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# VAGUE SET THEORY BASED ON $d$-ALGEBRAS

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ABSTRACT. The notions of vague $d$-subalgebras, vague BCK-ideals, vague $d$-ideals, vague $d^\sharp$-ideals and vague $d^*$-ideals are introduced, and their properties are investigated. Relations between vague $d$-subalgebras, vague BCK-ideals, vague $d$-ideals, vague $d^\sharp$-ideals and vague $d^*$-ideals are established.

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## 1. Introduction

Several authors have made a number of generalizations of Zadeh's fuzzy set theory [10]. Of these, the notion of vague set theory introduced by Gau and Buehrer [3] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [2] studied vague groups. Jun and Park [5, 9] studied vague ideals and vague deductive systems in subtraction algebras. In this paper, we also use the notion of vague set in the sense of Gau and Buehrer to discuss the vague theory on $d$-algebras. We introduce the notion of vague $d$-subalgebras, vague BCK-ideals, vague $d$-ideals, vague $d^\sharp$-ideals and vague $d^*$-ideals, and then we investigate their properties. We give relations between vague $d$-subalgebras, vague BCK-ideals, vague $d$-ideals, vague $d^\sharp$-ideals and vague $d^*$-ideals.

## 2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau$. A *BCK-algebra* is a system $(X, *, 0) \in K(\tau)$, where $\tau = (2, 0)$, such that

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)\ (x * x = 0)$,

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(a4) $(\forall x \in X)\ (0 * x = 0)$,
(a5) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

A *d-algebra* is a system $(X, *, 0) \in K(\tau)$, where $\tau = (2, 0)$, that satisfies (a3), (a4) and (a5). We can define a relation $\leq$ on a $d$-algebra $X$ by $x \leq y$ if and only if $x * y = 0$. In a BCK-algebra $(X, *, 0)$ the following hold:

(b1) $(\forall x, y \in X)\ ((x * y) * x = 0)$,
(b2) $(\forall x, y, z \in X)\ (((x * z) * (y * z)) * (x * y) = 0)$.

A $d$-algebra $(X, *, 0)$ is called a *$d^*$-algebra* (see [7]) if it satisfies the identity (b1).

A nonempty subset $I$ of a BCK-algebra $X$ is called a *BCK-ideal* of $X$ if it satisfies the following axioms:

(I1) $0 \in I$,
(I2) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

Let $(X, *, 0)$ be a $d$-algebra and $\emptyset \neq I \subseteq X$. Then $I$ is called a *d-subalgebra* of $X$ if $x * y \in I$ whenever $x, y \in I$, and $I$ is called a *BCK-ideal* of $X$ if it satisfies (I1) and (I2).

A nonempty subset $I$ of a $d$-algebra $(X, *, 0)$ is called a *d-ideal* of $X$ (see [7]) it satisfies (I2) and

$$(\forall x, y \in X)\ (x \in I \Rightarrow x * y \in I). \tag{1}$$

A nonempty subset $I$ of a $d$-algebra $(X, *, 0)$ is called a *$d^\sharp$-ideal* of $X$ (see [7]) if it is a $d$-ideal of $X$ that satisfies the following axiom:

$$(\forall x, y, z \in X)\ (x * y \in I,\ y * z \in I \Rightarrow x * z \in I). \tag{2}$$

If a $d^\sharp$-ideal of a $d$-algebra $X$ satisfies:

$$x * y \in I,\ y * x \in I \Rightarrow (x * z) * (y * z) \in I,\ (z * x) * (z * y) \in I \tag{3}$$

for all $x, y, z \in X$, then we say that $I$ is a *$d^*$-ideal* of $X$ (see [7]).

**Definition 1.** [2] A *vague set* $A$ in the universe of discourse $U$ is characterized by two membership functions given by:

(1) A true membership function

$$t_A : U \to [0, 1],$$

and

(2) A false membership function

$$f_A : U \to [0, 1],$$

where $t_A(u)$ is a lower bound on the grade of membership of $u$ derived from the "evidence for $u$", $f_A(u)$ is a lower bound on the negation of $u$ derived from the "evidence against $u$", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of $u$ in the vague set $A$ is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0,1]$. This indicates that if the actual grade of membership of $u$ is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set $A$ is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle \mid u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of $u$ in $A$, denoted by $V_A(u)$.

Recall that if $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0,1]$, we can define a relation between $I_1$ and $I_2$ by $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$. For $\alpha, \beta \in [0,1]$ we now define $(\alpha,\beta)$-cut and $\alpha$-cut of a vague set.

**Definition 2.** [2] Let $A$ be a vague set of a universe $X$ with the true-membership function $t_A$ and the false-membership function $f_A$. The $(\alpha,\beta)$*-cut* of the vague set $A$ is a crisp subset $A_{(\alpha,\beta)}$ of the set $X$ given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The $(\alpha,\beta)$-cuts of the vague set $A$ are also called *vague-cuts* of $A$.

**Definition 3.** [2] The $\alpha$*-cut* of the vague set $A$ is a crisp subset $A_\alpha$ of the set $X$ given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha,\beta)} = A_\alpha$. Equivalently, we can define the $\alpha$-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

## 3. Vague $d$-algerbas

In this section we first define the notion of vague $d$-subalgebras. For our discussion, we shall use the following notations on interval arithmetic:

Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. We define the term "imax" to mean the maximum of two intervals as

$$\text{imax}(I_1, I_2) := [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2] \in I[0,1]$. Similarly we define "imin". The concepts of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $I[0,1]$.

It is obvious that $L = \{I[0,1], \text{isup}, \text{iinf}, \succeq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$ (see [2]).

In what follows let $X$ denote a $d$-algebra unless specified otherwise.

**Definition 4.** A vague set $A$ of $X$ is called a *vague d-subalgebra* of $X$ if the following condition is true:

$$(\forall x, y \in X)\ (V_A(x * y) \succeq \text{imin}\{V_A(x), V_A(y)\}), \tag{4}$$

that is,

$$\begin{aligned} &t_A(x * y) \geq \min\{t_A(x), t_A(y)\}, \\ &1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \end{aligned} \tag{5}$$

for all $x, y \in X$.

**Example 1.** Consider a $d$-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $0$ | $a$ |
| $b$ | $b$ | $b$ | $0$ | $0$ |
| $c$ | $c$ | $c$ | $c$ | $0$ |

Let $A$ be the vague set in $X$ defined as follows:

$$A = \{\langle 0, [0.7, 0.03]\rangle, \langle a, [0.7, 0.03]\rangle, \langle b, [0.3, 0.08]\rangle, \langle c, [0.6, 0.08]\rangle\}.$$

It is routine to verify that $A$ is a vague $d$-subalgebra of $X$.

**Example 2.** Consider a $d$-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $0$ | $a$ |
| $b$ | $b$ | $b$ | $0$ | $0$ |
| $c$ | $c$ | $c$ | $a$ | $0$ |

Let $A$ be the vague set in $X$ defined as follows:

$$A = \{\langle 0, [0.5, 0.02]\rangle, \langle a, [0.5, 0.03]\rangle, \langle b, [0.3, 0.07]\rangle, \langle c, [0.5, 0.03]\rangle\}.$$

It is routine to verify that $A$ is a vague $d$-subalgebra of $X$.

**Lemma 1.** *Every vague d-subalgebra $A$ of $X$ satisfies:*

$$(\forall x \in X)\ (V_A(0) \succeq V_A(x)), \tag{6}$$

*that is,*

$$t_A(0) \geq t_A(x) \text{ and } 1 - f_A(0) \geq 1 - f_A(x)$$

*for all $x \in X$.*

*Proof.* Let $x \in X$. Then

$$t_A(0) = t_A(x * x) \geq \min\{t_A(x), t_A(x)\} = t_A(x)$$

and

$$1 - f_A(0) = 1 - f_A(x * x) \geq \min\{1 - f_A(x), 1 - f_A(x)\} = 1 - f_A(x).$$

This shows that $V_A(0) \succeq V_A(x)$. □

**Theorem 1.** *Let $A$ be a vague $d$-subalgebra of $X$. Then for any $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ of $A$ is a crisp $d$-subalgebra of $X$.*

*Proof.* Let $x, y \in A_{(\alpha,\beta)}$. Then $V_A(x) \succeq [\alpha, \beta]$, that is, $t_A(x) \geq \alpha$ and $1-f_A(x) \geq \beta$; and $V_A(y) \succeq [\alpha, \beta]$, that is, $t_A(y) \geq \alpha$ and $1 - f_A(y) \geq \beta$. It follows from (5) that

$$t_A(x * y) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$$

and

$$1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \geq \beta,$$

which mean that $V_A(x * y) \succeq [\alpha, \beta]$. Hence $x * y \in A_{(\alpha,\beta)}$. This completes the proof. □

**Definition 5.** Let $S$ be a $d$-subalgebra of $X$. A vague set $A$ of $X$ is called a *vague $d$-subalgebra* of $X$ related to $S$ if it satisfies the following condition:

$$(\forall x, y \in S)\ (V_A(x * y) \succeq \operatorname{imin}\{V_A(x), V_A(y)\}), \tag{7}$$

that is, (5) holds for all $x, y \in S$.

Note that a vague $d$-subalgebra of $X$ related to $X$ means a vague $d$-subalgebra of $X$, and every vague $d$-subalgebra of $X$ is also a vague $d$-subalgebra of $X$ related to $S$ for any $d$-subalgebra $S$ of $X$.

**Example 3.** Consider the $d$-algebra $(X, *, 0)$ which is given in Example 2. Let $A$ be the vague set in $X$ defined by

$$A = \{\langle 0, [0.5, 0.03]\rangle, \langle a, [0.5, 0.02]\rangle, \langle b, [0.3, 0.07]\rangle, \langle c, [0.5, 0.02]\rangle\},$$

and let $S$ be a $d$-subalgebra of $X$. If $S = \{0\}$ or $\{0, b\}$, then $A$ is a vague $d$-subalgebra of $X$ related to $S$. Otherwise, $A$ is not a vague $d$-subalgebra of $X$ related to $S$ since

$$1 - f_A(a * a) = 1 - f_A(0) = 0.97 < 0.98 = \min\{1 - f_A(a), 1 - f_A(a)\}$$

and

$$1 - f_A(c * c) = 1 - f_A(0) = 0.97 < 0.98 = \min\{1 - f_A(c), 1 - f_A(c)\}.$$

**Definition 6.** A vague set $A$ of $X$ is called a *vague BCK-ideal* of $X$ if the following conditions are true:

(c1) $(\forall x \in X)\ (V_A(0) \succeq V_A(x))$,

(c2) $(\forall x, y \in X)\ (V_A(x) \succeq \text{imin}\{V_A(x * y), V_A(y)\})$,

that is,

$$t_A(0) \geq t_A(x),\ 1 - f_A(0) \geq 1 - f_A(x), \tag{8}$$

and

$$\begin{aligned} &t_A(x) \geq \min\{t_A(x * y), t_A(y)\}, \\ &1 - f_A(x) \geq \min\{1 - f_A(x * y), 1 - f_A(y)\} \end{aligned} \tag{9}$$

for all $x, y \in X$.

**Definition 7.** A vague set $A$ of $X$ is called a *vague d-ideal* of $X$ if it satisfies (c2) and

(c3) $(\forall x, y \in X)\ (V_A(x * y) \succeq V_A(x)\})$,

that is,

$$t_A(x * y) \geq t_A(x) \text{ and } 1 - f_A(x * y) \geq 1 - f_A(x) \tag{10}$$

for all $x, y \in X$.

**Example 4.** Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $a$ | $0$ | $a$ |
| $b$ | $b$ | $b$ | $0$ | $c$ | $0$ |
| $c$ | $c$ | $c$ | $b$ | $0$ | $c$ |
| $d$ | $c$ | $c$ | $a$ | $a$ | $0$ |

Then $(X, *, 0)$ is a $d$-algebra. Let $A$ be the vague set in $X$ defined as follows:

$$A = \{\langle 0, [0.6, 0.2]\rangle, \langle a, [0.6, 0.2]\rangle, \langle b, [0.2, 0.3]\rangle, \langle c, [0.2, 0.3]\rangle, \langle d, [0.2, 0.3]\rangle\}.$$

It is routine to verify that $A$ is both a vague BCK-ideal of $X$ and a vague $d$-ideal of $X$.

**Example 5.** Consider a $d$-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $0$ | $a$ |
| $b$ | $b$ | $b$ | $0$ | $0$ |
| $c$ | $c$ | $c$ | $a$ | $0$ |

Let $A$ be the vague set in $X$ defined as follows:

$$A = \{\langle 0, [0.7, 0.2]\rangle, \langle a, [0.7, 0.2]\rangle, \langle b, [0.5, 0.3]\rangle, \langle c, [0.7, 0.2]\rangle\}.$$

It is routine to verify that $A$ is a vague $d$-subalgebra of $X$, but neither a vague BCK-ideal of $X$ nor a vague $d$-ideal of $X$ since

$$t_A(b) = 0.5 < 0.7 = \min\{t_A(b * c), t_A(c)\}$$

and/or

$$1 - f_A(b) = 1 - 0.3 = 0.7 < 0.8 = 1 - 0.2 = \min\{1 - f_A(b * c), 1 - f_A(c)\}.$$

In a $d$-algebra, a vague BCK-ideal need not be a vague $d$-subalgebra, and also a vague $d$-subalgebra need not be a vague BCK-ideal as seen in the following example.

**Example 6.** Let $X = \{0, a, b, c\}$ be a $d$-algebra with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $0$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $0$ |
| $c$ | $c$ | $c$ | $c$ | $0$ |

Then $X$ is not a BCK-algebra. Let $A$ be the vague set in $X$ defined as follows:

$$A = \{\langle 0, [0.8, 0.07]\rangle, \langle a, [0.8, 0.07]\rangle, \langle b, [0.8, 0.07]\rangle, \langle c, [0.09, 0.3]\rangle\}.$$

Then $A$ is a vague BCK-ideal which is not a vague $d$-subalgebra of $X$. The vague set $B$ in $X$ defined by

$$B = \{\langle 0, [0.8, 0.07]\rangle, \langle a, [0.09, 0.2]\rangle, \langle b, [0.09, 0.2]\rangle, \langle c, [0.8, 0.07]\rangle\}$$

is a vague $d$-subalgebra which is not a vague BCK-ideal of $X$.

From (c2) and (c3) it follows that every vague $d$-ideal of $X$ is a vague $d$-subalgebra of $X$, but the converse need not be true.

**Example 7.** Let $X = \{0, a, b, c\}$ be a $d$-algebra with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $0$ | $b$ |
| $b$ | $b$ | $b$ | $0$ | $0$ |
| $c$ | $c$ | $c$ | $c$ | $0$ |

Then $X$ is not a BCK-algebra. Let $A$ be the vague set in $X$ defined as follows:

$$A = \{\langle 0, [0.8, 0.07]\rangle, \langle a, [0.8, 0.07]\rangle, \langle b, [0.09, 0.3]\rangle, \langle c, [0.09, 0.3]\rangle\}.$$

Then $A$ is the vague $d$-subalgebra of $X$, but not a vague $d$-ideal of $X$ since $t_A(a * c) = t_A(b) < t_A(a)$ and/or $1 - f_A(a * c) = 1 - f_A(b) < 1 - f_A(a)$.

**Corollary 1.** *If $A$ is a vague $d$-ideal of $X$, then $V_A(0) \succeq V_A(x)$ for all $x \in X$, i.e.,*

$$t_A(0) \geq t_A(x) \text{ and } 1 - f_A(0) \geq 1 - f_A(x)$$

*for all $x \in X$.*

*Proof.* Since every vague $d$-ideal of $X$ is also a vague $d$-subalgebra of $X$, it is straightforward by Lemma 1. □

Note that every vague $d$-ideal of $X$ is always a vague BCK-ideal of $X$, but the converse need not be true. In Example 7, the vague set $A$ in $X$ is a vague BCK-ideal of $X$, but not a vague $d$-ideal of $X$.

**Proposition 1.** *Every vague BCK-ideal $A$ of $X$ satisfies:*

$$(\forall x, y \in X)\ (x \leq y \Rightarrow V_A(x) \succeq V_A(y)). \tag{11}$$

*Proof.* Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$t_A(x) \geq \min\{t_A(x * y), t_A(y)\} = \min\{t_A(0), t_A(y)\} = t_A(y)$$

and

$$\begin{aligned} 1 - f_A(x) &\geq \min\{1 - f_A(x * y), 1 - f_A(y)\} \\ &= \min\{1 - f_A(0), 1 - f_A(y)\} \\ &= 1 - f_A(y). \end{aligned}$$

This shows that $V_A(x) \succeq V_A(y)$. □

**Corollary 2.** *Every vague $d$-ideal $A$ of $X$ satisfies (11).*

**Proposition 2.** *Every vague BCK-ideal $A$ of $X$ satisfies:*

(i) $(\forall x, y \in X)\ (x * y = 0 \Rightarrow V_A(x) \succeq V_A(y))$.

(ii) $(\forall x, y, z \in X)\ ((x * y) * z = 0 \Rightarrow V_A(x) \succeq \text{imin}\{V_A(y), V_A(z)\})$.

*Proof.* (i) is straightforward by Proposition 1.

(ii) Let $x, y, z \in X$ be such that $(x * y) * z = 0$. By (i), we have $V_A(x * y) \succeq V_A(z)$. It follows from (c2) that

$$V_A(x) \succeq \text{imin}\{V_A(x * y), V_A(y)\} \succeq \text{imin}\{V_A(y), V_A(z)\}.$$

This completes the proof. □

**Corollary 3.** *Every vague $d$-ideal $A$ of $X$ satisfies:*

(i) $(\forall x, y \in X)\ (x * y = 0 \Rightarrow V_A(x) \succeq V_A(y))$.

(ii) $(\forall x, y, z \in X)\ ((x * y) * z = 0 \Rightarrow V_A(x) \succeq \text{imin}\{V_A(y), V_A(z)\})$.

**Definition 8.** A vague set $A$ of $X$ is called a *vague $d^\sharp$-ideal* of $X$ if it is a vague $d$-ideal of $X$ that satisfies:

(c4) $(\forall x, y, z \in X)\ (V_A(x * z) \succeq \text{imin}\{V_A(x * y), V_A(y * z)\})$,

that is,

$$\begin{array}{l} t_A(x * z) \geq \min\{t_A(x * y), t_A(y * z)\}, \\ 1 - f_A(x * z) \geq \min\{1 - f_A(x * y), 1 - f_A(y * z)\} \end{array} \tag{12}$$

for all $x, y, z \in X$.

**Example 8.** Consider the $d$-algebra $X$ which is given in Example 7. Let $A$ be the vague set in $X$ defined as follows:

$$A = \{\langle 0, [0.8, 0.07]\rangle, \langle a, [0.8, 0.07]\rangle, \langle b, [0.8, 0.07]\rangle, \langle c, [0.09, 0.4]\rangle\}.$$

Then $A$ is a vague $d^\sharp$-ideal of $X$.

Obviously, every vague $d^\sharp$-ideal of $X$ is a vague $d$-ideal of $X$, but the converse may not be true as seen in the following example.

**Example 9.** Let $A$ be the vague set of $X$ which is described in Example 4. Then $A$ is a vague $d$-ideal of $X$ (see Example 4), but $A$ is not a vague $d^\sharp$-ideal of $X$ since

$$V_A(b * c) = V_A(c) \not\succeq V_A(0) = \text{imin}\{V_A(b * d), V_A(d * c)\}.$$

We now give a condition for a vague BCK-ideal of $X$ to be a vague $d$-ideal of $X$.

**Theorem 2.** *In a $d^*$-algebra, every vague BCK-ideal is a vague $d$-ideal.*

*Proof.* Let $A$ be a vague BCK-ideal of a $d^*$-algebra $X$ and let $x, y \in X$. Then

$$t_A(x * y) \geq \min\{t_A((x * y) * x), t_A(x)\} = \min\{t_A(0), t_A(x)\} = t_A(x)$$

and

$$\begin{aligned} 1 - f_A(x * y) &\geq \min\{1 - f_A((x * y) * x), 1 - f_A(x)\} \\ &= \min\{1 - f_A(0), 1 - f_A(x)\} = 1 - f_A(x). \end{aligned}$$

Hence $A$ is a vague $d$-ideal of $X$. □

**Corollary 4.** *In a $d^*$-algebra, every vague BCK-ideal is a vague $d$-subalgebra.*

**Definition 9.** If a vague $d^\sharp$-ideal $A$ of $X$ satisfies

$$\text{(c5)}\ (\forall x, y, z \in X)\ (\operatorname{imin}\{V_A((x*z)*(y*z)), V_A((z*x)*(z*y))\} \succeq \operatorname{imin}\{V_A(x*y), V_A(y*x)\}),$$

that is,

$$\min\{t_A((x*z)*(y*z)), t_A((z*x)*(z*y))\} \geq \min\{t_A(x*y), V_A(y*x)\}$$

and

$$\min\{1-f_A((x*z)*(y*z)), 1-f_A((z*x)*(z*y))\} \geq \min\{1-f_A(x*y), 1-f_A(y*x)\}$$

for all $x, y, z \in X$, then we say that $A$ is a *vague $d^*$-ideal* of $X$.

**Example 10.** Consider the $d$-algebra $X$ which is given in Example 5. Then $X$ is not a BCK-algebra. Let $A$ be the vague set in $X$ defined as follows:

$$A = \{\langle 0, [0.7, 0.2]\rangle, \langle a, [0.7, 0.2]\rangle, \langle b, [0.05, 0.6]\rangle, \langle c, [0.05, 0.6]\rangle\}.$$

Then $A$ is a vague $d^*$-ideal of $X$.

Obviously, every vague $d^*$-ideal in a $d$-algebra is a vague $d^\sharp$-ideal, but the converse does not hold in general.

**Example 11.** Consider a $d$-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $0$ | $a$ |
| $b$ | $c$ | $b$ | $0$ | $c$ |
| $c$ | $c$ | $b$ | $b$ | $0$ |

Then $X$ is not a BCK-algebra. Let $A$ be the vague set in $X$ defined as follows:

$$A = \{\langle 0, [0.7, 0.03]\rangle, \langle a, [0.7, 0.03]\rangle, \langle b, [0.07, 0.8]\rangle, \langle c, [0.07, 0.8]\rangle\}.$$

Then $A$ is a vague $d^\sharp$-ideal of $X$, but not a vague $d^*$-ideal of $X$ since

$$\begin{aligned}&\operatorname{imin}\{V_A(0*a), V_A(a*0)\} = V_A(a)\\ &\succeq V_A(b) = \operatorname{imin}\{V_A(0), V_A(b)\}\\ &= \operatorname{imin}\{V_A((0*c)*(a*c)), V_A((c*0)*(c*a))\}.\end{aligned}$$

Since every BCK-algebra is also a $d$-algebra, we immediately obtain the following result from Proposition 2(i).

**Lemma 2.** *Every vague BCK-ideal $A$ of a BCK-algebra $X$ satisfies:*

$$(\forall x, y \in X)\ (x*y=0 \Rightarrow V_A(x) \succeq V_A(y)). \tag{13}$$

**Theorem 3.** *If $X$ is a BCK-algebra, then every vague BCK-ideal of $X$ is a vague $d^*$-ideal of $X$.*

*Proof.* Let $A$ be a vague BCK-ideal of a BCK-algebra $X$ and let $x, y, z \in X$. Since $(x * y) * x = 0$ by (b1), it follows from Lemma 2 that $V_A(x * y) \succeq V_A(x)$, proving (c3). Note that $((x*z)*(y*z))*(x*y) = 0$, and hence $V_A((x*z)*(y*z)) \succeq V_A(x*y)$ by Lemma 2. Using (c2), we have

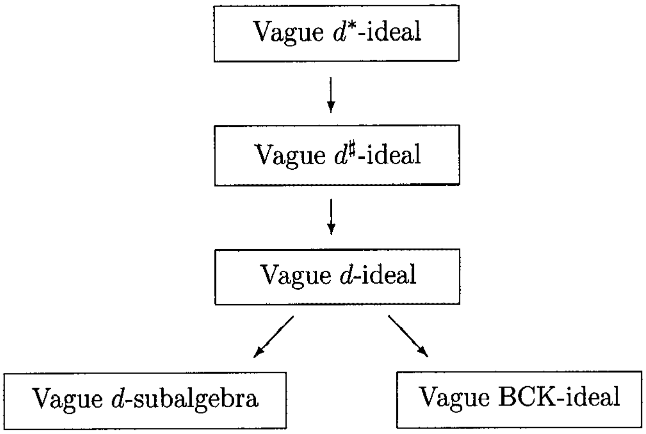
$$V_A(x * z) \succeq \mathrm{imin}\{V_A((x * z) * (y * z)), V_A(y * z)\} \succeq \mathrm{imin}\{V_A(x * y), V_A(y * z)\}.$$

This proves (c4). By (a1) and (b2), we get $((z * x) * (z * y)) * (y * x) = 0$ and $((x * z) * (y * z)) * (x * y) = 0$. It follows from Lemma 2 that $V_A((z * x) * (z * y)) \succeq V_A(y * x)$ and $V_A((x * z) * (y * z)) \succeq V_A(x * y)$. Hence

$$\mathrm{imin}\{V_A((x * z) * (y * z)), V_A((z * x) * (z * y))\} \succeq \mathrm{imin}\{V_A(x * y), V_A(y * x)\},$$

showing (c5). This completes the proof. □

**Remark 1.** (1) We have the following diagram in which reverse implications is not valid.




(2) In a $d^*$-algebra, the concepts of vague $d$-ideal and vague BCK-ideal coincide.

(3) In a BCK-algebra, the concepts of vague $d$-ideal, vague $d^{\sharp}$-ideal, vague $d^*$-ideal and vague BCK-ideal coincide.

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