

SHARP ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR

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ABSTRACT. In this paper, we prove the sharp estimates for multilinear commutator related to Littlewood-Paley operator. By using the sharp estimates, we obtained the weighted L^p -norm inequality for the multilinear commutator for $1 < p < \infty$.

1. Introduction

As the development of singular integral operators, their commutators have been well studied (see [1]-[4]). Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rocherberg and Weiss (see [3]) states that commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [8]-[10], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The Littlewood-Paley operator is an important operator in harmonic analysis (see [12]). The main purpose of this paper is to prove the sharp inequality for the multilinear commutator related to the Littlewood-Paley operator. By using the sharp inequality, we obtain the weighted L^p -norm inequality for the multilinear commutator with $1 < p < \infty$.

2. Preliminaries and theorems

First let us introduce some notations (see [4], [9], [11]). In this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function b , let $b_Q = |Q|^{-1} \int_Q b(x)dx$, the sharp function of b is defined by

$$b^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy.$$

It is well-known that (see [4])

$$b^\#(x) = \sup_{x \in Q} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |b(y) - c| dy.$$

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We say that b belongs to $BMO(R^n)$ if $b^\#$ belongs to $L^\infty(R^n)$ and define $\|b\|_{BMO} = \|b^\#\|_{L^\infty}$. It has been known that (see [11])

$$\|b - b_{2^k Q}\|_{BMO} \leq Ck\|b\|_{BMO}.$$

For $b_j \in BMO$ ($j = 1, \dots, m$), set $\tilde{b} = (b_1, \dots, b_m)$ and

$$\|\tilde{b}\|_{BMO} = \prod_{j=1}^m \|b_j\|_{BMO}.$$

Given a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\tilde{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\tilde{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\tilde{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy;$$

we write that $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$.

We denote the Muckenhoupt weights by A_1 (see [4]), that is,

$$A_1 = \{w : M(w)(x) \leq Cw(x), a.e.\}.$$

Throughout this paper, we will study some multilinear commutators as following.

Definition. Suppose b_j ($j = 1, \dots, m$) are the fixed locally integrable functions on R^n . Let $\mu > 1$, $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

- (1) $\int_{R^n} \psi(x) dx = 0$,
- (2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1)}$,
- (3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

The Littlewood-Paley multilinear commutator is defined by

$$g_\mu^{\tilde{b}}(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x-y|} \right)^{n\mu} |F_t^{\tilde{b}}(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^{\tilde{b}}(f)(x, y) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y-z) f(z) dz$$

and $\psi_t(x) = t^{-n} \psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = \int_{R^n} \psi_t(x-y) f(y) dy$, we also define that

$$g_\mu(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x-y|} \right)^{n\mu} |F_t(f)(y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

which is the Littlewood-Paley operator (see [12]).

Let H be the space $H = \left\{ h : \|h\| = (\int \int_{R_+^{n+1}} |h(y, t)|^2 dy dt / t^{n+1})^{1/2} < \infty \right\}$, then, for each fixed $x \in R^n$, $F_t^{\tilde{b}}(f)(x, y)$ may be viewed as a mapping from $[0, +\infty)$ to H , and it is clear that

$$g_\mu(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t(f)(y) \right\|$$

and

$$g_\mu^{\tilde{b}}(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^{\tilde{b}}(f)(x, y) \right\|.$$

Note that when $b_1 = \dots = b_m$, $g_\mu^{\tilde{b}}$ is just the m order commutator (see [1], [6], [7]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1]-[3], [5]-[10]). Our main purpose is to establish the sharp inequality for the multilinear commutator.

Now we state our theorems as following.

Theorem 1. *Let $\mu > 3 + 1/n$, $b_j \in BMO$ for $j = 1, \dots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$(g_\mu^{\tilde{b}}(f))^\#(x) \leq C \|\tilde{b}\|_{BMO} \left(M_r(f)(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_r(g_\mu^{\tilde{b}_{\sigma^c}}(f))(x) \right).$$

Theorem 2. *Let $\mu > 3 + 1/n$, $b_j \in BMO$ for $j = 1, \dots, m$. Then $g_\mu^{\tilde{b}}$ is bounded on $L^p(w)$ for $w \in A_1$ and $1 < p < \infty$.*

3. Proof of theorems

To prove the theorems, we need the following lemmas.

Lemma 1. (see [12]) *Let $w \in A_p$ and $1 < p < \infty$. Then g_μ is bounded on $L^p(w)$.*

Lemma 2. *Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \dots, k$ and $k \in N$. Then*

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \leq C \prod_{j=1}^k \|b_j\|_{BMO}$$

and

$$\left(\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy \right)^{1/r} \leq C \prod_{j=1}^k \|b_j\|_{BMO}.$$

Proof. Choose $1 < p_j < \infty$, $j = 1, \dots, m$ such that $1/p_1 + \dots + 1/p_m = 1$, we obtain, by the Hölder's inequality,

$$\begin{aligned} \frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy &\leq \prod_{j=1}^k \left(\frac{1}{|Q|} \int_Q |b_j(y) - (b_j)_Q|^{p_j} dy \right)^{1/p_j} \\ &\leq C \prod_{j=1}^k \|b_j\|_{BMO} \end{aligned}$$

and

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy \right)^{1/r} &\leq \prod_{j=1}^k \left(\frac{1}{|Q|} \int_Q |b_j(y) - (b_j)_Q|^{p_j r} dy \right)^{1/p_j r} \\ &\leq C \prod_{j=1}^k \|b_j\|_{BMO}. \end{aligned}$$

□

Lemma 3. Let $\mu > 3 + 1/n$. Then g_μ is bounded on $L^p(R^n)$, $1 < p < \infty$.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 , the following inequality holds:

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |g_\mu^{\tilde{b}}(f)(x) - C_0| dx \\ &\leq C \|b\|_{BMO} \left(M_r(f)(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_r(g_\mu^{\tilde{b}_{\sigma^c}}(f)(x)) \right). \end{aligned}$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Set $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{2Q^c}$.

We first consider the Case $m = 1$. Write

$$\begin{aligned} F_t^{b_1}(f)(x, y) &= (b_1(x) - (b_1)_{2Q})F_t(f)(y) \\ &\quad - F_t((b_1 - (b_1)_{2Q})f_1)(y) - F_t((b_1 - (b_1)_{2Q})f_2)(y). \end{aligned}$$

Then,

$$\begin{aligned} &|g_\mu^{b_1}(f)(x) - g_\mu(((b_1)_{2Q} - b_1)f_2)(x_0)| \\ &= \left| \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^{b_1}(f)(x, y) \right\| \right. \\ &\quad \left. - \left\| \left(\frac{t}{t + |x_0 - y|} \right)^{n\mu/2} F_t(((b_1)_{2Q} - b_1)f_2)(y) \right\| \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^{b_1}(f)(x, y) \right. \\
&\quad \left. - \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} F_t(((b_1)_{2Q} - b_1)f_2)(y) \right\| \\
&\leq \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} (b_1(x) - (b_1)_{2Q})F_t(f)(y) \right\| \\
&\quad + \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t((b_1 - (b_1)_{2Q})f_1)(y) \right\| \\
&\quad + \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t((b_1 - (b_1)_{2Q})f_2)(y) \right. \\
&\quad \left. - \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} F_t((b_1 - (b_1)_{2Q})f_2)(y) \right\| \\
&= A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by the Hölder's inequality with exponent $1/r + 1/r' = 1$, we get

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q A(x) dx \\
&= \frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_{2Q}| |g_\mu(f)(x)| dx \\
&\leq C \left(\frac{1}{|2Q|} \int_{2Q} |b_1(x) - (b_1)_{2Q}|^{r'} dx \right)^{1/r'} \left(\frac{1}{|Q|} \int_Q |g_\mu(f)(x)|^r dx \right)^{1/r} \\
&\leq C \|b_1\|_{BMO} M_r(g_\mu(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, choose $1 < p < r$, by the boundedness of g_μ on $L^p(R^n)$ and the Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q B(x) dx \\
&\leq \left(\frac{1}{|Q|} \int_{R^n} |g_\mu((b_1 - (b_1)_{2Q})f_1)(x)|^p dx \right)^{1/p} \\
&\leq C \left(\frac{1}{|Q|} \int_{2Q} |(b_1(x) - (b_1)_{2Q})f(x)|^p dx \right)^{1/p} \\
&\leq C \left(\frac{1}{|2Q|} \int_{2Q} |f(x)|^r dx \right)^{1/r} \left(\frac{1}{|2Q|} \int_{2Q} |b_1(x) - (b_1)_{2Q}|^{rp/(r-p)} dx \right)^{(r-p)/rp} \\
&\leq C \|b_1\|_{BMO} M_r(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, by the Minkowski's inequality and by the inequality $a^{1/2} - b^{1/2} \leq (a - b)^{1/2}$ for $a \geq b \geq 0$, we obtain

$$\begin{aligned}
& C(x) \\
& \leq \left[\int \int_{R_+^{n+1}} \left(\int_{(2Q)^c} \left| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} \right. \right. \right. \\
& \quad \left. \left. \left. - \left(\frac{t}{t + |x_0 - y|} \right)^{n\mu/2} \left| b_1(z) - (b_1)_{2Q} \right| |\psi_t(y - z)| |f(z)| dz \right|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2} \\
& \leq C \int_{(2Q)^c} \\
& \quad \times \left[\int \int_{R_+^{n+1}} \left(\frac{t^{n\mu/2} |x - x_0|^{1/2} |b_1(z) - (b_1)_{2Q}| |\psi_t(y - z)| |f(z)|}{(t + |x - y|)^{(n\mu+1)/2}} \right)^2 \frac{dy dt}{t^{n+1}} \right]^{1/2} dz \\
& \leq C \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)| |x - x_0|^{1/2} \\
& \quad \times \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \frac{t^{-n} dy dt}{(t + |y - z|)^{2n+2}} \right)^{1/2} dz,
\end{aligned}$$

note that

$$\begin{aligned}
& t^{-n} \int_{R^n} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \frac{dy}{(t + |y - z|)^{2n+2}} \\
& \leq t^{-n} \left(\int_{B(x,t)} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \frac{dy}{(t + |y - z|)^{2n+2}} \right. \\
& \quad \left. + \sum_{k=1}^{\infty} \int_{2^k B \setminus 2^{k-1} B} \left(\frac{t}{t + |x - y|} \right)^{n\mu+1} \frac{dy}{(t + |y - z|)^{2n+2}} \right) \\
& \leq Ct^{-n} \left(\int_{B(x,t)} \frac{2^{2n+2} dy}{(2t + |y - z|)^{2n+2}} \right. \\
& \quad \left. + \sum_{k=1}^{\infty} \int_{2^k B \setminus 2^{k-1} B} \left(\frac{t}{t + 2^{k-1} t} \right)^{n\mu+1} \frac{2^{(k+1)(2n+2)} dy}{(2^{k+1} t + |y - z|)^{2n+2}} \right) \\
& \leq Ct^{-n} \left(\int_{B(x,t)} \frac{dy}{(t + |x - z|)^{2n+2}} \right. \\
& \quad \left. + \sum_{k=1}^{\infty} 2^{(1-k)(n\mu+1)} \int_{2^k B} \frac{2^{(k+1)(2n+2)} dy}{(t + |x - z|)^{2n+2}} \right) \\
& \leq Ct^{-n} \left(t^n + \sum_{k=1}^{\infty} 2^{-k(n\mu+1)} 2^{k(2n+2)} (2^k t)^n \right) \frac{1}{(t + |x - z|)^{2n+2}}
\end{aligned}$$

$$\begin{aligned} &\leq C \left(1 + \sum_{k=1}^{\infty} 2^{k(3n-n\mu+1)} \right) \frac{1}{(t+|x-z|)^{2n+2}} \\ &\leq C \frac{1}{(t+|x-z|)^{2n+2}} \end{aligned}$$

and $|x-z| \sim |x_0 - z|$ for $x \in Q$ and $z \in R^n \setminus 2Q$. We obtain

$$\begin{aligned} &C \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)| |x - x_0|^{1/2} \\ &\quad \times \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu+1} \frac{t^{-n} dy dt}{(t+|y-z|)^{2n+2}} \right)^{1/2} dz \\ &\leq C \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)| |x - x_0|^{1/2} \left(\int_0^\infty \frac{dt}{(t+|x-z|)^{2n+2}} \right)^{1/2} dz \\ &\leq C \int_{(2Q)^c} \frac{|b_1(z) - (b_1)_{2Q}| |f(z)| |x - x_0|^{1/2}}{|x_0 - z|^{n+1/2}} dz \\ &\leq C \sum_{k=1}^{\infty} 2^{-k/2} |2^{k+1}Q|^{-1} \int_{2^{k+1}Q} |b_1(z) - (b_1)_{2Q}| |f(z)| dz \\ &\leq C \sum_{k=1}^{\infty} 2^{-k/2} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)|^r dz \right)^{1/r} \\ &\quad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_1(z) - (b_1)_{2Q}|^{r'} dz \right)^{1/r'} \\ &\leq C \sum_{k=1}^{\infty} 2^{-k/2} k \|b_1\|_{BMO} M_r(f)(\tilde{x}) \\ &\leq C \|b_1\|_{BMO} M_r(f)(\tilde{x}). \end{aligned}$$

Thus

$$\frac{1}{|Q|} \int_Q C(x) dx \leq C \|b_1\|_{BMO} M_r(f)(\tilde{x}).$$

Now, we consider the Case $m \geq 2$, we have known that, for $b = (b_1, \dots, b_m)$,

$$\begin{aligned} &F_t^{\vec{b}}(f)(x, y) \\ &= \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y-z) f(z) dz \\ &= \int_{R^n} \prod_{j=1}^m [(b_j(x) - (b_j)_{2Q}) - (b_j(z) - (b_j)_{2Q})] \psi_t(y-z) f(z) dz \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^m \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(z) - (b)_{2Q})_{\sigma^c} \psi_t(y-z) f(z) dz \\
&= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(y) \\
&\quad + (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f)(y) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(z) - (b)_{2Q})_{\sigma^c} \psi_t(y-z) f(z) dz \\
&= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(y) \\
&\quad + (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f)(y) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} c_{m,j} (b(x) - (b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x, y),
\end{aligned}$$

thus,

$$\begin{aligned}
&|g_\mu^{\tilde{b}}(f)(x) - g_\mu(((b_1)_{2Q} - b_1) \cdots ((b_m)_{2Q} - b_m) f_2)(x_0)| \\
&\leq \left\| \left(\frac{t}{t + |x-y|} \right)^{n\mu/2} F_t^{\tilde{b}}(f)(x, y) \right. \\
&\quad \left. - \left(\frac{t}{t + |x_0-y|} \right)^{n\mu/2} F_t(((b_1)_{2Q} - b_1) \cdots ((b_m)_{2Q} - b_m) f_2)(y) \right\| \\
&\leq \left\| \left(\frac{t}{t + |x-y|} \right)^{n\mu/2} (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(y) \right\| \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left\| \left(\frac{t}{t + |x-y|} \right)^{n\mu/2} (\tilde{b}(x) - (b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x, y) \right\| \\
&\quad + \left\| \left(\frac{t}{t + |x-y|} \right)^{n\mu/2} F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_1)(y) \right\| \\
&\quad + \left\| \left(\frac{t}{t + |x-y|} \right)^{n\mu/2} F_t \left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2 \right)(y) \right. \\
&\quad \left. - \left(\frac{t}{t + |x_0-y|} \right)^{n\mu/2} F_t \left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2 \right)(y) \right\| \\
&= I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}$$

For $I_1(x)$, by the Hölder's inequality with exponent $1/p_1 + \cdots + 1/p_m + 1/r = 1$, where $1 < p_j < \infty$, $j = 1, \dots, m$, we get

$$\frac{1}{|Q|} \int_Q I_1(x) dx$$

$$\begin{aligned}
&= \frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_{2Q}| \cdots |b_m(x) - (b_m)_{2Q}| |g_\mu(f)(x)| dx \\
&\leq \left(\frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_{2Q}|^{p_1} dx \right)^{1/p_1} \cdots \left(\frac{1}{|Q|} \int_Q |b_m(x) - (b_m)_{2Q}|^{p_m} dx \right)^{1/p_m} \\
&\quad \times \left(\frac{1}{|Q|} \int_Q |g_\mu(f)(x)|^r dx \right)^{1/r} \\
&\leq C \|\tilde{b}\|_{BMO} M_r(g_\mu(f))(\tilde{x}).
\end{aligned}$$

For $I_2(x)$, by the Minkowski's inequality and Lemma 1, we get

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q I_2(x) dx \\
&\leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_Q |(\tilde{b}(x) - (b)_{2Q})_\sigma| |g_\mu^{\tilde{b}_\sigma}(f)(x)| dx \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|2Q|} \int_{2Q} |(\tilde{b}(x) - (b)_{2Q})_\sigma|^{r'} dx \right)^{1/r'} \\
&\quad \times \left(\frac{1}{|Q|} \int_Q |g_\mu^{\tilde{b}_\sigma}(f)(x)|^r dx \right)^{1/r} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\tilde{b}_\sigma\|_{BMO} M_r(g_\mu^{\tilde{b}_\sigma}(f))(\tilde{x}).
\end{aligned}$$

For $I_3(x)$, choose $1 < p < r$, $1 < q_j < \infty$, $j = 1, \dots, m$ such that $1/p_1 + \cdots + 1/p_m + p/r = 1$, by the boundedness of g_μ on $L^p(R^n)$ and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q I_3(x) dx \\
&\leq \left(\frac{1}{|Q|} \int_{R^n} |g_\mu((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_1)(x)|^p dx \right)^{1/p} \\
&\leq C \left(\frac{1}{|Q|} \int_{R^n} |(b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q})|^p |f_1(x)|^p dx \right)^{1/p} \\
&\leq C \left(\frac{1}{|2Q|} \int_{2Q} |f(x)|^r dx \right)^{1/r} \\
&\quad \times \left(\frac{1}{|2Q|} \int_{2Q} |b_1(x) - (b_1)_{2Q}|^{p_1} dx \right)^{1/p_1} \cdots \\
&\quad \left(\frac{1}{|2Q|} \int_{2Q} |b_m(x) - (b_m)_{2Q}|^{p_m} dx \right)^{1/p_m}
\end{aligned}$$

$$\leq C\|\tilde{b}\|_{BMO}M_r(f_1)(\tilde{x}).$$

For $I_4(x)$, similar to the proof of $C(x)$ in Case $m = 1$, we obtain

$$I_4(x) \leq C \int_{(2Q)^c} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2)} \left| \prod_{j=1}^m (b_j(z) - (b_j)_{2Q}) \right| |f(z)| dz,$$

taking $1 < p_j < \infty$ $j = 1, \dots, m$ such that $1/p_1 + \dots + 1/p_m + 1/r = 1$, then, for $x \in Q$,

$$\begin{aligned} I_4(x) &\leq C \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2)} \\ &\quad \times \left| \prod_{j=1}^m (b_j(z) - (b_j)_{2Q}) \right| |f(z)| dz \\ &\leq C \sum_{k=1}^{\infty} 2^{-k/2} |2^{k+1}Q|^{-1} \int_{2^{k+1}Q} \left| \prod_{j=1}^m (b_j(z) - (b_j)_{2Q}) \right| |f(z)| dz \\ &\leq C \sum_{k=1}^{\infty} 2^{-k/2} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)|^r dz \right)^{1/r} \\ &\quad \times \prod_{j=1}^m \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_j(z) - (b_j)_{2Q}|^{p_j} dz \right)^{1/p_j} \\ &\leq C \sum_{k=1}^{\infty} k^m 2^{-km} \prod_{j=1}^m \|b_j\|_{BMO} M_r(f)(\tilde{x}) \\ &\leq C\|\tilde{b}\|_{BMO} M_r(f)(\tilde{x}), \end{aligned}$$

thus

$$\frac{1}{|Q|} \int_Q I_4(x) dx \leq C\|\tilde{b}\|_{BMO} M_r(f)(\tilde{x}).$$

This completes the proof of the theorem. \square

Proof of Theorem 2. Choose $1 < r < p$ in Theorem 1 and by using Lemma 1, we may get the conclusion of Theorem 2. This finishes the proof. \square

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