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Estimation for the Half-Triangle Distribution Based on Progressively Type-II Censored Samples

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Abstract

We derive some approximate maximum likelihood estimators (AMLEs) and maximum likelihood estimator (MLE) of the scale parameter in the half-triangle distribution based on progressively Type-II censored samples. We compare the proposed estimators in the sense of the mean squared error for various censored samples.

We also obtain the approximate maximum likelihood estimators of the reliability function using the proposed estimators. We compare the proposed estimators in the sense of the mean squared error.

Keywords: Approximate maximum likelihood estimator; Half-triangle distribution; Progressive Type-II censoring; Reliability.

1. Introduction

The half-triangle distribution has the following cumulative distribution function (cdf)

$$F(x) = 1 - \left(1 - \frac{x}{\theta}\right)^2, \quad 0 < x < \theta, \tag{1.1}$$

and probability density function (pdf)

$$f(x) = \frac{2}{\theta}\left(1 - \frac{x}{\theta}\right), \quad 0 < x < \theta. \tag{1.2}$$

A triangle distribution was applied to a kernel function in nonparametric density estimation. Johnson (1997) studied the possibility of using the more intuitively obvious triangular distribution as a proxy for the beta distribution. Some

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properties of the triangular distribution was studied by Balakrishnan and Nevzorov (2003).

In progressive Type-II censoring case, the first failure in the sample is observed and a random sample of size R_1 is immediately drawn from the remaining $n-1$ unfailed items and removed from the test, leaving $n-1-R_1$ items in test. When the second item has failed, R_2 of the still unfailed items are removed, and so on.

Wu and Chang (2003) studied the parameter estimation of a Pareto distribution under progressive Type-II censoring with random removals. They also obtained the maximum likelihood estimator (MLE) of the parameter and derived the expectation and variance of the MLE. Ng (2005) studied the estimation of parameters based on a progressively Type-II censored sample from a modified Weibull distribution. Wu et al. (2006) obtained Bayes estimators of the scale parameter and reliability function of the Rayleigh distribution based on progressively Type-II censored samples. Han et al. (2007) derived the approximate maximum likelihood estimators (AMLEs) of the scale parameter and the location parameter in an exponentiated logistic distribution under multiply Type-II censoring. Kang (2007) derived some explicit estimators of the scale parameter in a half-triangle distribution under multiply Type-II censoring by several approximate maximum likelihood estimation methods.

Recently, Lee et al. (2008) proposed some explicit estimators of scale parameter in the triangular distribution under multiply Type-II censoring by the approximate maximum likelihood estimation methods.

In this paper, we derive the AMLEs and the MLE of the scale parameter θ under progressively Type-II censored sample. The scale parameter is estimated by the approximate maximum likelihood estimation method using two different types of Taylor series expansions. We also obtain the AMLEs of the reliability function using the proposed estimators. We compare the proposed estimators in the sense of the mean squared error (MSE).

2. Maximum Likelihood Estimation

We will discuss the maximum likelihood estimation of the scale parameter based on progressively Type-II censored samples. Let $X_{1:m:n},\ldots,X_{m:m:n}$ denote such a progressively Type-II censored sample with $(R_1,\ldots,R_m)$ being the progressive censoring scheme.

The likelihood function based on the progressively Type-II censored sample is given by

$$\begin{aligned} L &= C\prod_{i=1}^{m} f(x_{i:m:n};\theta)\left[1-F(x_{i:m:n};\theta)\right]^{R_i} \\ &= C\left(\frac{2}{\theta}\right)^{m}\prod_{i=1}^{m}\left(1-\frac{x_{i:m:n}}{\theta}\right)^{2R_i+1} \end{aligned} \tag{2.1}$$

where $C = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-m+1-R_1-\cdots-R_{m-1})$.

From the equation (2.1), the log-likelihood function may be written as

$$\ln L = K - m\ln\theta + \sum_{i=1}^{m}(2R_i+1)\ln\left(1-\frac{x_{i:m:n}}{\theta}\right), \quad (2.2)$$

where K is a constant.

The random variable $Z_{i:m:n} = X_{i:m:n}/\theta$ has a standard half-triangle distribution with pdf and cdf;

$$f(z_{i:m:n}) = 2(1-z_{i:m:n}) \text{ and } F(z_{i:m:n}) = 1-(1-z_{i:m:n})^2.$$

On differentiating the log-likelihood function with respect to θ in turn and equation to zero, we obtain the estimating equations as

$$\frac{\partial\ln L}{\partial\theta} = -\frac{1}{\theta}\left[m - \sum_{i=1}^{m}(2R_i+1)\frac{Z_{i:m:n}}{1-Z_{i:m:n}}\right] = 0. \quad (2.3)$$

We can find the MLE of θ as values $\hat{\theta}$ that maximize the log-likelihood function in the equation (2.2) by solving the equation $\partial\ln L/\partial\theta = 0$. Since the equation (2.3) cannot be solved explicitly, some numerical method must be employed.

3. Approximate Maximum Likelihood Estimators

Since the log-likelihood equation do not admit explicit solutions, it will be desirable to consider an approximation to the likelihood equation which provide us with explicit estimator for the scale parameter.

Let

$$\xi_{i:m:n} = F^{-1}(p_{i:m:n}) = 1-\sqrt{q_{i:m:n}},$$

where $q_{i:m:n} = 1-p_{i:m:n}$ is given by [see Balakrishnan and Aggarwala (2000)]

$$p_{i:m:n} = 1-\prod_{j=m-i+1}^{m}\frac{j+R_{m-j+1}+\cdots+R_m}{j+1+R_{m-j+1}+\cdots+R_m} \quad \text{for } i=1,\dots,m.$$

We consider first two terms in Taylor series expansions of $Z_{i:m:n}/(1-Z_{i:m:n})$ and $1/(1-Z_{i:m:n})$.

First, we can approximate the function by

$$\frac{Z_{i:m:n}}{1-Z_{i:m:n}} \approx -\frac{\xi^2_{i:m:n}}{q_{i:m:n}} + \frac{1}{q_{i:m:n}}Z_{i:m:n}. \quad (3.1)$$

By substituting the equation (3.1) into the equation (2.3), we may approximate the likelihood equation in (2.3) by

$$\frac{\partial\ln L}{\partial\theta} \approx -\frac{1}{\theta}\left[m + \sum_{i=1}^{m}(2R_i+1)\frac{\xi^2_{i:m:n}}{q_{i:m:n}} - \sum_{i=1}^{m}(2R_i+1)\frac{1}{q_{i:m:n}}Z_{i:m:n}\right] = 0. \quad (3.2)$$

We can derive an estimator of θ as follows;

$$\tilde{\theta}_1 = \frac{B_1}{A_1} \tag{3.3}$$

where

$$A_1 = m + \sum_{i=1}^{m}(2R_i+1)\frac{\xi_{i:m:n}^2}{q_{i:m:n}} \quad \text{and} \quad B_1 = \sum_{i=1}^{m}(2R_i+1)\frac{1}{q_{i:m:n}}X_{i:m:n}.$$

Since $\xi^2_{i:m:n}/q_{i:m:n} > 0$ and $1/q_{i:m:n} > 0$, the estimator $\tilde{\theta}_1$ is always positive.

Second, we can approximate the function by

$$\frac{1}{1-Z_{i:m:n}} \approx \frac{(1-2\xi_{i:m:n})}{q_{i:m:n}} + \frac{1}{q_{i;m:n}} Z_{i:n:m}. \tag{3.4}$$

By substituting the equation (3.4) into the equation (2.3), we may approximate the likelihood equation in (2.3) by

$$\frac{\partial \ln L}{\partial \theta} \approx -\frac{1}{\theta}\left[m - \sum_{i=1}^{m}(2R_i+1)\frac{1-2\xi_{i:m:n}}{q_{i:m:n}}Z_{i:m:n} - \sum_{i=1}^{m}(2R_i+1)\frac{1}{q_{i:m:n}}Z_{i:m:n}^2\right] = 0. \tag{3.5}$$

The equation (3.5) is a quadratic equation in θ, with its roots given by

$$\tilde{\theta}_2 = \frac{-A_2 \pm \sqrt{A_2^2 - 4mB_2}}{2m} \tag{3.6}$$

where

$$A_2 = -\sum_{i=1}^{m}(2R_i+1)\frac{1-2\xi_{i:m:n}}{q_{i:m:n}}X_{i:m:n} \quad \text{and} \quad B_2 = -\sum_{i=1}^{m}(2R_i+1)\frac{1}{q_{i:m:n}}X_{i:m:n}^2.$$

Since $B_2 < 0$, only one root is admissible, and hence another AMLE of θ is given by

$$\tilde{\theta}_2 = \frac{-A_2 + \sqrt{A_2^2 - 4mB_2}}{2m}. \tag{3.7}$$

4. Estimation of the Reliability

The reliability function of the half-triangle distribution is

$$R(t) = 1 - F(t) = P[X > t] = \left(1 - \frac{t}{\theta}\right)^2, \quad t > 0. \tag{4.1}$$

For the progressively Type-II censored data, we now propose the AMLEs and MLE of the reliability function $R(t)$ using the proposed AMLEs $\tilde{\theta}_i$ and MLE $\hat{\theta}$ that can be used for progressively Type-II censored sample as follows.

$$\tilde{R}_i(t) = \left(1 - \frac{t}{\tilde{\theta}_i}\right)^2, \quad i = 1, 2, \text{ and } \hat{R}(t) = \left(1 - \frac{t}{\hat{\theta}}\right)^2 \tag{4.2}$$

From the equation (4.2), the mean squared errors of these estimators are simulated by Monte Carlo method (based on 10,000 Monte Carlo runs) for sample size $n=20$, $m=0$ and $m=10$ (2*0,1,0,2,0,2,2*0,5) (see Fig. 1).

5. Simulation Results

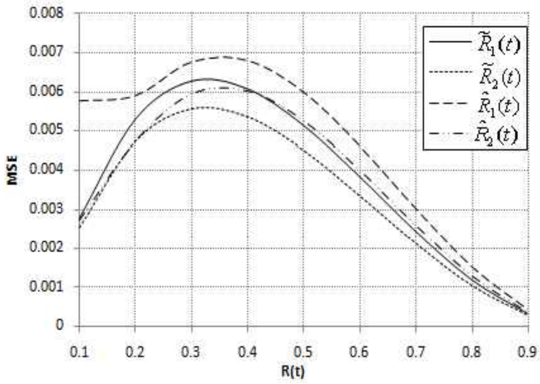
The simulations were carried out for sample size $n=10(5)20, 30$, different choices of the effective sample size m. For simplicity in notation, we will denote the schemes $(0,0,...,0,n-m)$ by $((m-1)*0,n-m)$, for example, $(10*0)$ denotes the progressive censoring scheme $(0,0,...,0)$ and $(3*0,2,2,0)$ denotes the progressive censoring scheme $(0,0,0,2,2,0)$.

The convergence of the Newton-Raphson method depended on the choice of the initial values. For this reason, the proposed AMLEs (3.3) and (3.6) were used as starting values for the iterations, and the MLE was obtained by solving the nonlinear equation (2.3). The mean squared errors of the proposed AMLEs and MLE are simulated by Monte Carlo method (based on 10,000 Monte Carlo runs) for sample size $n=10(5)20, 30$, and various choices of censoring under progressively Type-II censored sample. These values are given in Table 1.

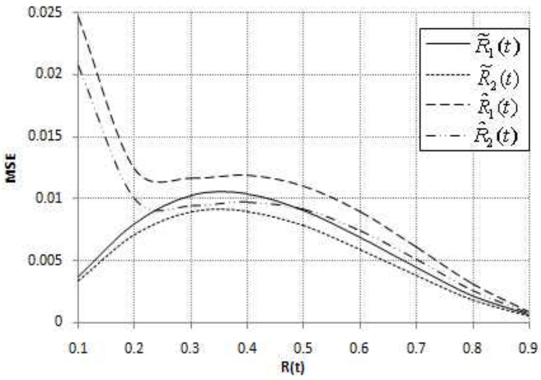
From Table 1, $\tilde{\theta}_2$ and $\hat{\theta}_2$ are more efficient than $\tilde{\theta}_1$ and $\hat{\theta}_1$ in the sense of the MSE. The estimator $\tilde{\theta}_1$ is a linear function of available order statistics. As expected, the MSE of all estimators decreases as sample size n increases.

For $m=20$, $\tilde{R}_2(t)$ and $\hat{R}_2(t)$ are more efficient than $\tilde{R}_1(t)$ and $\hat{R}_1(t)$ when $R(t)<0.4$. But $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are more efficient than $\hat{R}_1(t)$ and $\hat{R}_2(t)$ when $R(t)>0.4$.

For $m=10$, $\tilde{R}_2(t)$ and $\hat{R}_2(t)$ are more efficient than $\tilde{R}_1(t)$ and $\hat{R}_1(t)$ when $0.23<R(t)<0.5$. But $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are more efficient than $\hat{R}_1(t)$ and $\hat{R}_2(t)$ when $R(t)>0.5$ or $R(t)<0.23$. $\tilde{R}_2(t)$ is generally more efficient than the other estimators in both cases.



(a) $n=20, m=20(20*0)$



(b) $n=20, m=10(2*0,1,0,2,0,2,2*0,5)$

Fig. 1. The relative mean squared errors of $\widetilde{R}_i(t)$ and $\hat{R}_i(t)$

Table 1. The relative mean squared errors for the estimators of the scale parameter θ.

n	m	Scheme	$\widetilde{\theta}_1$	$\widetilde{\theta}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$
10	10	(10*0)	0.046029	0.040190	0.052039	0.042887
	6	(10*3,2,2,0)	0.094620	0.088703	0.105721	0.090680
	6	(2*0,4,3*0)	0.089927	0.079841	0.100202	0.075552
	5	(5,4*0)	0.101202	0.088464	0.105806	0.078065
15	15	(15*0)	0.028627	0.025297	0.031341	0.027386
	10	(5,9*0)	0.047006	0.041026	0.052740	0.043839
	10	(4*0,3,3*0,2,0)	0.053689	0.050140	0.059218	0.053627
	10	(0,3,6*0,2,0)	0.051220	0.047743	0.056315	0.051068
	10	(2*0,1,0,2,0,2,3*0)	0.053126	0.047321	0.059710	0.050451
20	20	(20*0)	0.020151	0.017996	0.021531	0.019191
	15	(3*0,2,4*0,3,6*0)	0.032143	0.028503	0.035546	0.030524
	10	(5,2*0,5,6*0)	0.051178	0.044817	0.057406	0.047530
	10	(2*0,1,0,2,0,2,2*0,5)	0.062804	0.062983	0.063067	0.063067
	10	(2*0,3,0,2,0,2,2*0,3)	0.058383	0.058368	0.058418	0.058418
30	30	(30*0)	0.012501	0.011345	0.013042	0.011695
	20	(3*0,5,3*0,5,12*0)	0.022256	0.019830	0.023866	0.021220
	20	(2*0,10,17*0)	0.020997	0.018723	0.022410	0.019946
	20	(9*0,10,10*0)	0.023801	0.021244	0.025877	0.022655
	15	(5,6*0,10,7*0)	0.034872	0.030915	0.038871	0.033065
	15	(10,6*0,5,7*0)	0.032794	0.029034	0.036418	0.031055

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