posterior probability for several configurations of (θ_1, μ) and n. In particular, for fixed (θ_1, μ), we take 1,000 independent random samples of X_i with sample size n from the model (10). In our simulation, we put $\mu = 1$. We want to testing the hypotheses $H_1 : \theta_1 < 2$ versus $H_2 : \theta_1 \geq 2$.

The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. <Table 1> shows the results of the averages and the standard deviations in parentheses of posterior probabilities. From <Table 1>, the EIBF and the MIBF give fairly reasonable answers for all of the θ_1. Also the EIBF and the MIBF give a similar behavior for all sample sizes, and the MIBF slightly favours the hypothesis H_1 than the EIBF. However the FBF clearly favours the hypothesis H_2 than the EIBF and the MIBF. That is, the FBF has considerable bias toward the hypothesis H_2. This fact does not a surprising result. Berger and Mortera (1999) showed that the FBF has considerable bias toward one of the hypotheses in nonsymmetric situations, and so the FBF should not be used in clearly nonsymmetric testing situations.

<Table 1> The Averages and the Standard Deviations in Parentheses of Posterior Probabilities

θ_1	n	$P^F(H_1 \mid \boldsymbol{x})$	$P^{EI}(H_1 \mid \boldsymbol{x})$	$P^{MI}(H_1 \mid \boldsymbol{x})$
1.0	5	0.480 (0.211)	0.717 (0.235)	0.696 (0.229)
	10	0.579 (0.269)	0.806 (0.215)	0.802 (0.211)
	15	0.655 (0.278)	0.854 (0.193)	0.853 (0.190)
	20	0.735 (0.255)	0.901 (0.156)	0.901 (0.153)
1.5	5	0.388 (0.182)	0.617 (0.238)	0.602 (0.232)
	10	0.406 (0.242)	0.661 (0.258)	0.661 (0.252)
	15	0.439 (0.263)	0.696 (0.258)	0.697 (0.252)
	20	0.458 (0.268)	0.719 (0.242)	0.723 (0.235)
1.8	5	0.349 (0.165)	0.570 (0.234)	0.565 (0.225)
	10	0.325 (0.213)	0.575 (0.262)	0.579 (0.254)
	15	0.330 (0.232)	0.587 (0.270)	0.592 (0.264)
	20	0.349 (0.245)	0.611 (0.268)	0.617 (0.263)
2.0	5	0.326 (0.156)	0.538 (0.232)	0.527 (0.224)
	10	0.296 (0.194)	0.543 (0.263)	0.548 (0.256)
	15	0.287 (0.200)	0.547 (0.255)	0.553 (0.249)
	20	0.285 (0.215)	0.542 (0.270)	0.550 (0.265)
2.2	5	0.310 (0.153)	0.514 (0.231)	0.508 (0.225)
	10	0.260 (0.180)	0.493 (0.257)	0.498 (0.251)
	15	0.242 (0.180)	0.484 (0.261)	0.492 (0.256)
	20	0.214 (0.177)	0.447 (0.260)	0.457 (0.257)
2.5	5	0.285 (0.140)	0.478 (0.224)	0.476 (0.218)
	10	0.215 (0.153)	0.430 (0.246)	0.439 (0.242)
	15	0.192 (0.151)	0.409 (0.245)	0.418 (0.241)
	20	0.182 (0.156)	0.399 (0.251)	0.410 (0.249)
3.0	5	0.263 (0.130)	0.445 (0.213)	0.449 (0.209)
	10	0.175 (0.138)	0.361 (0.237)	0.372 (0.233)
	15	0.122 (0.107)	0.286 (0.214)	0.296 (0.213)
	20	0.103 (0.102)	0.251 (0.210)	0.261 (0.211)

Example. The following data set is a certain test data on the endurance of 23 deep groove ball bearings by Lieblein and Zelen (1956).

17.88, 28.92, 33.0, 41.52, 42.12, 45.6, 48.48, 51.84, 51.96, 54.12, 55.56, 67.8, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4

This data set was analyzed by Chhikara and Folks (1989) who judged it as being well descriptive of an inverse Gaussian model. For this data set, the maximum likelihood estimator for θ_1 is 3.21. We want to testing the hypotheses $H_1 : \theta_1 < 2$ versus $H_2 : \theta_1 \geq 2$.

The values of the Bayes factor and the posterior probabilities of H_1 are given in <Table 2>. From the results of <Table 2>, the EIBF, the MIBF and the FBF give the same answer, and select the hypothesis H_2. But the FBF favours the hypothesis H_2 than the EIBF and the MIBF.

<Table 2> Bayes Factor Values and Posterior Probabilities

H_1	B_{21}^F	$P^F(H_1 \mid x)$	B_{21}^{EI}	$P^{EI}(H_1 \mid x)$	B_{21}^{MI}	$P^{MI}(H_1 \mid x)$
$\theta_1 < 2$	13.964	0.067	4.321	0.188	4.030	0.199

5. Concluding Remarks

In inverse Gaussian population, we developed the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factor for the one-sided hypotheses of the shape parameter under the reference prior. From our numerical results, the developed hypothesis testing procedures give fairly reasonable answers for all parameter configurations. However the FBF clearly favours the hypothesis H_2 than the EIBF and the MIBF. That is, the FBF considerable biased to the hypothesis H_2, and so the FBF should not be used in clearly nonsymmetric testing situations. Therefore from our results of simulation and example, we recommend the use of the EIBF and the MIBF than the FBF in practical application.

References

1. Berger, J.O. and Bernardo, J.M. (1989). Estimating a Product of Means : Bayesian Analysis with Reference Priors. *Journal of the American Statistical Association,* 84, 200-207.
2. Berger, J.O. and Bernardo, J.M. (1992). On the Development of Reference

Priors (with discussion). *Bayesian Statistics* IV, J.M. Bernardo, et. al., Oxford University Press, Oxford, 35-60.

3. Berger, J.O. and Mortera, J. (1999). Default Bayes Factors for Nonnested Hypothesis Testing. *Journal of the American Statistical Association*, 94, 542-554.
4. Berger, J.O. and Pericchi, L.R. (1996). The Intrinsic Bayes Factor for Model Selection and Prediction. *Journal of the American Statistical Association*, 91, 109-122.
5. Berger, J.O. and Pericchi, L.R. (1998). Accurate and Stable Bayesian Model Selection: the Median Intrinsic Bayes Factor. *Sankya* B, 60, 1-18.
6. Berger, J.O. and Pericchi, L.R. (2001). Objective Bayesian Methods for Model Selection: Introduction and Comparison (with discussion). *In Model Selection, Institute of Mathematical Statistics Lecture Notes-Monograph Series,* Vol 38, Ed. P. Lahiri, 135-207, Beachwood Ohio.
7. Chhikara, R.S. (1972). *Statistical Inference Related to the Inverse Gaussian Distribution*. Ph.D. Dissertation, Oklahoma State University.
8. Chhikara, R.S. and Folks, L. (1977). The Inverse Gaussian Distribution as a Lifetime Model. *Technometrics*, 19, 461-468.
9. Chhikara, R.S. and Folks, L. (1989). *The Inverse Gaussian Distribution; Theory, Methodology and Applications*, Marcel Dekker, New York.
10. Dmochowski, J. (1996). Intrinsic Priors Via Kullback-Leibler Geometry. *Bayesian Statistics* V, J.M. Bernardo, et. al., Oxford University Press, Oxford, 543-549.
11. Folks, L. and Chhikara, R.S. (1978). The Inverse Gaussian Distribution and Its Statistical Application-A Review. *Journal of Royal Statistical Society,* B, 40, 263-289.
12. Hsieh, H.K. (1990). Inferences on the Coefficient of Variation of an Inverse Gaussian Distribution. *Communications in Statistics-Theory and Methods,* 19, 1589-1605.
13. Kang, S.G., Kim, D.H. and Lee, W.D. (2004). Noninformative Priors for the Ratio of Parameters in Inverse Gaussian Distribution. *The Korean Journal of Applied Statistics,* 17, 49-60.
14. Kang, S.G., Kim, D.H. and Lee, W.D. (2005). Bayesian Analysis for the Difference of Exponential Means. *Journal of Korean Data & Information Science Society*, 16, 1067-1078.
15. Kang, S.G., Kim, D.H. and Lee, W.D. (2006). Bayesian One-Sided Testing for the Ratio of Poisson Means. *Journal of Korean Data & Information Science Society,* 17, 619-631.
16. Kang, S.G., Kim, D.H. and Lee, W.D. (2008). Noninformative Priors for the Coefficient of Variation in Two Inverse Gaussian Distributions. *Communications of The Korean Statistical Society*, 15, 429-440.

17. Lieblein, J. and Zelen, M. (1956). Statistical Investigation of the Fatigue Life of Deep-Groove Ball Bearings. *Journal of Research of the National Bureau of Standards*, 57, 273-316.
18. Mudholkar, G. and Natarajan, R. (2002). The Inverse Gaussian Models: Analogues of Symmetry, Skewness and Kurtoss, *Annals of the Institute Statistical Mathematics,* 54, 138-154.
19. O′ Hagan, A. (1995). Fractional Bayes Factors for Model Comparison (with discussion). *Journal of Royal Statistical Society*, B, 57, 99-118.
20. O′ Hagan, A. (1997). Properties of Intrinsic and Fractional Bayes Factors. *Test,* 6, 101-118.
21. Seshadri, V. (1999). *The Inverse Gaussian Distribution: Statistical Theory and Applications*, Springer, New York.
22. Spiegelhalter, D.J. and Smith, A.F.M. (1982). Bayes Factors for Linear and Log-Linear Models with Vague Prior Information. *Journal of Royal Statistical Society,* B, 44, 377-387.
23. Tweedie, M.C.K. (1957a). Statistical Properties of Inverse Gaussian Distributions I. *The Annals of Mathematical Statistics*, 28, 362-377.
24. Tweedie, M.C.K. (1957b). Statistical Properties of Inverse Gaussian Distributions II. *The Annals of Mathematical Statistics,* 28, 696-705.
25. Whitmore, G.A. (1979). An Inverse Gaussian Model for Labour Turnover. *Journal of Royal Statistical Society*, A, 142, 468-478.

[Received July 29, 2008, Accepted August 22, 2008]