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NUMERICAL INTEGRATION METHOD FOR SINGULAR PERTURBATION PROBLEMS WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT. In this paper, the numerical integration method for general singularly perturbed two point boundary value problems with mixed boundary conditions of both left and right end boundary layer is presented. The original second order differential equation is replaced by an approximate first order differential equation with a small deviating argument. By using the trapezoidal formula we obtain a three term recurrence relation, which is solved using Thomas Algorithm. To demonstrate the applicability of the method, we have solved four linear (two left and two right end boundary layer) and one nonlinear problems. From the results, it is observed that the present method approximates the exact or the asymptotic expansion solution very well.

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1. Introduction

The numerical treatment of singular perturbation problems has been far from trivial, because of the boundary layer behavior of the solutions. However, the area of singular perturbation is of increasing interest to applied mathematicians. The survey paper of Kadalbajoo and Reddy [5] and Kadalbajoo and Patidar[6], gives an erudite outline of the singular perturbation problems. For detailed discussion on the analytic theory of general singular perturbation problems, one may refer to Bender and Orsazag [1], Kevorkian and Cole [3], Nayfeh [7-8], O'Malley [9], Reddy[11-12] and Van Dyke [13]. The numerical integration method developed by Y.N.Reddy and K.Anantha Reddy [10] is extended for general singularly perturbed two point boundary value problems with mixed

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boundary conditions of both left and right end boundary layer. The motivation impulse for this method was to provide the practicing engineer or applied mathematician withy a means of solving more general class of singular perturbation problems in a routine manner. As a part of continuing effort to determine the applicability and the limitations of the integration method, we have been attempting to solve more general singularly perturbed two point boundary value problems in ordinary differential equations. Typically, these problems arise very frequently in fluid mechanics, elasticity, chemical reactor theory and many other allied areas. For example: the singular perturbation problems with mixed boundary conditions of the form:

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x), x \in [0,1]$$

with $y'(0) - ay(0) = \alpha$ and $y'(1) + by(1) = \beta$
arise in the study of adiabatic tubular chemical flow reactors with axial diffusion. O'Malley [9] obtained the asymptotic solution $y(x,\varepsilon)$, which converges to $y_0(x)$, $x \in [0,1]$, of the reduced problem, while $y'(x,\varepsilon)$ converges non-uniformly as $\varepsilon \to 0$ either at $x = 0$,(i.e $p(x) \geq M > 0$) or at $x = 1$(i.e., $p(x) \leq M < 0$). In this method, the original second order differential equation is replaced by an approximate first order differential equation with a small deviating argument. By using the trapezoidal formula we obtain a three term recurrence relation, which is solved using Thomas Algorithm. To demonstrate the applicability of the method, we have solved four linear (two left and two right end boundary layer) and one nonlinear problems. From the results, it is observed that the present method approximates the exact or the asymptotic expansion solution very well.

2. Left boundary layer

To describe the method we considered the following singular perturbation problem with mixed boundary conditions:

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x), x \in [0,1] \tag{1}$$

with

$$a_1 y(0) + a_2 y'(0) = \alpha, \tag{2a}$$

$$and, \quad a_3 y(1) + a_4 y'(1) = \beta \tag{2b}$$

where ε is a small positive parameter ($0 < \varepsilon << 1$)and $a_i, i = 1,..,4, \alpha, \beta$ are known constants. We assume that $a(x), b(x)$ *and* $f(x)$ are sufficiently continuously differentiable functions in [0,1]. Further more, we assume that $a(x) \geq M > 0$ throughout the interval [0,1], where M is some positive constant. This assumptions merely implies that the boundary layer will be in the neighborhood of x=0. Let δ be a small deviating argument ($0 < \delta << 1$). By using Taylor series expansion of order 2 in the neighborhood of the point x, we get

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x). \tag{3}$$

Substituting equation (3) in to equation (1), we get

$$2\varepsilon y(x-\delta) - 2\varepsilon\delta y'(x) + \delta^2 a(x)y'(x) + \delta^2 b(x)y(x) = \delta^2 f(x) \tag{4}$$

we rewrite equation (4) in the form

$$y'(x) = p(x)y(x-\delta) + q(x)y(x) + r(x), for \quad \delta \le x \le 1 \tag{5}$$

where

$$p(x) = \frac{-2\varepsilon}{2\varepsilon\delta + \delta^2 a(x)}, \tag{6a}$$

$$q(x) = \frac{2\varepsilon - \delta^2 b(x)}{2\varepsilon\delta + \delta^2 a(x)}, \tag{6b}$$

$$and, r(x) = \frac{\delta^2 f(x)}{2\varepsilon\delta + \delta^2 a(x)}. \tag{6c}$$

Equation (5) is a first order differential equation replacing the second order differential equation (1) with a small deviating argument. Transition from (1) to (5) is admitted, because of the condition that δ is small ($0 < \delta << 1$). This replacement is significant from the computational point of view. Further theoretical discussion and details on the validity of this transition can be referred in Elsgolts and Norkin[4]. Now we divide the interval [0,1] in to N equal subintervals of mesh size h=1/N so that $x_i = ih, i = 0, 1, 2, ..N$. Integrating equation (5) in the subinterval$[x_i, x_{i+1}], i = 0, 1, 2, , N$, we get

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} [p(x)y(x-\delta) + q(x)y(x) + r(x)]\, dx. \tag{7}$$

Using the Trapezoidal formula for evaluating the integral approximately, we get

$$y(x_{i+1}) - y(x_i) = \frac{h}{2}\Big[p(x_{i+1})y(x_{i+1}-\delta) + p(x_i)y(x_i-\delta)\Big]$$

$$+\frac{h}{2}[q(x_{i+1})y(x_{i+1}) + q(x_i)y(x_i)] + \frac{h}{2}[r(x_{i+1}) + r(x_i)] \tag{8}$$

Again, we make use of the Taylor series expansion of order 1 on $y(x-\delta)$ and we get

$$y(x-\delta) = y(x) - \delta y'(x). \tag{9}$$

Approximating $y'(x)$ by linear interpolation, (9) can be described as

$$y(x_i-\delta) \approx y(x_i) - \delta\left[\frac{y(x_i) - y(x_{i-1})}{h}\right] = \Big(1 - \frac{\delta}{h}\Big)y(x_i) + \frac{\delta}{h}y(x_{i-1}) \tag{10}$$

Similarly

$$y(x_{i+1}-\delta) \approx y(x_{i+1}) - \delta\left[\frac{y(x_{i+1}) - y(x_i)}{h}\right] = \Big(1 - \frac{\delta}{h}\Big)y(x_{i+1}) + \frac{\delta}{h}y(x_i) \tag{11}$$

Substituting equations (9) and (10) in (8) and rearranging, we get

$$\begin{aligned} & y(x_{i+1}) - y(x_i) \\ & = \frac{h}{2}p(x_{i+1})\left[\Big(1-\frac{\delta}{h}\Big)y(x_{i+1}) + \frac{\delta}{h}y(x_i)\right] \\ & \quad +\frac{h}{2}p(x_i)\left[(1-\frac{\delta}{h})y(x_i) + \frac{\delta}{h}y(x_{i-1})\right] \\ & \quad +\frac{h}{2}[q(x_{i+1})y(x_{i+1}) + q(x_i)y(x_i)] + \frac{h}{2}[r(x_{i+1}) + r(x_i)], \end{aligned}$$

$$\begin{aligned} y(x_{i+1}) - y(x_i) = & \frac{h}{2}(1-\frac{\delta}{h})p(x_{i+1})y(x_{i+1}) + \frac{\delta}{2}p(x_{i+1})y(x_i) \\ & +\frac{h}{2}(1-\frac{\delta}{h})p(x_i)y(x_i) + \frac{\delta}{2}p(x_i)y(x_{i-1}) \end{aligned}$$

$$+\frac{h}{2}q(x_{i+1})y(x_{i+1}) + \frac{h}{2}q(x_i)y(x_i) + \frac{h}{2}[r(x_{i+1}) + r(x_i)]. \tag{12}$$

Equation (12) can be rewritten in a three-term recurrence relationship as follows:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, i = 0,1,2,..,N \tag{13}$$

where

$$E_i = -\frac{\delta}{2}p_i \tag{14a}$$

$$F_i = 1 + \frac{\delta}{2}p_{i+1} + \frac{h}{2}(1-\frac{\delta}{h})p_i + \frac{h}{2}q_i \tag{14b}$$

$$G_i = 1 - \frac{h}{2}(1-\frac{\delta}{h})p_{i+1} - \frac{h}{2}q_{i+1} \tag{14c}$$

$$H_i = \frac{h}{2}(r_i + r_{i+1}) \tag{14d}$$

and $y_i = y(x_i), p_i = p(x_i), q_i = q(x_i)$ *and* $r_i = r(x_i)$. Equation (13) gives a system of N+ 1 equations with N+3 unknowns' y_0 *to* y_N and the unwanted unknowns' y_{-1} *and* y_{N+1}. To eliminate the unknowns, y_{-1} *and* y_{N+1}, we make use of the equations in (2) given as boundary conditions in mixed form. By employing the second order central difference approximation in (2),we get

$$a_1 y(0) + a_2\Big(\frac{y_1 - y_{-1}}{2h}\Big) = \alpha \tag{15a}$$

$$and,\quad a_3 y(1) + a_4\Big(\frac{y_{N+1} - y_{N-1}}{2h}\Big) = \beta \tag{15b}$$

From (15) we have

$$y_{-1} = \frac{2ha_i}{a_2}y_0 + y_1 - \frac{2h\alpha}{a_2}, \tag{16a}$$

$$and \quad y_{N+1} = \frac{2h\beta}{a_4}y_{N-1} - \frac{2ha_3}{a_4}y_N. \tag{16b}$$

Making use of (16a) in the first equation of the recurrence relation (13) at $i = 0$ and (16b) in the last equation of the recurrence relation (13) at $i = N$,respectively, we get,

$$\Big(\frac{2ha_1}{a_2}E_0 - F_0\Big)y_0 + (E_0 + G_0)y_1 = H_0 + \frac{2h\alpha}{a_2}E_0 \tag{17a}$$

$$and\;\; (E_N + G_N)y_{N-1} - \Big(\frac{2ha_3}{a_4}G_N + F_N\Big)y_N = H_N - \frac{2h\beta}{a_4}G_N \tag{17b}$$

Now, equations (13) and (17) give an N+1 by N+1 tri-diagonal system which can be solved by using Thomas Algorithm. Repeat the numerical scheme for different choice of δ(deviating argument, satisfying the condition $0 < \delta << 1$), until the solution profile do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criterion, namely

$$|y^{m+1}(x) - y^m(x)| \leq \sigma \; for \; 0 \leq x \leq 1 \tag{18}$$

Where y^m is the solution for the mth iterate of δ andσ is the prescribed tolerance bound.

3. Linear examples

To illustrate the present method we have chosen two linear singular perturbation problems with left-end boundary layer which are widely discussed in literature.

Example 3.1. Consider the following singular perturbation problem from Dorr et al([2], page 80).

$$\varepsilon y''(x) + y'(x) - y(x) = 0; x \in [0, 1]$$

with $-y'(0) = 0$ and $y(1) + \varepsilon y'(1) = 1$. The exact solution is given by :

$$y(x) = \frac{m_2 e^{m_1 x} - m_1 e^{m_2 x}}{m_2(1 + \varepsilon * m_1)e^{m_1} - m_1(1 + \varepsilon * m_2)e^{m_2}}$$

where $m_1 = \dfrac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}$ and $m_2 = \dfrac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon}$.

The numerical results are given in tables 1(a), 1(b) for $\varepsilon = 10^{-3} and\ \varepsilon = 10^{-4}$ respectively.

Example 3.2. Consider the following singular perturbation problem from Dorr et al([2],page80).

$$\varepsilon y''(x) + y'(x) = -1 - 2x; x \in [0, 1]$$

with $-y'(0) = 1$ and $y(1) + \varepsilon y'(1) = 0$. The exact solution is given by :

$$y(x) = 2 - x(1 + x) + \varepsilon\Big[1 - 2\Big(\varepsilon\Big[1 - exp(-\frac{x}{\varepsilon})\Big] - x\Big)\Big]$$

The numerical results are given in tables 2(a), 2(b) for $\varepsilon = 10^{-3} and\ \varepsilon = 10^{-4}$ respectively.

4. Non-linear examples

We have applied the present method on non-linear singularly perturbed problem with left end boundary layer by using the method of quasilinearization.

Example 4.1. Consider the following semi-linear boundary value problem from Dorr et al[[2], page 80].

$$\varepsilon y''(x) + y'(x) - y^2(x) = 0; x \in [0, 1]$$

with $-y'(0) = 0$ and $y(1) + \varepsilon y'(1) = 0$. The linear problem concerned is:

$$y''(x) + \frac{1}{\varepsilon}y'(x) + \frac{2}{\varepsilon(x+c))}y(x) = -\frac{1}{\varepsilon(x+c)^2},\ where\ c = \frac{-3-\sqrt{1+4\varepsilon}}{2}.$$

The asymptotic expansion solution is given by:

$$y(x) = \frac{1}{2-x} + \frac{\varepsilon}{4}exp(\frac{-x}{\varepsilon}) + O(\varepsilon)$$

The numerical results are given in tables 3(a), 3(b) for $\varepsilon = 10^{-3} and\ \varepsilon = 10^{-4}$ respectively.

5. Right boundary layer

Finally, we considered the following singular perturbation problem with mixed boundary conditions:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); x \in [0, 1] \tag{19}$$

with

$$a_1y(0) + a_2y'(0) = \alpha, \tag{20a}$$

$$and,\ \ a_3y(1) + a_4y'(1) = \beta \tag{20b}$$

where ε is a small positive parameter($0 < \varepsilon << 1$) and $a_i, i = 1, .., 4, \alpha, \beta$ are known constants. We assume that $a(x), b(x)\ and\ f(x)$ are sufficiently continuously differentiable functions in [0,1]. Further more, we assume that $a(x) \leq M < 0$throughout the interval [0,1], where M is some positive constant. This assumptions merely implies that the boundary layer will be in the neighborhood of $x = 1$.

The evaluation of the right-end boundary layer problem (19)-(20) is similar to that of the left-end boundary layer but there are some differences worth noting. Let δ be a small deviating argument ($0 < \delta << 1$). By using Taylor series expansion of order 2 in the neighborhood of the point x, we get

$$y(x+\delta) \approx y(x) + \delta y'(x) + \frac{\delta^2}{2}y''(x). \tag{21}$$

Substituting equation (21) in to equation (19), we get

$$2\varepsilon y'(x+\delta) - 2\varepsilon y(x) - 2\varepsilon\delta y'(x) + \delta^2 a(x)y'(x) + \delta^2 b(x)y(x) = \delta^2 f(x) \tag{22}$$

Now we rewrite equation (22) in the form

$$y'(x) = p(x)y(x+\delta) + q(x)y(x) + r(x),\ \ for\ 0 \leq x \leq (1-\delta) \tag{23}$$

where

$$p(x) = \frac{-2\varepsilon}{\delta^2 a(x) - 2\varepsilon\delta}, \tag{24a}$$

$$q(x) = \frac{2\varepsilon - \delta^2 b(x)}{\delta^2 a(x) - 2\varepsilon\delta}, \tag{24b}$$

$$andr(x) = \frac{\delta^2 f(x)}{\delta^2 a(x) - 2\varepsilon\delta}. \tag{24c}$$

Equation (23) is a first order differential equation replacing the second order differential equation (19) with a small deviating argument. Transition from (19) to (23) is admitted, because of the condition that δ is small$(0 < \delta << 1)$. This replacement is significant from the computational point of view. Further theoretical discussion and details on the validity of this transition can be referred in Elsgolts and Norkin[4].

Now we divide the interval [0,1] in to N equal subintervals of mesh size h=1/N so that $x_i = ih, i = 0,1,2,..N$. Integrating equation (23) in the subinterval $[x_{i-1}, x_i], i = 0,1,2,,N$, we get

$$y(x_i) - y(x_{i-1}) = \int_{x_{i-1}}^{x_i} [p(x)y(x+\delta) + q(x)y(x) + r(x)]\, dx. \tag{25}$$

Using the Trapezoidal formula for evaluating the integral approximately, we get

$$y(x_i) - y(x_{i-1}) = \frac{h}{2}[p(x_i)y(x_i+\delta)] + p(x_{i-1})y(x_{i-1}+\delta)$$

$$+\frac{h}{2}[q(x_i)y(x_i) + q(x_{i-1})y(x_{i-1})] + \frac{h}{2}[r(x_{i-1}) + r(x_i)] \tag{26}$$

Again, we make use of the Taylor series expansion of order 1 on $y(x+\delta)$ and we get

$$y(x+\delta) = y(x) + \delta y'(x) \tag{27}$$

Approximating $y'(x)$ by linear interpolation, (27) can be described as

$$y(x_i+\delta) \approx y(x_i) + \delta[\frac{y(x_{i+1}) - y(x_i)}{h}] = (1 - \frac{\delta}{h})y(x_i) + \frac{\delta}{h}y(x_{i+1}) \tag{28}$$

Similarly

$$y(x_{i-1}+\delta) \approx y(x_{i-1}) + \delta[\frac{y(x_i) - y(x_{i-1})}{h}] = (1 - \frac{\delta}{h})y(x_{i-1}) + \frac{\delta}{h}y(x_i) \tag{29}$$

Substituting equations (28) and (29) in (26) and rearranging, we get

$$y(x_i) - y(x_{i-1})$$
$$= \frac{h}{2}p(x_{i-1})[(1-\frac{\delta}{h})y(x_{i-1}) + \frac{\delta}{h}y(x_i)] + \frac{h}{2}p(x_i)[(1-\frac{\delta}{h})y(x_i) + \frac{\delta}{h}y(x_{i+1})]$$
$$+\frac{h}{2}[q(x_{i-1})y(x_{i-1}) + q(x_i)y(x_i)] + \frac{h}{2}[r(x_{i-1}) + r(x_i)]$$

$$y(x_i) - y(x_{i-1})$$
$$= \frac{h}{2}(1-\frac{\delta}{h})p(x_{i-1})y(x_{i-1}) + \frac{\delta}{2}p(x_{i-1})y(x_{i-1}) + \frac{h}{2}(1-\frac{\delta}{h})p(x_i)y(x_i)$$
$$+\frac{\delta}{2}p(x_i)y(x_{i+1}) + \frac{h}{2}q(x_{i-1})y(x_{i-1}) + \frac{h}{2}q(x_i)y(x_i) + \frac{h}{2}[r(x_{i-1}) + r(x_i)] \quad (30)$$

Equation (30) can be rewritten in a three-term recurrence relationship as follows:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, i = 0, 1, 2, .., N \quad (31)$$

where

$$E_i = -1 - \frac{h}{2}(1-\frac{\delta}{h})p_{i-1} - \frac{h}{2}q_{i-1}, \quad (32a)$$
$$F_i = -1 + \frac{\delta}{2}p_{i-1} + \frac{h}{2}(1-\frac{\delta}{h})p_i + \frac{h}{2}q_i, \quad (32b)$$
$$G_i = -\frac{\delta}{2}p_i, \quad (32c)$$
$$H_i = \frac{h}{2}(r_i + r_{i-1}) \quad (32d)$$

and $y_i = y(x_i), p_i = p(x_i), q_i = q(x_i)$ *and* $r_i = r(x_i)$. Equation (31) gives a system of N+ 1 equations with N+3 unknowns' y_0 *to* y_N and the unwanted unknowns' y_{-1} *and* y_{N+1}. To eliminate the unknowns, y_{-1} *and* y_{N+1}, we make use of the equations in (19) given as boundary conditions in mixed form. By employing the second order central difference approximation in (19),we get

$$a_1 y(0) + a_2(\frac{y_1 - y_{-1}}{2h}) = \alpha, \quad (33a)$$
$$and \quad a_3 y(1) + a_4(\frac{y_{N+1} - y_{N-1}}{2h}) = \beta. \quad (33b)$$

From (33) we have

$$y_{-1} = \frac{2ha_1}{a_2}y_0 + y_1 - \frac{2h\alpha}{a_2}, \quad (34a)$$
$$and \quad y_{N+1} = \frac{2h\beta}{a_4}y_{N-1} - \frac{2ha_3}{a_4}y_N. \quad (34b)$$

Making use of (34a) in the first equation of the recurrence relation (31) at $i = 0$,and(34b) in the last equation of the recurrence relation (31) at $i =$

N,respectively, we get

$$\left(\frac{2ha_1}{a_2}E_0 - F_0\right) y_0 + (E_0 + G_0)y_1 = H_0 + \frac{2h\alpha}{a_2}E_0 \tag{35a}$$

$$and \ \ (E_N + G_N)y_{N-1} - \left(\frac{2ha_3}{a_4}G_N + F_N\right) y_N = H_N - \frac{2h\beta}{a_4}G_N \tag{35b}$$

Now, equations (31) and (35) give an N+1 by N+1 tri-diagonal system which can be solved by using Thomas Algorithm.
Repeat the numerical scheme for different choice of δ(deviating argument, satisfying the condition $0 < \delta < 1$), until the solution profile do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criterion, namely

$$|y^{m+1}(x) - y^m(x)| \leq \sigma \ \ for \ \ 0 \leq x \leq 1 \tag{36}$$

where y^m is the solution for the mth iterate of δ andσ is the prescribed tolerance bound.

6. Examples with right boundary layer

The applicability of the present method is demonstrated by solving two right-end boundary layer problems.

Example 6.1. Consider the following singular perturbation problem

$$\varepsilon y''(x) + y'(x) = 3 - 2x; x \in [0,1]$$

with $y(0) - \varepsilon y'(0) = 1$ and $y'(1) = 1$. The exact solution is given by :

$$y(x) = x(3 - x - 2\varepsilon) + \varepsilon\left[3 - 2\varepsilon\left[1 - exp\left(-\frac{(x-1)}{\varepsilon}\right)\right]\right]$$

The numerical results are given in tables 4(a), 4(b) for $\varepsilon = 10^{-3} and \ \varepsilon = 10^{-4}$ respectively.

Example 6.2. Consider the following singular perturbation problem from Dorr et al([2],page80 with a=1,n=1).

$$\varepsilon y''(x) - y'(x) - y(x) = 0; x \in [0,1]$$

with $y(0) - \varepsilon y'(0) = 1$ and $y'(1) = 0$. The exact solution is given by :

$$y(x) = \frac{m_1 e^{m_1 x} - m_2 e^{(m_1(x-1)+m_2)}}{m_1(1 - \varepsilon m_2) - m_2(1 - \varepsilon m_1)e^{(m_2 - m_1)}}$$

where $m_1 = \dfrac{1 + \sqrt{(1 + 4\varepsilon)}}{2\varepsilon}$ and $m_2 = \dfrac{1 - \sqrt{(1 + 4\varepsilon)}}{2\varepsilon}$.

The numerical results are given in tables 5(a), 5(b) for $\varepsilon = 10^{-3} and \ \varepsilon = 10^{-4}$ respectively.

7. Discussion and conclusions

The numerical integration method developed by Y.N. Reddy and K.A. Reddy [10] is extended for general singularly perturbed two point boundary value problems with mixed boundary conditions of both left and right end boundary layer. The original second order differential equation is replaced by an approximate first order differential equation with a small deviating argument. By using the trapezoidal formula we obtain a three term recurrence relation, which is solved using Thomas Algorithm To demonstrate the applicability of the method, we have solved four linear (two left and two right end boundary layer) and one nonlinear problems. From the results, it is observed that the present method approximates the exact or the asymptotic expansion solution very well.

Table 1a Numerical Results of Example 3.1, $\epsilon = 10^{-3}, h = 10^{-2}$

x	y(x)(δ=0.008)	y(x)(δ=0.009)	y(x)(δ=0.01)	Exact Solution
0.00	0.3687875	0.3686524	0.3685414	0.3682464
0.02	0.3753995	0.3753455	0.3753010	0.3753034
0.04	0.3829635	0.3829122	0.3828695	0.3828774
0.06	0.3906901	0.3906389	0.3905964	0.3906043
0.08	0.3985727	0.3985217	0.3984792	0.3984870
0.10	0.4066144	0.4065635	0.4065210	0.4065289
0.20	0.4493231	0.4492731	0.4492313	0.4492391
0.30	0.4965177	0.4964693	0.4964289	0.4964364
0.40	0.5486692	0.5486234	0.5485852	0.5485923
0.50	0.6062986	0.6062564	0.6062212	0.6062276
0.60	0.6699811	0.6699437	0.6699126	0.6699183
0.70	0.7403525	0.7403215	0.7402957	0.7403002
0.80	0.8181153	0.8180925	0.8180734	0.8180766
0.90	0.9040459	0.9040332	0.9040226	0.9040241
1.00	0.9990022	0.9990020	0.9990019	0.9990014

Table1b Numerical Results of Example3.1, $\epsilon = 10^{-4}, h = 10^{-2}$

x	y(x)(δ=0.0008)	y(x)(δ=0.0009)	y(x)(δ=0.001)	Exact Solution
0.00	0.3691142	0.3689860	0.3688707	0.3679162
0.02	0.3757321	0.3756852	0.3756364	0.3753104
0.04	0.3833029	0.3832588	0.3832117	0.3828913
0.06	0.3910365	0.3909924	0.3909455	0.3906255
0.08	0.3989262	0.3988822	0.3988353	0.3985159
0.10	0.4069751	0.4069313	0.4068844	0.4065656
0.20	0.4497223	0.44967920	0.4496332	0.4493200
0.30	0.4969594	0.4969178	0.4968733	0.4965704
0.40	0.5491581	0.5491188	0.5490766	0.5487896
0.50	0.6068397	0.6068034	0.6067645	0.6065002
0.60	0.6705798	0.6705477	0.6705134	0.6702796
0.70	0.7410150	0.7409884	0.7409600	0.7407661
0.80	0.8188484	0.8188289	0.8188080	0.8186648
0.90	0.9048572	0.9048464	0.9048349	0.9047555
1.00	0.9999001	0.9999000	0.9999001	0.9998993

Table 2a Numerical Results of Example 3.2,$\epsilon = 10^{-3}, h = 10^{-2}$

x	y(x)(δ=0.008)	y(x)(δ=0.009)	y(x)(δ=0.01)	Exact Solution
0.00	2.0005026	2.0007796	2.0010011	2.0009999
0.02	1.9801470	1.9804195	1.9806374	1.9806379
0.04	1.9589970	1.9592639	1.9594774	1.9594780
0.06	1.9370470	1.9373084	1.9375174	1.9375180
0.08	1.9142970	1.9145528	1.9147575	1.9147580
0.10	1.8907470	1.8909973	1.8911974	1.8911980
0.20	1.7609971	1.7612195	1.7613975	1.7613980
0.30	1.6112471	1.6114417	1.6115975	1.6115980
0.40	1.4414971	1.4416640	1.4417975	1.4417982
0.50	1.2517470	1.2518862	1.2519976	1.2519983
0.60	1.0419971	1.0421085	1.0421976	1.0421985
0.70	0.8122472	0.8123308	0.8123977	0.8123989
0.80	0.5624973	0.5625531	0.5625978	0.5625992
0.90	0.2927473	0.2927755	0.2927979	0.2927996
1.00	0.0029975	0.0029978	0.0029980	0.0030000

Table 2b Numerical Results of Example 3.2,$\epsilon = 10^{-4}, h = 10^{-2}$

x	y(x)(δ=0.0008)	y(x)(δ=0.0009)	y(x)(δ=0.001)	Exact Solution
0.00	1.9978049	1.9980813	1.9983028	2.0000999
0.02	1.9774493	1.9777213	1.9779392	1.9797039
0.04	1.9562993	1.9565657	1.9567792	1.9585080
0.06	1.9343493	1.9346102	1.9348192	1.9365120
0.08	1.9115993	1.9118546	1.9120593	1.9137160
0.10	1.8880492	1.8882991	1.8884993	1.8901199
0.20	1.7582994	1.7585213	1.7586993	1.7601399
0.30	1.6085494	1.6087435	1.6088994	1.6101600
0.40	1.4387994	1.4389658	1.4390993	1.4401802
0.50	1.2490493	1.2491881	1.2492994	1.2502004
0.60	1.0392994	1.0394104	1.0394994	1.0402205
0.70	0.8095495	0.8096327	0.8096995	0.8102409
0.80	0.5597996	0.5598550	0.5598996	0.5602612
0.90	0.2900496	0.2900774	0.2900997	0.2902816
1.00	0.0002997	0.0002998	0.0002998	0.0003019

Table 3a Numerical Results of Example 4.1, $\epsilon = 10^{-3}, h = 10^{-2}$

x	y(x)(δ=0.001)	y(x)(δ=0.008)	y(x)(δ=0.01)	Dorr Solution
0.00	0.4997527	0.4997451	0.4997449	0.5002500
0.02	0.5047930	0.5047898	0.5047898	0.5050505
0.04	0.5099413	0.5099381	0.5099382	0.5102041
0.06	0.5151957	0.5151925	0.5151926	0.5154639
0.08	0.5205596	0.5205563	0.5205564	0.5208333
0.10	0.5260363	0.5260330	0.5260331	0.5263158
0.20	0.5552444	0.5552410	0.5552409	0.5555556
0.30	0.5878865	0.5878832	0.5878832	0.5882353
0.40	0.6246065	0.6246033	0.6246030	0.6249999
0.50	0.6662191	0.6662161	0.6662157	0.6666666
0.60	0.7137724	0.7137697	0.7137693	0.7142856
0.70	0.7686363	0.7686335	0.7686333	0.7692305
0.80	0.8326367	0.8326349	0.8326344	0.8333330
0.90	0.9082636	0.9082624	0.9082624	0.9090905
1.00	0.9990018	0.9990019	0.9990019	0.9999993

Table 3b Numerical Results of Example 4.1, $\epsilon = 10^{-4}, h = 10^{-2}$

x	y(x)(δ=0.0001)	y(x)(δ=0.0008)	y(x)(δ=0.001)	Dorr Solution
0.00	0.4999696	0.4999691	0.4999692	0.5000250
0.02	0.5050191	0.5050189	0.5050191	0.5050505
0.04	0.5101721	0.5101719	0.5101721	0.5102041
0.06	0.5154313	0.5154312	0.5154313	0.5154639
0.08	0.5208001	0.5207999	0.5208001	0.5208333
0.10	0.5262818	0.5262817	0.5262820	0.5263158
0.20	0.5555179	0.5555178	0.5555184	0.5555556
0.30	0.5881938	0.5881934	0.5881942	0.5882353
0.40	0.6249537	0.6249532	0.6249543	0.6249999
0.50	0.6666148	0.6666143	0.6666154	0.6666666
0.60	0.7142276	0.7142271	0.7142276	0.7142856
0.70	0.7691647	0.7691648	0.7691647	0.7692305
0.80	0.8332583	0.8332582	0.8332574	0.8333330
0.90	0.9090045	0.9090048	0.9090042	0.9090905
1.00	0.9999000	0.9999000	0.9999000	0.9999993

Table 4a Numerical Results of Example 6.1, $\epsilon = 10^{-3}, h = 10^{-2}$

x	y(x)(δ=0.008)	y(x)(δ=0.009)	y(x)(δ=0.01)	Exact Solution
0.00	0.0029709	0.0029712	0.0029714	0.0029980
0.10	0.2925916	0.2926173	0.2926378	0.2927980
0.20	0.5623413	0.5623950	0.5624378	0.5625980
0.30	0.8120908	0.8121728	0.8122379	0.8123980
0.40	1.0418402	1.0419507	1.0420380	1.0421977
0.50	1.2515895	1.2517287	1.2518380	1.2519976
0.60	1.4413388	1.4415061	1.4416381	1.4417975
0.70	1.6110878	1.6112844	1.6114382	1.6115974
0.80	1.7608367	1.7610619	1.7612383	1.7613974
0.90	1.8905855	1.8908401	1.8910384	1.8911973
0.92	1.9141355	1.9143956	1.9145985	1.9147574
0.94	1.9368851	1.9371511	1.9373585	1.9375173
0.96	1.9588349	1.9591066	1.9593185	1.9594773
0.98	1.9799848	1.9802620	1.9804785	1.9806373
1.00	2.0003402	2.0006220	2.0008421	2.0009992

Table 4b Numerical Results of Example 6.1, $\epsilon = 10^{-4}, h = 10^{-2}$

x	y(x)(δ=0.0008)	y(x)(δ=0.0009)	y(x)(δ=0.001)	Exact Solution
0.00	0.0002953	0.0002957	0.0002959	0.0003000
0.10	0.2899159	0.2899417	0.2899623	0.2902800
0.20	0.5596657	0.5597195	0.5597622	0.5602601
0.30	0.8094152	0.8094972	0.8095623	0.8102400
0.40	1.0391645	1.0392751	1.0393623	1.0402197
0.50	1.2489139	1.2490530	1.2491623	1.2501996
0.60	1.4386631	1.4388305	1.4389625	1.4401795
0.70	1.6084121	1.6086087	1.6087625	1.6101594
0.80	1.7581611	1.7583864	1.7585627	1.7601392
0.90	1.8879100	1.8881645	1.8883628	1.8901193
0.92	1.9114598	1.9117200	1.9119228	1.9137152
0.94	1.9342095	1.9344755	1.9346828	1.9365113
0.96	1.9561592	1.9564310	1.956643	1.9585073
0.98	1.9773091	1.9775866	1.9778029	1.9797033
1.00	1.9976645	1.9979465	1.9981666	2.0000994

Table 5a Numerical Results of Example 6.2, $\epsilon = 10^{-3}, h = 10^{-2}$

x	y(x)(δ=0.008)	y(x)(δ=0.009)	y(x)(δ=0.01)	Exact Solution
0.00	0.9990109	0.9990106	0.9990104	0.9990020
0.10	0.9041042	0.9040911	0.9040794	0.9040246
0.20	0.8181680	0.8181447	0.8181243	0.8180769
0.30	0.7404002	0.7403688	0.7403412	0.7403005
0.40	0.6700244	0.6699865	0.6699534	0.6699185
0.50	0.6063378	0.6062952	0.6062578	0.6062279
0.60	0.5487046	0.5486584	0.5486181	0.5485924
0.70	0.4965496	0.4965009	0.4964584	0.4964366
0.80	0.4493520	0.4493017	0.4492577	0.4492393
0.90	0.4066406	0.4065894	0.4065446	0.4065291
0.92	0.3985984	0.3985471	0.3985022	0.3984872
0.94	0.3907152	0.3906639	0.3906189	0.3906045
0.96	0.3829882	0.3829367	0.3828917	0.3828776
0.98	0.3754236	0.3753694	0.3753226	0.3753037
1.00	0.3688113	0.3686759	0.3685626	0.3682464

Table 5b Numerical Results of Example 6.2, $\epsilon = 10^{-4}, h = 10^{-2}$

x	y(x)(δ=0.0008)	y(x)(δ=0.0009)	y(x)(δ=0.001)	Exact Solution
0.00	0.9999009	0.9999010	0.9999009	0.9999000
0.10	0.9049085	0.9048971	0.9048852	0.9047559
0.20	0.8188949	0.8188747	0.8188534	0.8186653
0.30	0.7410570	0.7410300	0.7410012	0.7407664
0.40	0.6706179	0.6705853	0.6705508	0.6702799
0.50	0.6068740	0.6068375	0.6067982	0.6065005
0.60	0.5491892	0.5491496	0.5491071	0.5487899
0.70	0.4969875	0.4969457	0.4969009	0.4965706
0.80	0.4497477	0.4497045	0.4496581	0.4493202
0.90	0.4069981	0.4069541	0.4069070	0.4065291
0.92	0.3989487	0.3989047	0.3988574	0.3984872
0.94	0.3910585	0.3910144	0.3909671	0.3906045
0.96	0.3833245	0.3832803	0.3832330	0.3828776
0.98	0.3757532	0.3757064	0.3756571	0.3753037
1.00	0.3691349	0.3690068	0.3688911	0.3682464

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