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OSCULATORY WFI-ALGEBRAS

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Abstract. The notions of mote, beam and osculatory WFI-algebra are introduced, and several properties are investigated. Relations between osculatory WFI-algebra and associative WFI-algebra are provided. Characterizations of osculatory WFI-algebra are given.

1. Introduction

In 1990, W. M. Wu [7] introduced the notion of fuzzy implication algebra (FI-algebra, for short), and investigated several properties. In [6], Z. Li and C. Zheng introduced the notion of distributive (resp. regular, commutative) FI-algebra, and investigated the relations between such FI-algebra and MV-algebra. In [1], Y. B. Jun discussed several aspects of WFI-algebra. He introduced the notion of associative (resp. normal, medial) WFI-algebra, and investigated several properties. He gave conditions for a WFI-algebra to be associative/medial, and provided characterizations of associative/medial WFI-algebra, and showed that every associative WFI-algebra is a group in which every element is an involution. He also verified that the class of all medial WFI-algebras is a variety. Y. B. Jun and S. Z. Song [5] introduced the notions of simulative and/or mutant WFI-algebra and investigated some properties. They established characterizations of a simulative WFI-algebra, and gave a relation between an associative WFI-algebra and a simulative WFI-algebra. They also found some types for a simulative WFI-algebra to be mutant. Jun et al. [4] introduced the concept of ideals of WFI-algebra, and gave relations between a filter and an ideal. Moreover, they provided characterizations of an ideal, and established an extension property for an ideal. In [2] and [3], the present author discussed perfect (resp. weak and concrete) filters of WFI-algebra. In this paper, we introduce the notions of mote, beam and osculatory WFI-algebra, and investigate several properties. We give relations between osculatory WFI-algebra and associative WFI-algebra. We provide characterizations of osculatory WFI-algebra.

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2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *WFI-algebra* we mean a system $\mathfrak{X} := (X, \ominus, 1) \in K(\tau)$ in which the following axioms hold:

(a1) $(x \in X)\ (x \ominus x = 1)$,
(a2) $(x, y \in X)\ (x \ominus y = y \ominus x = 1 \Rightarrow x = y)$,
(a3) $(x, y, z \in X)\ (x \ominus (y \ominus z) = y \ominus (x \ominus z))$,
(a4) $(x, y, z \in X)\ ((x \ominus y) \ominus \big((y \ominus z) \ominus (x \ominus z)\big) = 1)$.

We call the special element 1 the *unit*. For the convenience of notation, we shall write $[x, y_1, y_2, \ldots, y_n]$ for

$$(\cdots((x \ominus y_1) \ominus y_2) \ominus \cdots) \ominus y_n.$$

We define $[x, y]^0 = x$, and for $n > 0$, $[x, y]^n = [x, y, y, \ldots, y]$, where y occurs n-times. We use the notation $x^n \ominus y$ instead of $x \ominus (\cdots (x \ominus (x \ominus y)) \cdots)$ in which x occurs n-times.

Proposition 2.1 ([1]). *In a WFI-algebra $\mathfrak{X}$, the following are true:*

(b1) $x \ominus [x, y]^2 = 1$,
(b2) $1 \ominus x = 1 \Rightarrow x = 1$,
(b3) $1 \ominus x = x$,
(b4) $x \ominus y = 1 \Rightarrow [y, z, x \ominus z] = 1\ \&\ [z, x, z \ominus y] = 1$,
(b5) $[x, y, 1] = [x, 1, y \ominus 1]$,
(b6) $[x, y]^3 = x \ominus y$.

A nonempty subset S of a WFI-algebra $\mathfrak{X}$ is called a *subalgebra* of $\mathfrak{X}$ if $x \ominus y \in S$ whenever $x, y \in S$. A nonempty subset F of a WFI-algebra $\mathfrak{X}$ is called a *filter* of $\mathfrak{X}$ if it satisfies:

(c1) $1 \in F$,
(c2) $(\forall x \in F)\ (\forall y \in X)\ (x \ominus y \in F \Rightarrow y \in F)$.

A filter F of a WFI-algebra $\mathfrak{X}$ is said to be *closed* [1] if F is also a subalgebra of $\mathfrak{X}$.

Proposition 2.2 ([1]). *Let F be a filter of a WFI-algebra $\mathfrak{X}$. Then F is closed if and only if $x \ominus 1 \in F$ for all $x \in F$.*

Proposition 2.3 ([1]). *In a finite WFI-algebra, every filter is closed.*

We now define a relation "$\preceq$" on $\mathfrak{X}$ by $x \preceq y$ if and only if $x \ominus y = 1$. It is easy to verify that a WFI-algebra is a partially ordered set with respect to $\preceq$. A WFI-algebra $\mathfrak{X}$ is said to be *associative* [1] if it satisfies $[x, y, z] = x \ominus (y \ominus z)$ for all $x, y, z \in X$. For a WFI-algebra $\mathfrak{X}$, the set

$$\mathcal{S}(\mathfrak{X}) := \{x \in X \mid x \preceq 1\}$$

is called the *simulative part* of $\mathfrak{X}$. A WFI-algebra $\mathfrak{X}$ is said to be *simulative* [5] if it satisfies

(S) $x \preceq 1 \Rightarrow x = 1$.

Note that the condition (S) is equivalent to $\mathcal{S}(\mathfrak{X}) = \{1\}$.

Proposition 2.4 ([5]). *The simulative part $\mathcal{S}(\mathfrak{X})$ of a WFI-algebra $\mathfrak{X}$ is a filter of $\mathfrak{X}$.*

3. Osculatory WFI-algebra

We begin with the following definition.

Definition 3.1. A WFI-algebra $\mathfrak{X}$ is said to be *osculatory* if it satisfies:

$$(\forall x, y \in X)\,(y \ominus x = 1 \;\Rightarrow\; [x, y]^2 = x). \tag{3.1}$$

Example 3.2. Let $X = \{1, a, b, c\}$ be a set with the following Cayley table.

$\ominus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	c	c	1	1
c	c	b	a	1

Then $\mathfrak{X} := (X, \ominus, 1)$ is an osculatory WFI-algebra.

Let $\mathfrak{X}$ be a WFI-algebra. Consider the following equation:

$$[x, y]^2 = [y, x, [y, x]^2]. \tag{3.2}$$

Example 3.3. Let $X = \{1, a, b\}$ be a set with the following Cayley table.

$\ominus$	1	a	b
1	1	a	b
a	1	1	b
b	b	b	1

Then $\mathfrak{X} := (X, \ominus, 1)$ is a WFI-algebra which satisfies the equation (3.2). But $\mathfrak{X}$ is not associative since $[a, a]^2 = a \neq 1 = a^2 \ominus a$.

Example 3.4. Let $X = \{1, a, b\}$ be a set with the following Cayley table.

$\ominus$	1	a	b
1	1	a	b
a	b	1	a
b	a	b	1

Then $\mathfrak{X} := (X, \ominus, 1)$ is a WFI-algebra which does not satisfy the equation (3.2) since $[a, b]^2 = a \neq 1 = [b, a, [b, a]^2]$.

Lemma 3.5 ([1]). *Let $\mathfrak{X}$ be a WFI-algebra. Then the following are equivalent.*

(i) *$\mathfrak{X}$ is associative.*
(ii) $(\forall x \in X)\ (x \ominus 1 = x)$.
(iii) $(\forall x, y \in X)\ (x \ominus y = y \ominus x)$.

Proposition 3.6. *Every associative WFI-algebra satisfies the equation* (3.2).

Proof. Let $\mathfrak{X}$ be an associative WFI-algebra and let $x, y \in X$. Using the associativity of $\mathfrak{X}$, (a1), (b3) and Lemma 3.5, we have

$$\begin{aligned}[y, x, [y, x]^2] &= [y, x, y \ominus x, x] = 1 \ominus x = x \\ &= x \ominus 1 = x \ominus (y \ominus y) = [x, y]^2.\end{aligned}$$

This completes the proof. □

Proposition 3.7. *If a WFI-algebra $\mathfrak{X}$ satisfies the equation* (3.2), *then $\mathfrak{X}$ is osculatory.*

Proof. Let $x, y \in X$ be such that $y \ominus x = 1$. Using (b3), it is straightforward. □

Corollary 3.8. *Every associative WFI-algebra is osculatory.*

Definition 3.9. If an element m of a WFI-algebra $\mathfrak{X}$ is maximal in $(\mathfrak{X}, \preceq)$, we say that m is a *mote* of $\mathfrak{X}$.

Denote by $M(\mathfrak{X})$ the set of all motes of $\mathfrak{X}$. For any $m \in M(\mathfrak{X})$, the set

$$B(m) := \{x \in X \mid x \ominus m = 1\}$$

is called a *beam* of $\mathfrak{X}$ with respect to m (briefly, m-*beam* of $\mathfrak{X}$). Obviously, $1 \in M(\mathfrak{X})$ and $B(1) = \mathcal{S}(\mathfrak{X})$.

Example 3.10. Let $X = \{1, a, b, c, d\}$ be a set with the following Cayley table.

$\ominus$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	c
c	c	c	d	1	b
d	c	c	c	1	1

Then $\mathfrak{X} := (X, \ominus, 1)$ is a WFI-algebra and $M(\mathfrak{X}) = \{1, c\}$. Hence $B(1) = \{1, a, b\}$ and $B(c) = \{c, d\}$.

Proposition 3.11. *Every mote m of a WFI-algebra $\mathfrak{X}$ is represented by the following identity:*

$$(\forall x \in X)\,(m = [m, x]^2). \tag{3.3}$$

Proof. Since $m \ominus [m, x]^2 = 1$ for all $x \in X$, we have $m = [m, x]^2$ for all $x \in X$. □

We give a condition for an element of $\mathfrak{X}$ to be a mote of $\mathfrak{X}$.

Theorem 3.12. *If an element m of a WFI-algebra $\mathfrak{X}$ satisfies the following equation:*

$$(\forall x, y \in X)\,(x \ominus m = [x, m, y, y]) \tag{3.4}$$

then m is a mote of $\mathfrak{X}$.

Proof. Let $x \in X$ be such that $m \ominus x = 1$. Then

$$m = 1 \ominus m = [1, m, x, x] = [m, x]^2 = 1 \ominus x = x,$$

and so m is a mote of $\mathfrak{X}$. ◻

The following is more simple condition for an element to be a mote.

Theorem 3.13. *If an element m of a WFI-algebra $\mathfrak{X}$ satisfies the following identity:*

$$[m, 1]^2 = m, \tag{3.5}$$

then m is a mote of $\mathfrak{X}$.

Proof. Using (a3), (b1), (b4), (b5) and (3.5), we have

$$[m, x, 1] = [m, 1, x \ominus 1] = x \ominus [m, 1]^2 = x \ominus m \tag{3.6}$$

for all $x \in X$, and

$$[x, m, 1, 1] = [x, 1, m \ominus 1] \ominus 1 = [x, 1]^2 \ominus [m, 1]^2 = [x, 1]^2 \ominus m \preceq x \ominus m. \tag{3.7}$$

for all $x \in X$. Using (b1), we have $x \ominus m \preceq [x, m, 1, 1]$. Hence

$$x \ominus m = [x, m, 1, 1], \tag{3.8}$$

and so

$$\begin{aligned} [x, m, y, y] &\preceq [y, 1, [x, m, y, 1]] = [y, [x, m, y], 1] \\ &= [x, m, y \ominus y, 1] = [x, m, 1, 1] = x \ominus m \end{aligned} \tag{3.9}$$

for all $x, y \in X$. Combining (a2), (b1) and (3.9), we get

$$x \ominus m = [x, m, y, y] \tag{3.10}$$

for all $x, y \in X$. It follows from Theorem 3.12 that m is a mote of $\mathfrak{X}$. ◻

In the following theorem, we show that every mote m of a WFI-algebra $\mathfrak{X}$ satisfies the condition (3.5).

Theorem 3.14. *Every mote m of a WFI-algebra $\mathfrak{X}$ satisfies the condition* (3.5).

Proof. Let m be a mote of a WFI-algebra $\mathfrak{X}$. Using (a3) and (3.3), we obtain

$$[m, x, y \ominus x] = y \ominus [m, x]^2 = y \ominus m \tag{3.11}$$

for all $x, y \in X$. Using (b1), (b4) and (3.11), we have

$$[m, x, z] \preceq [m, x, [z, x]^2] = [z, x, m]. \tag{3.12}$$

It follows from (a3) and (b4) that

$$[m, x, y \ominus z] = y \ominus [m, x, z] \preceq y \ominus [z, x, m] = [z, x, y \ominus m]. \tag{3.13}$$

Using (a1), (b5) and (3.13), we get

$$[m, 1, x \ominus 1] = [m, x, 1] = [m, x, 1 \ominus 1] \preceq [1, x, 1 \ominus m] = x \ominus m.$$

Obviously, $x \ominus m \preceq [m, 1, x \ominus 1]$, and hence

$$[m, 1, x \ominus 1] = x \ominus m. \tag{3.14}$$

This implies that $[m, 1]^2 = [m, 1, 1 \ominus 1] = 1 \ominus m = m$. This completes the proof. □

Corollary 3.15. *If m is a mote of a WFI-algebra $\mathfrak{X}$, then*

$$(\forall x \in X)\,([x \ominus m, 1]^2 = [x, 1]^2 \ominus m). \tag{3.15}$$

Proof. Using (b5) and Theorem 3.14, we have

$$[x \ominus m, 1]^2 = [x, 1]^2 \ominus [m, 1]^2 = [x, 1]^2 \ominus m.$$

This completes the proof. □

Corollary 3.16. *If p and q are motes of a WFI-algebra $\mathfrak{X}$, then so is $p \ominus q$.*

Proof. Let p and q be motes of a WFI-algebra $\mathfrak{X}$. Then $[p, 1]^2 = p$ and $[q, 1]^2 = q$. Hence

$$[p \ominus q, 1]^2 = [p, 1]^2 \ominus [q, 1]^2 = p \ominus q,$$

and so $p \ominus q$ is a mote of $\mathfrak{X}$ by Theorem 3.13. □

Proposition 3.17. *For any element x of a WFI-algebra $\mathfrak{X}$, the element $[x, 1]^2$ is a mote of $\mathfrak{X}$.*

Proof. Let $x \in X$ and $m = [x, 1]^2$. Then

$$[m, 1]^2 = [[x, 1]^2, 1]^2 = [x, 1]^3 \ominus 1 = [x, 1]^2 = m.$$

It follows from Theorem 3.13 that $[x, 1]^2$ is a mote of $\mathfrak{X}$. □

Theorem 3.18. *A WFI-algebra $\mathfrak{X}$ is osculatory if and only if it satisfies the following identity:*

$$(\forall x, y \in X)\,([[x, y]^2, x]^2 = [x, y]^2). \tag{3.16}$$

Proof. Assume that a WFI-algebra $\mathfrak{X}$ is osculatory. Since $x \ominus [x, y]^2 = 1$, it follows from (3.1) that

$$[x, y]^2 = [[x, y]^2, x]^2$$

which proves (3.16). Now let $\mathfrak{X}$ be a WFI-algebra in which the identity (3.16) is valid. Let $x, y \in X$ be such that $y \ominus x = 1$. Using (b3) and (3.16), we have

$$[x, y]^2 = [1 \ominus x, y]^2 = [[y, x]^2, y]^2 = [y, x]^2 = 1 \ominus x = x.$$

Hence $\mathfrak{X}$ is osculatory. □

Lemma 3.19. *For any motes p and q of $\mathfrak{X}$, we have*

(i) $(\forall x, y \in X)\ (x \in B(p)\ \&\ y \in B(q) \Rightarrow x \ominus y \in B(p \ominus q))$.
(ii) $(\forall x, y \in B(p))\ ([x, y, 1] = 1)$.

Proof. (i) Let $x \in B(p)$ and $y \in B(q)$. Then $x \ominus p = 1$ and $y \ominus q = 1$. Hence

$$\begin{aligned}[x, y, p \ominus q] &= [x, y, [p \ominus q, 1]^2] = [p, q, 1] \ominus [x, y, 1] \\ &= [p, q, 1] \ominus [x, 1, y \ominus 1] = [x, 1, y \ominus [p \ominus q, 1]^2] \\ &= [x, 1, y \ominus (p \ominus q)] = [x, 1, p \ominus (y \ominus q)] \\ &= [x, 1, p \ominus 1] = [x, p, 1] = 1 \ominus 1 = 1,\end{aligned}$$

and so $x \ominus y \in B(p \ominus q)$.

(ii) It is direct consequence of (i). □

Proposition 3.20. *Let $\mathfrak{X}$ be an osculatory WFI-algebra. Then*

$$(\forall m \in M(\mathfrak{X}))\, (\forall x, y \in X)\, (x, y \in B(m) \Rightarrow [x, y]^2 = [y, x]^2). \tag{3.17}$$

Proof. Let $m \in M(\mathfrak{X})$ and $x, y \in B(m)$. Then $[x, y, 1] = 1$ by Lemma 3.19(ii). Using Theorem 3.18, (a3) and (b6), we have

$$\begin{aligned}[y, x]^2 \ominus [x, y]^2 &= [y, x]^2 \ominus [[x, y]^2, x]^2 = [[x, y]^2, x, [y, x]^3] \\ &= [[x, y]^2, x, y \ominus x] = y \ominus [[x, y]^2, x]^2 \\ &= y \ominus [x, y]^2 = [x, y, y \ominus y] \\ &= [x, y, 1] = 1.\end{aligned}$$

Similarly, $[x, y]^2 \ominus [y, x]^2 = 1$. This completes the proof. □

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