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ON THE ORDERED n-PRIME IDEALS IN ORDERED Γ-SEMIGROUPS

Manoj Siripitukdet and Aiyared Iampan

Abstract. The motivation mainly comes from the conditions of the (ordered) ideals to be prime or semiprime that are of importance and interest in (ordered) semigroups and in (ordered) Γ-semigroups. In 1981, Sen [8] has introduced the concept of the Γ-semigroups. We can see that any semigroup can be considered as a Γ-semigroup. The concept of ordered ideal extensions in ordered Γ-semigroups was introduced in 2007 by Siripitukdet and Iampan [12]. Our purpose in this paper is to introduce the concepts of the ordered n-prime ideals and the ordered n-semiprime ideals in ordered Γ-semigroups and to characterize the relationship between the ordered n-prime ideals and the ordered ideal extensions in ordered Γ-semigroups.

1. Preliminaries

In 1981, the concept and notion of the Γ-semigroups was introduced by Sen [8]. In 1997, Kwon and Lee [5] introduced the concepts of the weakly prime ideals and the weakly semiprime ideals in ordered Γ-semigroups and gave some characterizations of the weakly prime ideals and the weakly semiprime ideals in ordered Γ-semigroups analogous to the characterizations of the weakly prime ideals and the weakly semiprime ideals in ordered semigroups considered by Kehayopulu [3]. In 1998, Kwon and Lee [4] introduced the ideals and the weakly prime ideals in ordered Γ-semigroups and gave some characterizations of the ideals and the weakly prime ideals in ordered Γ-semigroups analogous to the characterizations of the ideals and the weakly prime ideals in ordered semigroups considered by Kehayopulu [3]. In 1999, Lee and Kwon [6] gave two new characterizations of the weakly prime ideals in ordered semigroups. They proved two theorems as follow: Let a be a quasi-completely regular element of an ordered semigroup S. If there exists an ideal not containing a, then there exists a weakly prime ideal not containing a. Let P^* be the intersection of weakly prime ideals of an ordered semigroup S, $a \in P^*$ and I be any proper

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ideal of S. Then $a^n \in I$ for some positive integer n. P^* is an archimedean subsemigroup of an ordered semigroup S. In 2004, Dutta and Adhikari [1] introduced the concepts of the ordered Γ-semigroups and the intra-regular ordered Γ-semigroups and the concepts of the left ideals and the right ideals in ordered Γ-semigroups. The main results of their paper are the following: They proved that for an ordered Γ-semigroup M, the following statements are equivalent:

(1) $(A\Gamma A] = A$ for each ideal A of M.
(2) $(A\Gamma B] = A \cap B$ for all ideals A and B of M.
(3) $a \in (M\Gamma a\Gamma M\Gamma a\Gamma M]$ for all $a \in M$.

Let M be an ordered Γ-semigroup. The ideals of M are weakly prime if and only if they form a chain and one of the three equivalent conditions (1), (2) and (3) mentioned above holds in M. The ideals of M are prime if and only if they form a chain and M is intra-regular. In 2006, Siripitukdet and Iampan [11] characterized the relationship between the (ordered) s-prime ideals and the (ordered) semilattice congruences in ordered Γ-semigroups. They showed that for an ordered Γ-semigroup M, the congruence n on M is the intersection of σ_I for all s-prime ideals I of M and the congruence $\mathcal{N}$ on M is the intersection of σ_I for all ordered s-prime ideals I of M. In 2007, Siripitukdet and Iampan [12] introduced the concepts of the extensions of ordered s-prime ideals, prime ideals, ordered s-semiprime ideals and semiprime ideals in ordered Γ-semigroups and characterize the relationship between the extensions of ordered ideals and some congruences in ordered Γ-semigroups. They defined the equivalence relations on an ordered Γ-semigroup M as follows:

$$\begin{aligned}
\sigma_I &:= \{(x, y) \in M \times M : x, y \in I \text{ or } x, y \notin I\},\\
\Phi_I &:= \{(x, y) \in M \times M : \ll x, I \gg = \ll y, I \gg\},\\
\mathcal{N} &:= \{(x, y) \in M \times M : N(x) = N(y)\}
\end{aligned}$$

and showed that if I is an ordered s-prime ideal of M, then $\Phi_I = \sigma_I$ and $\mathcal{N} \subseteq \Phi_I$. So the concept of prime is the really interested and important thing about (ordered) semigroups and (ordered) Γ-semigroups.

Our aim in this paper is fourfold.

(1) To generalize the definitions of the ordered prime ideal and the ordered semiprime ideal in ordered Γ-semigroups.
(2) To introduce the concept of the ordered n-prime ideals in ordered Γ-semigroups and to study the ordered n-prime ideals in ordered Γ-semigroups.
(3) To generalize the ordered prime ideals in commutative ordered Γ-semigroups.
(4) To characterize the relationship between the ordered n-prime ideals and the ordered ideal extensions in commutative ordered Γ-semigroups.

To present the main theorems we first recall the definition of the Γ-semigroup which is important here.

Let Γ be any nonempty set. A nonempty set M is called a Γ-*semigroup* [7, 8, 9] if for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, we have (i) $a\alpha b \in M$ and (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$. A Γ-semigroup M is called a *commutative* Γ-*semigroup* if $a\gamma b = b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$. A nonempty subset K of a Γ-semigroup M is called a *sub*-Γ-*semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

For examples of Γ-semigroups, see [2, 10, 11, 12].

A partially ordered Γ-semigroup M is called an *ordered* Γ-*semigroup* (some author called *po*-Γ-semigroup) [5] if for any $a, b, c \in M$ and $\gamma \in \Gamma, a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$. An ordered Γ-semigroup M is called a *commutative ordered* Γ-*semigroup* if M is a commutative Γ-semigroup. For any nonempty subsets A and B of an ordered Γ-semigroup M and any nonempty subset Γ' of Γ, let $A\Gamma' B := \{a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $\{a\}\Gamma' B$ as $a\Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$. A nonempty subset I of an ordered Γ-semigroup M is called an *ordered ideal* of M if $M\Gamma I \subseteq I, I\Gamma M \subseteq I$ and for all $a \in I$ and $b \in M, b \leq a$ implies $b \in I$. An ordered ideal I of an ordered Γ-semigroup M is called an *ordered prime ideal* of M if for any $a, b \in M, a\Gamma b \subseteq I$ implies $a \in I$ or $b \in I$. Equivalently, for any subsets A and B of $M, A\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. An ordered ideal I of an ordered Γ-semigroup M is called an *ordered semiprime ideal* of M if for any $a \in M, a\Gamma a \subseteq I$ implies $a \in I$. Equivalently, for any subset A of $M, A\Gamma A \subseteq I$ implies $A \subseteq I$. Let n be any integer such that $n \geq 2$. For any subsets $A_1, A_2, \ldots, A_{n-1}$ and A_n of M and let i be an integer such that $2 \leq i \leq n-1$. We define the symbol as follows:

$$\begin{aligned} \widehat{A}_{(1;n)} &:= A_2\Gamma A_3 \cdots A_{n-1}\Gamma A_n, \\ \widehat{A}_{(i;n)} &:= A_1\Gamma A_2 \cdots A_{i-1}\Gamma A_{i+1}\Gamma A_{i+2} \cdots A_{n-1}\Gamma A_n, \\ \widehat{A}_{(n;n)} &:= A_1\Gamma A_2 \cdots A_{n-2}\Gamma A_{n-1}. \end{aligned}$$

An ordered ideal I of an ordered Γ-semigroup M is called an *ordered n-prime ideal* of M if for any subsets A_1,A_2,…,A_{n-1} and A_n of M, $A_1\Gamma A_2 \cdots A_{n-1}\Gamma A_n \subseteq I$ implies that there exists an integer i $(1 \leq i \leq n)$ such that

$$\widehat{A}_{(1;n)}, \widehat{A}_{(2;n)}, \ldots, \widehat{A}_{(i-1;n)}, \widehat{A}_{(i+1;n)}, \widehat{A}_{(i+2;n)}, \ldots, \widehat{A}_{(n;n)} \subseteq I.$$

An ordered ideal I of an ordered Γ-semigroup M is called an *ordered n-semiprime ideal* of M if for any subsets $A_1, A_2, \ldots, A_{n-1}$ and A_n of M with $A_1 = A_2 = \cdots = A_n, A_1\Gamma A_2 \cdots A_{n-1}\Gamma A_n \subseteq I$ implies $\widehat{A}_{(n;n)} \subseteq I$. Hence we have the following statements for ordered Γ-semigroups.

(1) Every ordered prime ideal is an ordered semiprime ideal.
(2) Every n-ordered prime ideal is an n-ordered semiprime ideal.
(3) The ordered prime ideals and the 2-ordered prime ideals coincide.

(4) The ordered semiprime ideals and the 2-ordered semiprime ideals coincide.

For a subset H of an ordered Γ-semigroup M, we denote $(H] := \{t \in M : t \leq h$ for some $h \in H\}$. If $H = \{a\}$, then we also write $(\{a\}]$ as $(a]$. We see that $H \subseteq (H], ((H]] = (H]$ and for any subsets A and B of M with $A \subseteq B$, we have $(A] \subseteq (B]$. For an ordered ideal I of an ordered Γ-semigroup M and a subset A of M. The set $\ll A, I \gg := \{x \in M : A\Gamma x \subseteq I\}$ is called the *extension* [12] of I by A. If $A = \{a\}$, then we also write $\ll \{a\}, I \gg$ as $\ll a, I \gg$.

We shall assume throughout this paper that M stands for a commutative ordered Γ-semigroup. Before the characterizations of the relationship between the ordered n-prime ideals and ordered ideal extensions in M for the main theorems, we give auxiliary results which are necessary in what follows.

Lemma 1.1 ([12]). *Let I be an ordered ideal of M, $A \subseteq M$ and $\gamma \in \Gamma$. Then we have the following statements.*

(a) *$\ll A, I \gg$ is an ordered ideal of M.*
(b) *$I \subseteq \ll A, I \gg \subseteq \ll A\Gamma A, I \gg \subseteq \ll A\gamma A, I \gg$.*
(c) *If $A \subseteq I$, then $\ll A, I \gg = M$.*

Lemma 1.2 ([12]). *Let I be an ordered ideal of M and $A \subseteq M$. Then*

$$\ll A, I \gg = \bigcap_{a \in A} \ll a, I \gg = \ll A \setminus I, I \gg.$$

2. Main theorems

In this section, we give the relationship between the ordered n-prime ideals and ordered ideal extensions in ordered Γ-semigroups.

The following theorem shows the important property that hold in every integer $n \geq 3$, the ordered n-prime ideals of M are a generalization of ordered $(n-1)$-prime ideals.

Theorem 2.1. *Every ordered $(n-1)$-prime ideal of M is an ordered n-prime ideal of M for all integers $n \geq 3$.*

Proof. Assume that I is an ordered $(n-1)$-prime ideal of M. Now, let $A_1, A_2, \ldots, A_n \subseteq M$ be such that $A_1\Gamma A_2 \cdots A_{n-1}\Gamma A_n \subseteq I$. Let $B_1 = A_1\Gamma A_2$ and $B_i = A_{i+1}$ for all $i = 2, 3, \ldots, n-1$. Then $B_1\Gamma B_2 \cdots B_{n-2}\Gamma B_{n-1} \subseteq I$. By hypothesis, it implies that there exists an integer i $(1 \leq i \leq n-1)$ such that

$$\widehat{B}_{(1;n-1)}, \widehat{B}_{(2;n-1)}, \ldots, \widehat{B}_{(i-1;n-1)}, \widehat{B}_{(i+1;n-1)}, \widehat{B}_{(i+2;n-1)}, \ldots, \widehat{B}_{(n-1;n-1)} \subseteq I.$$

Case 1: $\widehat{B}_{(1;n-1)} \not\subseteq I$.

Then $\widehat{B}_{(2;n-1)}, \widehat{B}_{(3;n-1)}, \ldots, \widehat{B}_{(n-1;n-1)} \subseteq I$, so $\widehat{A}_{(3;n)}, \widehat{A}_{(4;n)}, \ldots, \widehat{A}_{(n;n)} \subseteq I$. It follows from hypothesis that there exists an integer j $(1 \leq j \leq n-1)$ such that

$$\widehat{A}_{(1;n-1)}, \widehat{A}_{(2;n-1)}, \ldots, \widehat{A}_{(j-1;n-1)}, \widehat{A}_{(j+1;n-1)}, \widehat{A}_{(j+2;n-1)}, \ldots, \widehat{A}_{(n-1;n-1)} \subseteq I.$$

since $A_1\Gamma A_2\cdots A_{n-2}\Gamma A_{n-1}=\widehat{A}_{(n;n)}\subseteq I$. Then

$$A_2\Gamma A_3\cdots A_{n-2}\Gamma A_{n-1}=\widehat{A}_{(1;n-1)}\subseteq I \text{ or}$$
$$A_1\Gamma A_3\cdots A_{n-2}\Gamma A_{n-1}=\widehat{A}_{(2;n-1)}\subseteq I.$$

Thus, since I is an ordered ideal of M,

$$\widehat{A}_{(1;n)}=A_2\Gamma A_3\cdots A_{n-1}\Gamma A_n\subseteq I \text{ or } \widehat{A}_{(2;n)}=A_1\Gamma A_3\cdots A_{n-1}\Gamma A_n\subseteq I.$$

Hence $\widehat{A}_{(1;n)},\widehat{A}_{(3;n)},\widehat{A}_{(4;n)},\ldots,\widehat{A}_{(n;n)}\subseteq I$ or $\widehat{A}_{(2;n)},\widehat{A}_{(3;n)},\ldots,\widehat{A}_{(n;n)}\subseteq I$.

Case 2: $\widehat{B}_{(1;n-1)}\subseteq I$.
Then there exists an integer j $(2\leq j\leq n-1)$ such that

$$\widehat{B}_{(2;n-1)},\widehat{B}_{(3;n-1)},\ldots,\widehat{B}_{(j-1;n-1)},\widehat{B}_{(j+1;n-1)},\widehat{B}_{(j+2;n-1)},\ldots,\widehat{B}_{(n-1;n-1)}\subseteq I.$$

Thus

$$\widehat{A}_{(3;n)},\widehat{A}_{(4;n)},\ldots,\widehat{A}_{(j;n)},\widehat{A}_{(j+2;n)},\widehat{A}_{(j+3;n)},\ldots,\widehat{A}_{(n;n)}\subseteq I.$$

Since $A_3\Gamma A_4\cdots A_{n-1}\Gamma A_n=\widehat{B}_{(1;n-1)}\subseteq I$,

$$\widehat{A}_{(1;n)}=A_2\Gamma A_3\cdots A_{n-1}\Gamma A_n\subseteq I \text{ and } \widehat{A}_{(2,n)}=A_1\Gamma A_3\Gamma A_4\cdots A_{n-1}\Gamma A_n\subseteq I.$$

Thus $\widehat{A}_{(1;n)},\widehat{A}_{(2;n)},\ldots,\widehat{A}_{(j;n)},\widehat{A}_{(j+2;n)},\widehat{A}_{(j+3;n)},\ldots,\widehat{A}_{(n;n)}\subseteq I$.
Therefore I is an ordered n-prime ideal of M. Hence we complete the proof of the theorem. □

The ordered n-prime ideals are not ordered $(n-1)$-prime ideals in general for ordered Γ-semigroups and integers $n\geq 3$. We prove it by the following examples:

Example 1 ([11])**.** Let $M=\{a,b,c,d\}$ and $\Gamma=\{\gamma\}$ with the multiplication and the relation $\leq$ on M defined by

$$x\gamma y=\begin{cases} b & \text{if } x,y\in\{a,b\},\\ c & \text{otherwise.}\end{cases}$$
$$\leq:=\{(a,a),(b,b),(c,c),(d,d),(b,c),(b,d),(c,d)\}.$$

Then M is an ordered Γ-semigroup and $\{b,c\}$ is an ordered ideal of M. We can prove that $\{b,c\}$ is a 3-prime ideal of M but not a 2-prime ideal of M since $\{a\}\Gamma\{d\}\subseteq\{b,c\}$ while $\{a\}\nsubseteq\{b,c\}$ and $\{d\}\nsubseteq\{b,c\}$.

Example 2. Let $S=\{a,b,c,d\}$ be the ordered semigroup defined by the following multiplication and relation $\leq$ on S as follows:

$*$	a	b	c	d
a	b	b	d	d
b	b	b	d	d
c	d	d	c	d
d	d	d	d	d

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b),(d,b),(d,c)\}.$$

Let $M = S$ and $\Gamma = \{*\}$. Then M is an ordered Γ-semigroup and $\{d\}$ is an ordered ideal of M. We can prove that $\{d\}$ is a 3-prime ideal of M but not a 2-prime ideal of M since $\{b\}\Gamma\{c\} \subseteq \{d\}$ while $b \neq d$ and $c \neq d$.

Immediately from Theorem 2.1, we have Corollary 2.2.

Corollary 2.2. *Every ordered prime ideal of M is an ordered n-prime ideal of M for all integers $n \geq 2$.*

Theorem 2.3. *An ordered ideal I of M is an ordered n-prime ideal of M if and only if any extension of I is an ordered $(n-1)$-prime ideal of M for all integers $n \geq 3$.*

Proof. Assume that I is an ordered n-prime ideal of M. By Lemma 1.1 (a), we have that for any subset A of M, $\ll A, I \gg$ is an ordered ideal of M. For any subset B of M, let $A_1, A_2, \ldots, A_{n-1} \subseteq M$ be such that $A_1\Gamma A_2 \cdots A_{n-2}\Gamma A_{n-1} \subseteq \ll B, I \gg$. Then $B\Gamma A_1\Gamma A_2 \cdots A_{n-2}\Gamma A_{n-1} \subseteq I$. Let $B_1 = B$ and $B_i = A_{i-1}$ for all $i = 2, 3, \ldots, n$. Then $B_1\Gamma B_2 \cdots B_{n-1}\Gamma B_n \subseteq I$. Since I is an ordered n-prime ideal of M, there exists an integer i $(1 \leq i \leq n)$ such that

$$\widehat{B}_{(1;n)}, \widehat{B}_{(2;n)}, \ldots, \widehat{B}_{(i-1;n)}, \widehat{B}_{(i+1;n)}, \widehat{B}_{(i+2;n)}, \ldots, \widehat{B}_{(n;n)} \subseteq I.$$

Thus there exists an integer j $(2 \leq j \leq n)$ such that

$$\widehat{B}_{(2;n)}, \widehat{B}_{(3;n)}, \ldots, \widehat{B}_{(j-1;n)}, \widehat{B}_{(j+1;n)}, \widehat{B}_{(j+2;n)}, \ldots, \widehat{B}_{(n;n)} \subseteq I.$$

This implies that there exists an integer $k = j-1$ $(1 \leq k \leq n-1)$ such that

$$B\Gamma\widehat{A}_{(1;n-1)}, B\Gamma\widehat{A}_{(2;n-1)}, \ldots, B\Gamma\widehat{A}_{(k-1;n-1)}, B\Gamma\widehat{A}_{(k+1;n-1)},\\ B\Gamma\widehat{A}_{(k+2;n-1)}, \ldots, B\Gamma\widehat{A}_{(n-1;n-1)} \subseteq I.$$

Hence

$$\widehat{A}_{(1;n-1)}, \widehat{A}_{(2;n-1)}, \ldots, \widehat{A}_{(k-1;n-1)}, \widehat{A}_{(k+1;n-1)},\\ \widehat{A}_{(k+2;n-1)}, \ldots, \widehat{A}_{(n-1;n-1)} \subseteq \ll B, I \gg.$$

Therefore $\ll B, I \gg$ is an ordered $(n-1)$-prime ideal of M.

Conversely, assume that any extension of I is an ordered $(n-1)$-prime ideal of M. Let $A_1, A_2, \ldots, A_n \subseteq M$ be such that $A_1\Gamma A_2 \cdots A_{n-1}\Gamma A_n \subseteq I$. Then we get $A_1\Gamma A_2 \cdots A_{n-2}\Gamma A_{n-1} \subseteq \ll A_n, I \gg$. By hypothesis, it implies that there exists an integer i $(1 \leq i \leq n-1)$ such that

$$\widehat{A}_{(1;n-1)}, \widehat{A}_{(2;n-1)}, \ldots, \widehat{A}_{(i-1;n-1)}, \widehat{A}_{(i+1;n-1)},\\ \widehat{A}_{(i+2;n-1)}, \ldots, \widehat{A}_{(n-1;n-1)} \subseteq \ll A_n, I \gg.$$

We consider the following $(n-1)$ cases. Let $\widehat{A}_{(i;n-1)} \nsubseteq \ll A_n, I \gg$. Then

$$\widehat{A}_{(1;n-1)}, \widehat{A}_{(2;n-1)}, \ldots, \widehat{A}_{(i-1;n-1)}, \widehat{A}_{(i+1;n-1)},\\ \widehat{A}_{(i+2;n-1)}, \ldots, \widehat{A}_{(n-1;n-1)} \subseteq \ll A_n, I \gg.$$

Thus

$$\widehat{A}_{(1;n)}, \widehat{A}_{(2;n)}, \ldots, \widehat{A}_{(i-1;n)}, \widehat{A}_{(i+1;n)}, \widehat{A}_{(i+2;n)}, \ldots, \widehat{A}_{(n-1;n)} \subseteq I.$$

We now only prove that $\widehat{A}_{(i;n)} \subseteq I$ or $\widehat{A}_{(n;n)} \subseteq I$. For any integer j $(1 \leq j \leq n)$ and $j \neq i$, we have

$$A_1 \Gamma A_2 \ldots A_{j-1} \Gamma A_{j+1} \Gamma A_{j+2} \cdots A_{n-1} \Gamma A_n \subseteq \ll A_j, I \gg.$$

Let $B_k = A_k$ for all $k = 1, 2, \ldots, j-1$ and $B_k = A_{k+1}$ for all $k = j, j+1, \ldots, n-1$. Then

$$B_1 \Gamma B_2 \cdots B_{n-2} \Gamma B_{n-1} \subseteq \ll A_j, I \gg.$$

Hence there exists an integer k $(1 \leq k \leq n-1)$ such that

$$\begin{gathered} \widehat{B}_{(1;n-1)}, \widehat{B}_{(2;n-1)}, \ldots, \widehat{B}_{(k-1;n-1)}, \widehat{B}_{(k+1;n-1)}, \\ \widehat{B}_{(k+2;n-1)}, \ldots, \widehat{B}_{(n-1;n-1)} \subseteq \ll A_j, I \gg. \end{gathered}$$

This implies that there exists an integer l $(1 \leq l \leq n)$ and $l \neq j$ (assume $l < j$) such that

$$\begin{gathered} \widehat{A}_{(1;n)}, \widehat{A}_{(2;n)}, \ldots, \widehat{A}_{(l-1;n)}, \widehat{A}_{(l+1;n)}, \widehat{A}_{(l+2;n)}, \ldots, \widehat{A}_{(j-1;n)}, \widehat{A}_{(j+1;n)}, \\ \widehat{A}_{(j+2;n)}, \ldots, \widehat{A}_{(n;n)} \subseteq I. \end{gathered}$$

Since $j \neq i$, we get $\widehat{A}_{(i;n)} \subseteq I$ or $\widehat{A}_{(n;n)} \subseteq I$. Hence

$$\widehat{A}_{(1;n)}, \widehat{A}_{(2;n)}, \ldots, \widehat{A}_{(n-2;n)}, \widehat{A}_{(n-1;n)} \subseteq I$$

or

$$\widehat{A}_{(1;n)}, \widehat{A}_{(2;n)}, \ldots, \widehat{A}_{(i-1;n)}, \widehat{A}_{(i+1;n)}, \widehat{A}_{(i+2;n)}, \ldots, \widehat{A}_{(n;n)} \subseteq I.$$

Therefore I is an ordered n-prime ideal of M. Hence the proof of the theorem is completed. □

Theorem 2.4. *If $a \in (M\Gamma a]$ for all $a \in M$, then the ordered n-prime ideals and the ordered $(n-1)$-prime ideals of M coincide for all integers $n \geq 3$.*

Proof. Let I be an ordered n-prime ideal of M. By Theorem 2.3, $\ll M, I \gg$ is an ordered $(n-1)$-prime ideal of M. Let $a \in \ll M, I \gg$. Then $a \leq m\gamma a \in I$ for some $m \in M$ and $\gamma \in \Gamma$, so $a \in I$. Thus $\ll M, I \gg \subseteq I$. By Lemma 1.1 (b), $\ll M, I \gg = I$. By Lemma 2.1, the proof is completed. □

Theorem 2.5. *If I is an ordered semiprime ideal of M, then $I = \ll M, I \gg$.*

Proof. By Lemma 1.1 (b), $I \subseteq \ll M, I \gg$. Let $a \in \ll M, I \gg$. Then $a\Gamma a \subseteq M\Gamma a \subseteq I$. Since I is an ordered semiprime ideal of M, $a \in I$. Hence $\ll M, I \gg \subseteq I$, so we conclude that $I = \ll M, I \gg$. □

Theorem 2.6. *For any integer $n \geq 3$, let I be an ordered semiprime ideal and an ordered n-prime ideal of M and let*

$$\mathcal{P} = \{T : T \text{ is an ordered } (n-1)\text{-prime ideal of } M \text{ and } I \subseteq T\}.$$

Then $I = \bigcap_{T \in \mathcal{P}} T$.

Proof. Clearly, $I \subseteq \bigcap_{T \in \mathcal{P}} T$. By Lemma 1.2 and Theorem 2.5,

$$I = \bigcap_{x \in M} \ll x, I \gg .$$

By Lemma 1.1 (b) and Theorem 2.3, $I \subseteq \ll x, I \gg$ is an ordered $(n-1)$-prime ideal of M for all $x \in M$. Thus $\ll x, I \gg \in \mathcal{P}$ for all $x \in M$. Hence $\bigcap_{T \in \mathcal{P}} T \subseteq \bigcap_{x \in M} \ll x, I \gg = I$. Therefore $I = \bigcap_{T \in \mathcal{P}} T$. Hence the theorem is now completed. □

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Manoj Siripitukdet
Department of Mathematics, Faculty of Science
Naresuan University
Phitsanulok 65000, Thailand
E-mail address: manojs@nu.ac.th

Aiyared Iampan
Department of Mathematics, Faculty of Science
Naresuan University
Phitsanulok 65000, Thailand
E-mail address: aiyaredi@nu.ac.th