Commun. Korean Math. Soc. **23** (2008), No. 1, pp. 29–40

ON SOME CLASSES OF REGULAR ORDER SEMIGROUPS

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Abstract. Here, some classes of regular order semigroups are discussed. We shall consider that the problems of the existences of (multiplicative) inverse ${}^{\delta}$po-transversals for such classes of po-semigroups and obtain the following main results: (1) Giving the equivalent conditions of the existence of inverse ${}^{\delta}$po-transversals for regular order semigroups (2) showing the order orthodox semigroups with biggest inverses have necessarily a weakly multiplicative inverse ${}^{\delta}$po-transversal. (3) If the Green's relation $\mathcal{R}$ and $\mathcal{L}$ are strongly regular (see. sec.1), then any principally ordered regular semigroup (resp. ordered regular semigroup with biggest inverses) has necessarily a multiplicative inverse ${}^{\delta}$po-transversal. (4) Giving the structure theorem of principally ordered semigroups (resp. ordered regular semigroups with biggest inverses) on which $\mathcal{R}$ and $\mathcal{L}$ are strongly regular.

In T. S. Blyth and G. A. Pinto ([1]-[3]), the following concepts have been introduced and discussed.

A negative (resp. positive) ordered regular semigroup S means that S is an order semigroup (i.e. po-semigroup) in which for any $x \in S$ there is $s \in S$ such that $xsx \leq x$ (resp. $x \leq xsx$). A (negative) order regular semigroup S is said to be principally ordered, for short POR-semigroup S, if for any $x \in S$ there exists

$$x^* = \max\{y \in S \mid xyx \leq x\}.$$

We refer the reader to [1, 2], if S is a POR-semigroup, then every $x \in S$ has a biggest inverse, namely the element $x^0 = x^*xx^* \in V(x)$ (which is the inverses set of x). Thus S becomes an ordered regular semigroup with biggest inverses (for short ORB-semigroup S) (see [3]). In this case we always denote the set of all biggest inverse of S by S^0, i.e., S^0=$\{x^0 \mid \forall x \in S\}$.

Conversely, an ORB-semigroup S is necessarily not a POR-semigroup. For example, we can prove that a naturally ordered ORB-semigroup $(S, \leq)$ on which $\mathcal{R}$ and $\mathcal{L}$ are regular (see [7]) can not become a POR-semigroup. It need only to notice mapping $o : S \longrightarrow S^o$ denoted by $x \longrightarrow x^o$ is always antitone on a naturally ordered POR-semigroup $(S, \leq)$ (see [2, Theorem 3.3] and [1]).

Received September 25, 2007; Revised December 18, 2007.

2000 *Mathematics Subject Classification.* 06F05.

Key words and phrases. regular order semigroup, inverse ${}^{\delta}$po-transversals, POR-semigroups, ORB-semigroups.

This is in contradiction with the mapping o being isotone on $E(S)$ (which is the set of all idempotents of S).

Let S be a po-semigroup. If S is also regular (resp. orthodox, inverse and so on) then we call that S is a regular (resp. orthodox,inverse and so on) po-semigroup. Clearly, the POR- and ORB-semigroups are regular po-semigroups.

Let T be a po-subsemigroup of the regular po-semigroup S. If there is a surjective mapping δ from S to T denoted by $x \longrightarrow x^{\delta}$ such that

(R1) $(\forall x \in S)$ $x^{\delta} \in V_T(x)$(where $V_T(x) = V(x) \cap T$)

(O1) $(\forall x \in S)$ $|V_T(x)| = 1$ then T is called an inverse ${}^{\delta}$po-transversal of S. If T satisfies (R1),(O1) and

(O2) $(\forall x, y \in S)$ $x^{\delta}xyy^{\delta} \in E(T)$ (resp.$(x^{\delta}xyy^{\delta})^{\delta} \in E(T)$)

then S is said to be multiplicative (resp.weakly multiplicative). If T satisfies (R1), (O1) and

(O3) the maximum idempotent-separating congruence on S is the identity congruence on T,

then T is called a fundamental inverse ${}^{\delta}$po-transversal of S.

Clearly, any inverse ${}^{\delta}$po-transversal is an inverse transversal.

Let S be a ORB-semigroup. Green's relation $\mathcal{R}$ and $\mathcal{L}$ are called strongly regular if (S1) and (S2) hold :

(S1) $\mathcal{R}$ and $\mathcal{L}$ are regular on S (i.e. $x \le y \Longrightarrow xx^o \le yy^o, x^o x \le y^o y$)

(S2) $(\forall e, f \in E(S))$ $e \le_n f \Longrightarrow ee^o \le ff^o$ and $e^o e \le f^o f$

where the order "$\le_n$" is the natural order on $E(S)$.

Example 1. Using the method in [1, Example 1], let $k > 1$ be a fixed integer and for every positive integer n let n_k denote the biggest multiple of k that is less than or equal to n. Then integer set Z becomes a regular semigroup under the operation $+_k$ defined by $m +_k n = m + n_k$; Let $(L, \wedge)$ be a semilattice and $L^{[2]} = \{(x, y) \in L \times L \mid y \le x\}$. With the Cartesian order, let $M = L^{[2]} \times Z$. A typical element $((x, y), p)$ of M will be denoted by $[x, y, p]$. Define a binary operation on M by

$$[x, y, p][a, b, q] = [x \wedge a, y \wedge b, p + q_k].$$

Then M becomes a POR-semigroup and we have

$$[x, y, p]^* = [x, x, -p_k + k - 1]$$

$$[x, y, p]^0 = [x, y, (-p_k + k - 1)_k + k - 1];$$

$$E(M) = \{[x, y, p] \in M \mid \forall (x, y) \in L^{[2]}, p \in Z \text{ such that } p_k = 0\}.$$

Computing we know that $E(M)$ is a left zero subband of M (i.e. $E(M)$ is a subsemigroup of all idempotents of S). So M is orthodox. On M, Green's relation $\mathcal{R}$ and $\mathcal{L}$ are strongly regular. In fact, since the mapping k :$p \longrightarrow p_k$

is surjective on Z and has the property : if $p \leq q$ then $p + p_k \leq q + q_k$. So we have that if $[x, y, p] \leq [a, b, q]$ then

$$\begin{array}{rcl} [x,y,p][x,y,p]^0 & = & [x,y,p+((-p_k+k-1)_k+k-1)_k] \\ & \leq & [a,b,q+((-q_k+k-1)_k+k-1)_k] \\ & = & [a,b,q][a,b,q]^0. \end{array}$$

Similarly $[x, y, p]^0[x, y, p] \leq [a, b, q]^0[a, b, q]$, that is, $\mathcal{R}$ and $\mathcal{L}$ are regular. Let $[x, y, p], [a, b, q] \in E(M)$, and $[x, y, p] \leq_n [a, b, q]$, notice $p_k = q_k = 0$ and

$$[x, y, p] = [x, y, p][a, b, q] = [a, b, q][x, y, p] = [x, y, q]$$

so $p = q$. Thus we have

$$[x, y, p][x, y, p]^0 = [x, y, p][x, y, k-1] = [x, y, p] \leq [a, b, p] = [a, b, p][a, b, p]^0.$$

Similarly $[x, y, p]^0[x, y, p] \leq [a, b, p]^0[a, b, p]$, that is, (S2) holds, which shows that $\mathcal{R}$ and $\mathcal{L}$ are strongly regular.

The inverse transversal of regular semigroups was first introduced by Blyth and McFadden in [4]. In structure description of regular semigroup S, the inverse transversal S^δ play very important roles. Thus we think that the existence of an inverse $^\delta$transversal for a regular semigroup is of course very important problem. In this note, we study first the existence condition of inverse $^\delta$po-transversals. In Section 1, we obtain necessary and sufficient conditions for a po-subsemigroup T of the regular po-semigroup S to be an inverse $^\delta$po-transversal. Then in Section 2, we show that any order orthodox semigroup with biggest inverses S has necessarily a weakly multiplicative inverse $^\delta$po-transversal S^δ with the mapping δ is an anti-homomorphism. For an ORB-semigroup S on which $\mathcal{R}$ and $\mathcal{L}$ are strongly regular then we show that S has necessarily a multiplicative inverse $^\delta$po-transversal. Hence, in Section 3, we give immediately the structure theorem of the POR-(resp. ORB-)semigroup S on which $\mathcal{R}$ and $\mathcal{L}$ are strongly regular.

1. The existence conditions of inverse $^\delta$po-transversals

Since regular po-semigroups (resp. POR- and ORB-) are also the class of regular semigroups, so we list here some basic facts on a regular semigroup with inverse transversal used in this note, the reader can consult [4]-[7] for more details.

Lemma 1.1. *Let S be a regular semigroup with inverse transversal S^0. Then we have*

(1) $(\forall x, y \in S)(xy^o)^o = y^{oo}x^o$ *and* $(y^o x)^o = x^o y^{oo}$;
(2) $(\forall x, y \in S)(xy)^o = (x^o xy)^o x^o = y^o(xyy^o)^o = y^o(x^o xyy^o)^o x^o$;
(3) $(\forall x, y \in S)$ $x\ \mathcal{L}\ y$ *if and only if* $x^o x = y^o y$; $x\mathcal{R}y$ *if and only if* $xx^o = yy^o$;
(4) $I = \{xx^o \mid x \in S\}$ *and* $\wedge = \{x^o x \mid x \in S\}$ *are subbands of* S;
(5) $(\forall x \in S)$ $x^o = x^{ooo}$;

(6) *S^o is multiplicative if and only if S^o is a quasi-ideal of S (i.e, $S^oSS^o \subseteq S^o$) and $e^o \in E(S^o)$ for every $e \in E(S)$;*

(7) *If S is orthodox then $e^0 \in E(S)$ for any $e \in E(S)$ and S^0 is weakly multiplicative.*

We shall give the existence conditions of inverse ${}^{\delta}$po-transversals for the class of regular po-semigroups.

Theorem 1.2. *Let S^δ be a po-subsemigroup of the regular po-semigroup S with the mapping δ is an anti-homomorphism from S to S^δ denoted by $x \longmapsto x^\delta$ such that $x^\delta \in V_{S^\delta}(x)$. The following conditions are equivalent:*

(1) *S^δ is an inverse ${}^{\delta}$po-transversal of S;*

(2) *The equivalent relation*

$$\nu = \{(x, y) \in S \times S \mid V_{S^\delta}(x) = V_{S^\delta}(y)\}$$

is the smallest inverse semigroup congruence on S and the mapping $\delta\delta$ is a morphism on S^δ.

Proof. (1) $\Longrightarrow$(2) Let S^δ be an inverse ${}^{\delta}$po-transversal of S then S^δ is an inverse transversal of S as a regular semigroup. By Lemma 1.1 (5) $x^\delta = x^{\delta\delta\delta}$ for any $x \in S$. Notice that $|V_{S^\delta}(x)| = 1$ and $x^\delta \in V_{S^\delta}(x)$ for any $x \in S$. So we have that $(x, y) \in \nu$ if and only if $V_{S^\delta}(x) = V_{S^\delta}(y)$ if and only if $V_{S^\delta}(x^{\delta\delta}) = V_{S^\delta}(y^{\delta\delta})$ if and only if $x^\delta = y^\delta$ if and only if $x^{\delta\delta} = y^{\delta\delta}$. We denote the ν-class of containing x by $(x)_\nu$.

If $(x, y) \in \nu$ and $z \in S$ then $x^\delta = y^\delta$ and $z^\delta \in V_{S^\delta}(z)$. It follows from δ being an anti-homomorphism from S to S^δ then $(zx)^\delta = x^\delta z^\delta = y^\delta z^\delta = (zy)^\delta$, that is, $(zx, zy) \in \nu$. Similarly $(xz, yz) \in \nu$ and thus ν is a congruence on S.

To show that S/ν is inverse, notice first that it is certainly regular, since any homomorphic image of a regular semigroup is regular. Now, by Lallement's Lemma in [6, II.4.6] any idempotent of S/ν is of the form eν denoted by $e\nu = (e)_\nu$ for $e \in E(S)$. For any $e, f \in E(S), e^\delta \in V_{S^\delta}(e)$, $f^\delta \in V_{S^\delta}(f)$, by S^δ being inverse and δ being antihomomorphic we have $(ef)^\delta = f^\delta e^\delta = e^\delta f^\delta = (fe)^\delta$, that is, $(ef)_\nu = (fe)_\nu$ and $(e)_\nu(f)_\nu = (ef)_\nu = (fe)_\nu = (f)_\nu(e)_\nu$. Thus that S/ν is inverse.

Nextly, suppose that ξ is an inverse semigroup congruence on S. Let $(x, y) \in \nu$ for $x^\delta \in V_{S^\delta}(x) = V_{S^\delta}(y)$ and $(x^\delta)_\xi \in S/\xi$. Then it is clear that $(x^\delta)_\xi^{-1} = (x)_\xi$ and $(x^\delta)_\xi^{-1} = (y)_\xi$ by $x^\delta = y^\delta$. Thus we obtain that $(x)_\xi = (y)_\xi$, that is, $(x, y) \in \xi$. Therefore $\nu \subseteq \xi$ and so ν is as stated in (2), the smallest inverse semigroup congruence on S.

Finally, let $x^\delta, y^\delta \in S^\delta$ since S^δ is an inverse ${}^{\delta}$po-transversal of S so that $x^\delta = y^\delta$ if and only if $(x^\delta)^{\delta\delta} = (y^\delta)^{\delta\delta}$ by Lemma 1.1 (5), that is, the mapping $\delta\delta|_{S^\delta}$ is a morphism.

(2)$\Longrightarrow$(1) It need only to show that $|V_{S^\delta}(x)| = 1$ for any $x \in S$, because this means S^δ is an inverse ${}^{\delta}$po-transversal of S by Definition in Sec 1. Let the equivalence relation

$$\tau = \{(x, y) \in S \times S \mid x^\delta = y^\delta\}.$$

Then τ is a congruence on S by δ being anti-homomorphic and so $\xi = \tau \cap \nu$ is also. Clearly $(x)_\xi \subseteq (x)_\nu$ and $(x)_\nu = \bigcup_{y\in(x)_\nu}(y)_\xi$ for any $(x)_\nu \in S/\nu$. Since S/ν is inverse so there is $(y)_\xi^{-1} \in S/\xi$ such that $(y)_\xi^{-1}$ is the unique inverse of $(x)_\xi$ for any $(x)_\xi \in S/\xi$ and $y \in (x)_\nu$. Thus we show that S/ξ is also inverse. By the smallest property of ν we have $\xi = \nu$, that is, $\nu = \tau \cap \nu$.

Now, suppose that $y^\delta \in V_{S^\delta}(x)$ then $(x^\delta)_\nu$ and $(y^\delta)_\nu$ are both inverses in S/ν of $(x)_\nu$. By uniqueness of inverse in S/ν we conclude that $(x^\delta)_\nu = (y^\delta)_\nu$ and so $(x^\delta)^\delta = (y^\delta)^\delta$ by $\nu = \tau \cap \nu$. Thus we imply that $x^\delta = (x^\delta)^{\delta\delta} = (x^{\delta\delta})^\delta = (y^{\delta\delta})^\delta = (y^\delta)^{\delta\delta} = y^\delta$ by $\delta\delta|_{S^\delta}$ being a morphism, which shows that $|V_{S^\delta}(x)| = 1$ for any $x \in S$. □

Corollary 1.3. *Let S^δ be a po-subsemigroup of the ORB-(resp.POR-) semigroup S with the mapping δ is an anti-homomorphism from S to S^δ such that $x^\delta \in V_{S^\delta}(x)$ for any $x \in S$. The following conditions are equivalent;*

(1) *S^δ is an inverse ${}^\delta$po-transversal of S;*

(2) *The equivalence relation*

$$\nu = \{(x, y) \in S \times S \mid V_{S^\delta}(x) = V_{S^\delta}(y)\}$$

is the smallest inverse semigroup congruence on S and the mapping $\delta\delta$ is a morphism on S^δ.

Theorem 1.4. *Let S^δ be a po-subsemigroup of the regular po-semigroup S with mapping δ is surjective from S to S^δ such that $x^\delta \in V_{S^\delta}(x)$ for any $x \in S$. Then the following statements are equivalent:*

(1) *S^δ is a fundamental inverse ${}^\delta$po-transversal of S;*

(2) *The equivalent relation*

$$\mu = \{(x, y) \in \mathcal{H} \mid x^\delta e x^{\delta\delta} = y^\delta e y^{\delta\delta} \textit{ for any } e \in E(S^\delta)\}$$

has the following properties :

(μ1) *$\mu_\delta = \mu \cap (S^\delta \times S^\delta)$ is the maximum idempotent separating congruence on S^δ and μ_δ is the identity congruence on S^δ;*

(μ2) *μ is the smallest inverse semigroup congruence on S.*

Proof. (1)$\Longrightarrow$(2) Let S^δ be as stated in (1) and $(x, y) \in \mu$, by $(x, y) \in \mathcal{H}$ then $xx^\delta = yy^\delta$ and $x^\delta x = y^\delta y$ by Lemma 1.1 (3). Now, for any $z \in S, e \in E(S^\delta)$, we compute that

$$\begin{aligned}
(zx)^\delta e(zx)^{\delta\delta} &= x^\delta(zxx^\delta)^\delta e(x^\delta(zxx^\delta)^\delta)^\delta && \text{(by Lemma 1.1 (2))}\\
&= x^\delta(zxx^\delta)^\delta e(zxx^\delta)^{\delta\delta}x^{\delta\delta} && \text{(by Lemma 1.1 (1))}\\
&= x^\delta(zyy^\delta)^\delta e(zyy^\delta)^{\delta\delta}x^{\delta\delta} && \text{(by } xx^\delta = yy^\delta)\\
&= y^\delta(zyy^\delta)^\delta e(zyy^\delta)^{\delta\delta}y^{\delta\delta} && \text{(by } (zyy^\delta)^\delta e(zyy^\delta)^{\delta\delta} \in E(S^\delta))\\
&= (zy)^\delta e(zy)^{\delta\delta}
\end{aligned}$$

that is, $(zx, zy) \in \mu$. Similarly $(xz, yz) \in \mu$. Thus we obtain that μ is a congruence on S and S/μ is a regular (po-)semigroup as the proof of Theorem 1.2. By [6, V.Theorem 3.2], $\mu_\delta = \mu \cap (S^\delta \times S^\delta)$ is the maximum idempotent-separating congruence on S^δ. If S^δ is fundamental then μ_δ is the identity on S^δ. This is the statement $(\mu 1)$.

To show the statement $(\mu 2)$ we notice that for any $(x)_\mu \in S/\mu$, $(x)_\mu$ contains the idempotent $x^{\delta\delta}(x^\delta)^2 x^{\delta\delta} \in E(S^\delta)$. In fact, by $x \ \mathcal{L} \ x^\delta x$ and $x \ \mathcal{R} \ xx^\delta$ we have $xx^\delta = x^{\delta\delta}x^\delta = x^{\delta\delta}(x^{\delta\delta})^\delta$ and $x^\delta x = (x^{\delta\delta})^\delta x^{\delta\delta}$ so $x \ \mathcal{H} \ x^{\delta\delta}$. Clearly, $x^\delta e x^{\delta\delta} = (x^{\delta\delta})^\delta e (x^{\delta\delta})^{\delta\delta}$ by Lemma 1.1 (5), so that $x \ \mu \ x^{\delta\delta}$ and $x \ \mu \ x^{\delta\delta} \ \mu \ (x^{\delta\delta})^2 = (x^{\delta\delta})^2 (x^\delta)^2 (x^{\delta\delta})^2 \ \mu \ x^{\delta\delta}(x^\delta)^2 x^{\delta\delta}$, that is, the idempotent $x^{\delta\delta}(x^\delta)^2 x^{\delta\delta} \in (x)_\mu$. If $(x)_\mu$ and $(y)_\mu$ are idempotents of S/μ then $(x)_\mu = (e)_\mu$ and $(y)_\mu = (f)_\mu$ for some $e, f \in E(S^\delta)$. Thus we have

$$(x)_\mu (y)_\mu = (e)_\mu (f)_\mu = (ef)_\mu = (fe)_\mu = (f)_\mu (e)_\mu = (y)_\mu (x)_\mu$$

which shows that S/μ is inverse and clearly $(x)_\mu^{-1} = (x^\delta)_\mu$ for any $x \in S$. Now, suppose that ξ is an inverse semigroup congruence on S. Let $(x, y) \in \mu$ then $(x)_\mu^\delta = (x)_\mu^{-1} = (y)_\mu^{-1} = (y^\delta)_\mu$ and clearly $(x^\delta)_{\mu^\delta} = (y^\delta)_{\mu^\delta}$. Thus by $(\mu 1)$ we have that $x^\delta = y^\delta$. We consider that $(x)_\xi \in S/\xi$ since $x^\delta = y^\delta$ so that $(x^\delta)_\xi$ and $(y^\delta)_\xi$ are both inverse in S/μ of $(x)_\xi$ and so $(x^\delta)_\xi = (x)_\xi^{-1} = (y)_\xi^{-1} = (y^\delta)_\xi$ by the uniqueness of inverse in S/ξ. Thus we obtain $(x, y) \in \xi$, that is, $\mu \subseteq \xi$ which shows that μ is the smallest inverse semigroup congruence on S.

(2)$\Longrightarrow$(1) Suppose that the congruence μ has the properties $(\mu 1)$ and $(\mu 2)$ we shall prove that $|V_{S^\delta}(x)| = 1$ for any $x \in S$. Let $y^\delta \in V_{S^\delta}(x)$ then by $(\mu 2)$ we know that $(x^\delta)_\mu$ and $(y^\delta)_\mu$ are both inverse in S/μ of $(x)_\mu$. By the uniqueness of inverse in S/μ we obtain that $(x^\delta)_\mu = (y^\delta)_\mu$ and $(x^\delta)_{\mu_\delta} = (x^\delta)_{\mu_\delta}$. Since μ_δ is the identity on S^δ by $(\mu 1)$, so $x^\delta = y^\delta$, that is, $|V_{S^\delta}(x)| = 1$ for any $x \in S$ which shows S^δ is an inverse ${}^\delta$transversal of S. Since S^δ is a po-semigroup, so S^δ is also an inverse ${}^\delta$po-transversal of S and it is fundamental by $(\mu 1)$. $\square$

Corollary 1.5. *Let S^δ be a po-subsemigroup of the ORB-(resp.POR-) semigroup S with the mapping δ is surjective from S to S^δ such that $x^\delta \in V_{S^\delta}(x)$ for $x \in S$, then the following statements are equivalent:*

(1) *S^δ is a fundamental inverse ${}^\delta$po-transversal of S;*

(2) *The equivalence relation*

$$\mu = \{(x, y) \in \mathcal{H} \mid x^\delta e x^{\delta\delta} = y^\delta e y^{\delta\delta} \text{ for any } e \in E(S^\delta)\}$$

has the following properties:

$(\mu 1)$ *$\mu_\delta = \mu \cap (S^\delta \times S^\delta)$ is the maximum idempotent-separating congruence on S^δ and μ_δ is the identity congruence on S^δ;*

$(\mu 2)$ *μ is the smallest inverse semigroup congruence on S.*

2. The existence of inverse ${}^{\delta}$po-transversals for some classes of po-semigroups

In the proof of Theorem 1.2, we may see that the anti-homomorphism δ is very important. We think thus that if there exists an anti-homomorphism δ from some regular po-semigroup S to some po-subsemigroup S^{δ} of S then it is possible that S has an inverse ${}^{\delta}$po-transversal. In fact, we may show that it is true for some po-semigroup.

Theorem 2.1. *Let S be an order orthodox semigroup with the biggest inverses (for short, OOB-semigroup). Then S has the weakly multiplicative inverse ${}^{\delta}$po-transversal as following:*

$$S^{\delta} = \{x^{\delta} \in V(x) \mid x^{\delta} \hat{=} x^{o} \in (x)_{\nu}^{-1} \text{ for any } x \in S\}$$

with the mapping δ is an anti-homomorphism from S to S^{δ} and the mapping $\delta\delta$ denoted by $x^{\delta\delta} = (x^{\delta})^{\delta} \in (x)_{\nu}$ for any $x \in S$ is a homomorphism from S to S^{δ}. Here,

$$\nu = \{(x, y) \in S \times S \mid V(x) = V(y) \text{ and } x^{o} = y^{o}\}$$

is the smallest inverse semigroup congruence on S where $a^{o} \in V(a)$ for any $a \in S$ and a^{o} is the biggest inverse of a.

Proof. Let S be an OOB-semigroup. We know that equivalence relation

$$\nu = \{(x, y) \in S \times S \mid V(x) = V(y)\}$$

is the smallest inverse semigroup congruence on S by [6, VI.1 Theorem 1.12]. We denote the biggest inverse of x by x^{o} for $x \in S$. Let the equivalence relation

$$\tau = \{(x, y) \in S \times S \mid x^{o} = y^{o}\},$$

then we have the following results. (α) $\nu = \tau \cap \nu$.

Let $(x, y) \in \tau$ and $z \in S$ then by S being orthodox we have $(zx)^{o} = x^{o}z^{o} = y^{o}z^{o} = (zy)^{o}$ so $(zx, zy) \in \tau$. Similarly $(xz, yz) \in \tau$, that is, τ is a congruence on S. Let $\xi = \tau \cap \nu$ then ξ is also a congruence on S. We denote the $\xi-$class (resp. $\nu-$class) of containing x by $(x)_{\xi}$ (resp. $(x)_{\nu}$). Similar to Theorem 1.2, we may show that ξ is also an inverse semigroup congruence on S and $\xi \subseteq \nu$. By the smallest property of ν we have $\xi = \tau \cap \nu = \nu$.

(β) $(\forall x \in S)$ $(x)_{\nu}^{-1}{=}(x^{o})_{\nu}$, $(x)_{\nu} = (x^{oo})_{\nu}$.

Since $(x)_{\nu}^{-1}$ is the unique of $(x)_{\nu}$ and clearly $(x)_{\nu}^{-1} = (x^{o})_{\nu}$ by $x^{o} \in V(x)$. If $(x)_{\nu}^{-1} = (x^{o})_{\nu} = (y^{o})_{\nu}$ then $x^{o} = y^{o}$ by (α). By $(x)_{\nu}^{-1} = (x^{o})_{\nu}$ imply $(x)_{\nu} = ((x)_{\nu}^{-1})^{-1} = ((x^{o})_{\nu})^{-1} = (x^{oo})_{\nu}$ is also unique.

(γ) $(\forall x \in S)$ If $y \in (x)_{\nu}$ then $y^{o} = x^{o}$ and $y^{oo} = x^{oo}$.

By (β) for $x \in S$, $x^{oo} \in (x)_{\nu}$, if $y \in (x)_{\nu}$ then $(x)_{\nu} = (y)_{\nu}$, by (α) $\nu = \tau \cap \nu$ so $x^{o} = y^{o}$ and imply $x^{oo} = y^{oo}$.

(δ) $S^{\delta} = \{x^{\delta} \mid x^{\delta} \hat{=} x^{o} \in (x)_{\nu}^{-1}$, for any $x \in S\}$ is a po-semigroup with δ is an anti-homomorphism from S to S^{δ} and $\delta\delta$ is a homomorphism.

By (β) and (γ) for any $x \in S$ there is the inverse $x^{o} \in (x)_{\nu}^{-1}$. For the inverse $x^{o} \in (x)_{\nu}^{-1}$ we denote by $x^{\delta} \hat{=} x^{o} \in (x)_{\nu}^{-1}$ then δ is a mapping from S onto S^{δ}

and $x^{\delta}y^{\delta} = (yx)^{\delta} \in (yx)_{\nu}^{-1}$, $x^{\delta\delta}y^{\delta\delta} = (xy)^{\delta\delta} \in (xy)_{\nu}$ by (β). So we have that S^{δ} is a po-semigroup and statement (δ) holds.

(ϵ) S^{δ} is an inverse ${}^{\delta}$po-transversal of S.

Let the mapping $\phi : S^{\delta} \longrightarrow S/\nu$ denoted by $x^{\delta} \longmapsto (x)_{\nu}$, that is, $x^{\delta}\phi = (x)_{\nu}$. If $(x)_{\nu} = (y)_{\nu}$ then $x^{\delta} = y^{\delta}$ and for any $(x)_{\nu} \in S/\nu$ there is $x^{\delta} \in (x)_{\nu}$ such that $x^{\delta}\phi = (x)_{\nu}$. Let $x^{\delta}, y^{\delta} \in S^{\delta}$ then $(x^{\delta}y^{\delta})\phi = (yx)^{\delta}\phi = (yx)_{\nu} = (y)_{\nu} \cdot (x)_{\nu} = y^{\delta}\phi \cdot x^{\delta}\phi$. Thus we know that ϕ is an anti-isomorphism from S^{δ} to S/ν. So S^{δ} is also an inverse po-semigroup by S/ν being inverse. Clearly, x^{δ} is the unique element in $V_{S^{\delta}}(x)$, so S^{δ} is an inverse ${}^{\delta}$po-transversal of S. By lemma 1.1(7), S^{δ} is weakly multiplicative. □

Since the principally ordered orthodox semigroups (for short, *POO*-semigroups) are OOB-semigroups, we have

Corollary 2.2. *Let S be an POO-semigroup. Then S has the weakly multiplicative inverse ${}^{\delta}$po-transversal*

$$S^{\delta} = \{x^{\delta} \in V(x) \mid x^{\delta} \hat{=} x^{o} \in (x)_{\nu}^{-1} \text{ for any } x \in S\}$$

with the mapping δ is an anti-homomorphism from S to S^{δ} and the mapping $\delta\delta$ denoted by $x^{\delta\delta} = (x^{\delta})^{\delta} \in (x)_{\nu}$ for $x \in S$ is a homomorphism from S to S^{δ}. Here,

$$\nu = \{(x, y) \in S \times S \mid V(x) = V(y) \text{ and } x^{o} = y^{o}\}$$

is the smallest inverse semigroup congruence on S where x^{o} as above for $x \in S$.

It is well known that for *OOB*-semigroup S. S^{o} isn't necessarily an inverse ^{o}transversal of S. Theorem 2.1 and Corollary 2.2 solved the existence problems of inverse ${}^{\delta}$po-transversals of *OOB*-semigroups and *POO*-semigroups.

Here we list the following basic facts on ORB-semigroups, which will be used in following theorems, the reader can consult [1]-[3] for more details.

Lemma 2.3. *Let S be a ORB-semigroup and $S^{o} = \{x^{o} \mid \forall x \in S\}$ then we have*

(1) $(\forall e \in E(S))$ $e \le e^{o} \le (e^{o})^{2}$, $e = ee^{o}e^{oo}e = ee^{oo}e^{o}e$, $e^{oo} \in V(e) \cap E(S)$ *and* $e^{o} \in E(S) \Leftrightarrow e^{o} = e^{oo}$*;*

(2) $(\forall x \in S)$ $x \le x^{oo}, x^{o} = x^{ooo}$ *and* $(xx^{o})^{o} = x^{oo}x^{o}, (x^{o}x)^{o} = x^{o}x^{oo}$;

(3) $(\forall x, y \in S)$ $x \mathcal{R} y$ *if and only if* $xx^{o} = yy^{o}$; $x \mathcal{L} y$ *if and only if* $x^{o}x = y^{o}y$ *and $x^{o}x$ (resp. xx^{o}) is the biggest idempotent in $(x)_{\mathcal{L}}$ (resp. $(x)_{\mathcal{R}}$)*

(4) *if $\mathcal{R}$ and $\mathcal{L}$ are weakly regular* (*see* [1]) *then*

(α) $(\forall e \in E(S))$ $e^{o} = e^{oo} \in E(S)$;

(β) $(\forall x \in S)$ $V(x) \cap S^{o} = \{x^{o}\}$;

(ν) *if S^{o} is a po-subsemigroup of S then S^{0} is an inverse ^{o}po-transversal of S.*

Lemma 2.4. *Let S be ORB-semigroup on which $\mathcal{R}$ and $\mathcal{L}$ are strongly regular then*

$$(\forall x, y \in S)(xy)^{o} = (x^{o}xy)^{o}x^{o} = y^{o}(xyy^{o})^{o}.$$

Proof. Let $x, y \in S$ then $xy\ \mathcal{L}\ x^o xy$ so that $(xy)^o xy = (x^o xy)^o \cdot x^o xy$ by Lemma 2.3 (3). Since $(xy)(xy)^o\ \mathcal{R}\ xy(x^o xy)^o x^o$. Similarly,

$$xy(xy)^o = xy(xy)^o(xy(xy)^o)^o = xy(x^o xy)^o x^o (xy \cdot (x^o xy)^o x^o)^o.$$

Since $xx^o, xy(x^o xy)^o x^o \in E(S)$ and

$$xy(x^o xy)^o x^o = xx^o xy(x^o xy)^o x^o = xy(x^o xy)^o x^o xx^o$$

so that $xy(x^o xy)^o \cdot x^o \leq_n xx^o$. By $\mathcal{R}$ and $\mathcal{L}$ being strongly regular we have $xy(xy)^o = xy(x^o xy)\ ^o x^o (xy(x^o xy)^o x^o)^o \leq xx^o (xx^o)^o = xx^o$. Thus $x^o xy(xy)^o \leq x^o$. Since $(x^o xy)^o x^o \in V(xy)$, so $(x^o xy)^o x^o \leq (xy)^o$. Then we have

$$xy(xy)^o = xy(x^o xy)^o x^o xy(xy)^o \leq xy(x^o xy)^0 x^0 \leq xy(xy)^o.$$

Consequently,

$$xy(xy)^o = xy(x^o xy)^o x^o$$

so that

$$\begin{aligned}(xy)^o &= (xy)^o xy(xy)^o = (xy)^o xy(x^o xy)^o x^o = (x^o xy)^o x^o xy \cdot (x^o xy)^o x^o \\ &= (x^o xy)^o x^o.\end{aligned}$$

Similarly, we have $(xy)^o = y^o (xyy^o)^o$. □

Now we may prove the following theorem. This theorem will solve the existence of multiplicative inverse ${}^{\delta}$po-transversals for the ORB- and POR-semigroups on which $\mathcal{R}$ and $\mathcal{L}$ are strongly regular.

Theorem 2.5. *Let S be an ORB-semigroup on which $\mathcal{R}$ and $\mathcal{L}$ are strongly regular then S^o is necessarily a multiplicative inverse ^{o}po-transversal of S.*

Proof. We show first that S^o is a quasi-ideal of S and so that S^o is a po-subsemigroup of S since S is regular. Let $a, b \in S^0, x \in S$ and $y = axb$ then $a = a^{oo}$ and $b = b^{oo}$ by Lemma 2.3 (2). By Lemma 2.4 we have $yy^o = yy^o aa^o = aa^o yy^o$ so that $yy^o \leq_n aa^o$. Similarly, $y^o y \leq_n b^o b$. By $\mathcal{R}$ and $\mathcal{L}$ being strongly regular and Lemma 2.3 (2) we obtain $(yy^o)^o yy^o = y^{oo} y^o yy^o = y^{oo} y^o \leq (aa^o)^o aa^o = aa^o$. Therefore we have $y^{oo} y^o y \leq aa^o y = aa^o axb = axb = y = yy^o y \leq y^{oo} y^o y$ by Lemma 2.3 (2). Consequently, $y = y^{oo} y^o y$ and similarly $y = yy^o y^{oo}$. Thus we obtain $y = yy^o y = y^{oo} y^o yy^o y = y^{oo} y^o y = y^{oo} y^o yy^o y^{oo} = y^{oo} y^o y^{oo} = y^{oo} \in S^o$, which shows that S^o is a quasi-ideal of S.

Now we know immediately that S^o is an inverse ^{o}po-transversal of S with the mapping o is surjective from S to S^o by lemma 2.3(4)(ν). We may prove that S^o is multiplicative. In fact, by lemma 2.3 (4)(α) $e^o \in E(S)$ for any $e \in E(S)$, then by lemma 1.1(6) and upper result we have that S^o is multiplicative. □

Corollary 2.6. *Let S be a POR-semigroup on which $\mathcal{R}$ and $\mathcal{L}$ are strongly regular then $S^o = \{x^o \in V(x) \mid x^o = x^* x x^*$ for any $x \in S\}$ is necessarily a multiplicative inverse ^{o}po-transversal of S.*

3. The structures of ORB- and POR-semigroups

By the results in Section 2, on the understanding that $\mathcal{R}$ and $\mathcal{L}$ are strongly regular, the following structure theorem is may obtained by Blyth and McFadden in [4].

Theorem 3.1. *Let S be a ORB-semigroup on which $\mathcal{R}$ and $\mathcal{L}$ are strongly regular. Let $S^o = \{x^o \mid \forall x \in S\}$, $I = \{xx^o \mid \forall x \in S\}$ and $\wedge = \{x^o x \mid \forall x \in S\}$, then S is ordering ^{o}-isomorphic to W (i.e. $x^o\phi = (x\phi)^o$ for $x \in S$ and the ordering isomorphism ϕ) as following.*

$$W = \{(e, a, f) \in I \times S^0 \times \Lambda \mid e^0 = aa^{-1}, f^0 = a^{-1}a\}$$

with the Cartesian order

$$(e, a, f) \preceq (g, b, h) \Longleftrightarrow e \leq g, a \leq b, f \leq h$$

where multiplicative in W is defined by

$$(e, a, f)(g, b, h) = (eafga^{-1}, afgb, b^{-1}fgbh).$$

Proof. We need only to prove that S is ordering ^{0}-isomorphic to W. The rests are obtained by Blyth and McFadden in [4]. Let the mapping

$$\phi : S \longrightarrow W,\ x\phi = (xx^0, x^{00}, x^0x) = (e_x, x^{00}, f_x)$$

where $e_x = xx^0$ and $f_x = x^0x$ for any $x \in S$, then ϕ is an algebraic isomorphism by [4]. It's inverse ϕ^{-1} is given by $(e, a, f)\phi^{-1} = eaf$ for each $(e, a, f) \in W$. To show that ϕ is isotone, suppose that $x \leq y$ in S then

$$\begin{array}{rcll} x \leq y & \Longrightarrow & x^0x \leq y^0y & \text{(by } \mathcal{L} \text{ is regular)} \\ & \Longrightarrow & (x^0x)(x^0x)^0 \leq (y^0y)(y^0y)^0 & \text{(by } \mathcal{R} \text{ is regular)} \\ & \Longrightarrow & x^0x^{00} \leq y^0y^{00} & \text{(by lemma 1.1(1) and (2)).} \end{array}$$

Similarly, $xx^0 \leq yy^0$ and $x^{00}x^0 \leq y^{00}y^0$, therefore $x^{00} = x^{00}x^0xx^0x^{00} \leq y^{00}y^0yy^0y^{00} = y^{00}$. We conclude that $(e_x, x^{00}, f_x) = x\phi \preceq y\phi = (e_y, y^{00}, f_y)$, that is, ϕ is isotone and is therefore an isomorphism of po-semigroups.

By [4] we know $(x\phi)^0 = (e_x, x^{00}, f_x)^0 = (e_{x^0}, x^0, f_{x^0})$. By Lemma 1.1 (1) and (2) we compute that $x^0\phi = (e_{x^0}, x^0, f_{x^0})$ so imply $x^0\phi = (x\phi)^0$.

The following result is a generalization of Theorem 2.6 in [7]. □

Corollary 3.2. *Let S be a ORB-semigroup on which the order $\leq$ is natural. If $\mathcal{R}$ and $\mathcal{L}$ are regular then $S^o = \{x^o \in V(x) \mid \forall x \in S\}$ is a multiplicative inverse ^{o}po-transversal of S and S is ordering ^{o}-isomorphic to W as above.*

Proof. We need only to notice that when the order $\leq$ is natural then $\mathcal{R}$ and $\mathcal{L}$ are regular if and only if $\mathcal{R}$ and $\mathcal{L}$ are strongly regular. Thus we immediately obtain that the result holds by Theorem 3.1. □

For the POR-semigroups we have the following results.

Theorem 3.3. *Let S be a POR-semigroup on which $\mathcal{R}$ and $\mathcal{L}$ are strongly regular then*

(1) *S has the multiplicative inverse ^{0}po-transversal $S^0 = \{x^0 \in V(x) \mid \forall x \in S\}$ with the mapping o is surjective from S to S^0;*

(2) *As an ORB-semigroup then S is ordering ^{o}-isomorphic to W as above;*

(3) *Let ϕ is an ordering ^{o}-isomorphism from S to W, then W can be principally ordered such that W becomes a POR-semigroup and ϕ becomes an ordering *-isomorphism from S to POR-semigroup W.*

Proof. By Theorem 3.1 the statement (1) and (2) hold. We need only to show the statement (3). Since ϕ is an ordering isomorphism, then ϕ^{-1} is also and

$$xyx \leq x \Longleftrightarrow (xyx)\phi \preceq x\phi(\forall x, y \in S).$$

Since for each $x \in S$ there exists $x^* = \max\{y \in S | xyx \leq x\}$. So for each $(e_x, x^{oo}, f_x) \in W$ there also exists

$$\begin{aligned}(e_x, x^{oo}, f_x)^* &= \max\{(e_y, y^{oo}, f_y) \in W | (e_x, x^{oo}, f_x)(e_y, y^{oo}, f_y)(e_x, x^{00}, f_x) \\ &= (e_{xyx}, (xyx)^{00}, f_{xyx}) \preceq (e_x, x^{00}, f_x)\}.\end{aligned}$$

In fact, by Lemma in [2] we compute that $(e_x, x^{00}, f_x)^* = (e_{x^*}, (x^*)^{00}, f_{x^*})$ and $(e_x, x^{00}, f_x)^0 = (e_{x^0}, (x^0)^{00}, f_{x^o})$. Therefore we know that W can be principally ordered by the ordering isomorphism ϕ. In such a way, if $x \in S, x\phi = (e_x, x^{00}, f_x)$ then clearly $(x\phi)^* = (e_x, x^{00}, f_x)^* = (e_{x^*}, (x^*)^{00}, f_{x^*})$ and $x^*\phi = (e_{x^*}, (x^*)^{00}, f_{x^*})$, that is, $x^*\phi = (x\phi)^*$.

In closing this note, we point that the POR-semigroup M in Example 1 has multiplicative inverse ^{0}po-transversal

$$M^0 = \{[x, y, p_k + k - 1] \in M \mid (x, y) \in L^{[2]}, p \in Z\}.$$

In fact, $E(M^0) = \{[x, y, k-1] \mid (x, y) \in L^{[2]}\}$ is a semilattice and M^0 is an quasi-ideal of M. The proof here is omitted. □

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