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OBLIQUE PROJECTIONS AND SHIFT-INVARIANT SPACES

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ABSTRACT. We give an elementary proof of one of the main results in [H.O. Kim, R.Y. Kim. J.K. Lim, The infimum cosine angle between two finitely generated shift-invariant spaces and its applications, Appl. Comput. Harmon. Anal. 19 (2005) 253–281] concerning the existence of an oblique projection onto a finitely generated shift-invariant space along the orthogonal complement of another finitely generated shift-invariant space under the assumption that the generators generate Riesz bases.

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1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space over the complex field $\mathbb{C}$. Suppose that U and V are closed subspaces of $\mathcal{H}$. Then, the *oblique projection* $P_{U\perp V}$ of $\mathcal{H}$ on U along $V^\perp$ is well-defined if $\mathcal{H} = U \dot{+} V^\perp$, i.e., $\mathcal{H} = U + V^\perp$ and $U \cap V^\perp = \{0\}$ [1]. In this case, for each $f \in \mathcal{H}$, $f = u + v^\perp$ for unique $u \in U$ and $v^\perp \in V^\perp$. Hence $P_{U\perp V} f = u$ is a well-defined bounded operator. It is well-known that the existence of the oblique projection is closely related with the concept of the following *angle* $R(U, V)$ between the two closed spaces U and V [9]:

$$R(U,V) = \operatorname*{ess-inf}_{u \in U \setminus \{0\}} \frac{\|P_V u\|}{\|u\|},$$

where P_V denotes the orthogonal projection of $\mathcal{H}$ onto V. In general, $R(U,V) \neq R(V,U)$. It is shown in [9] that the oblique projection exists if and only if $R(U,V) > 0$ and $R(V,U) > 0$. In this article, we show, using elementary methods, that if U and V are finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$ with Riesz generators, then the angle condition $R(U,V)$ can be concretely

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realized as the essential supremum of the operator norm of certain family of matrices.

First, we need some definitions. A sequence $\{f_i\}_{i\in I} \subset \mathcal{H}$ is said to be a *Riesz basis* if there exist positive constants A and B, called *Riesz bounds*, such that, for each $\{c_i\}_{i\in I} \in \ell^2(I)$

$$A\sum_{i\in I}|c_i|^2 \le \left\|\sum_{i\in I} c_i f_i\right\|^2 \le B\sum_{i\in I}|c_i|^2.$$

It turns out that a Riesz basis is a bounded unconditional Schauder basis of $\mathcal{H}$. If, for each $f \in \mathcal{H}$,

$$A\|f\|^2 \le \sum_{i\in I}|\langle f, f_i\rangle|^2 \le B\|f\|^2$$

then we say that $\{f_i\}_{i\in I}$ is a *frame* for $\mathcal{H}$ with frame bounds A and B. It is well-known that a Riesz basis with bounds A and B is also a frame with bounds A and B [10].

A closed subspace S of $L^2(\mathbb{R}^d)$ is said to be *shift-invariant* if $f(x-\alpha) \in S$ for each $f(x) \in S$ and each $\alpha \in \mathbb{Z}^d$ [3, 8]. If S is the closed linear span of $\{f(x-\alpha) : f \in F\}$ for some $F \subset S$, then we say that F *generates* S. For $x \in \mathbb{T}^d = [0,1]^d$ and $f \in L^2(\mathbb{R}^d)$, we define

$$\hat{f}_{||x} = (\hat{f}(x-\alpha))_{\alpha\in\mathbb{Z}^d},$$

which is well-defined a.e. Here the Fourier transform $\hat{f}$ of f is defined via

$$\hat{f}(x) = \int_{\mathbb{R}^d} f(t)e^{-2\pi i x\cdot t}\,dt$$

for $f \in L^2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and extended to be an isometry of $L^2(\mathbb{R}^d)$ by a theorem of Plancherel. We let $\hat{S}_{||x} = \{\hat{f}_{||x} : f \in S\}$ for $x \in \mathbb{T}^d$. Then $f \perp S$ if and only if $\hat{f}_{||x} \perp \hat{s}_{||x}$ for a.e. $x \in \mathbb{T}^d$ and for each $s \in S$ [2].

Then the following facts are well-known [3, 5, 8]:

Proposition 1. *Suppose that S is a shift-invariant generated by $\Phi = \{\phi_1, \cdots, \phi_n\}$. For $x \in \mathbb{T}^d$, define the following Gramian of Φ at x via*

$$G_\Phi(x) = \left(\langle \hat{\phi}_{j_{||x}}, \hat{\phi}_{i_{||x}}\rangle\right)_{1\le i,j\le n},$$

which is easily seen to be a non-negative definite $n \times n$ matrix. Then $\left\{\phi(x-\alpha) : \alpha \in \mathbb{Z}^d, \phi \in \Phi\right\}$ is a Riesz basis of S with Riesz bound A and B if and only if, for a.e. $x \in \mathbb{T}^d$, $AI_n \le G_\Phi(x) \le BI_n$, where I_n is the $n \times n$ identity matrix. This, in turn, holds if and only if $\left\{\hat{\phi}_{j_{||x}} : 1 \le j \le r\right\}$ is a Riesz basis for $\hat{S}_{||x}$ a.e. with uniform Riesz bounds A and B. Note that, in this case, $G_\Phi(x)$ is invertible a.e.

Moreover, in this case, $f \in S$ if and only if there exist $a_1, \cdots, a_n \in L^2(\mathbb{T}^d)$ such that $\hat{f}_{||x} = \sum_{j=1}^{n} a_j(x)\hat{\phi}_{j\,||x}$ *a.e.*

For the applications of the theory of shift-invariant spaces to wavelets and Gabor systems, see [3, 4, 8].

In this article, we give an elementary proof of one of the main results in [7] concerning the existence of an oblique projection of $L^2(\mathbb{R}^d)$ onto a finitely generated shift-invariant space along the orthogonal complement of another finitely generated shift-invariant space in the case that the finitely generated shift-invariant spaces have the same number of Riesz generators. Even though Theorem 4.10 in [7] is nice and general, the proof is rather complicated. Hence it is not easy to grasp what really is behind the result. We believe our proof is much elementary and conveys the main idea of the proof the above-mentioned result.

2. Main results

We first characterize the existence of the oblique projection of two shift-invariant spaces via mixed Gramians under mild conditions.

Throughout the rest of this article we assume the following:
Let $\Phi = \{\phi_1, \phi_2, \cdots, \phi_r\}$, $\Psi = \{\psi_1, \psi_2, \cdots, \psi_r\} \subset L^2(\mathbb{R}^d)$, and let U be the shift-invariant space generated by Φ and V the shift-invariant space generated by Ψ. Suppose that $\{\phi(x-\alpha) : \phi \in \Phi, \alpha \in \mathbb{Z}^d\}$ is a Riesz basis of U and $\{\psi(x-\alpha) : \psi \in \Psi, \alpha \in \mathbb{Z}^d\}$ is a Riesz basis of V. We may also assume that their common Riesz bounds are A and B. Note that this situation is not uncommon in many applications [8]. Let

$$G(x) = G_{\Phi,\Psi}(x) = \left(\langle \hat{\phi}_{j\,||x}, \hat{\psi}_{i\,||x}\rangle\right)_{1 \le i,j \le r}, \quad x \in \mathbb{T}^d,$$

be the *mixed Gramian* of Φ and Ψ, and let $G_\Phi(x)$ and $G_\Psi(x)$ be the Gramians of Φ and Ψ, respectively. Recall that the three matrices are well-defined a.e. We need the following lemma:

Lemma 1. *If $P_{U\perp V}$ is well-defined, then so are $P_{\hat{U}_{||x}\perp \hat{V}_{||x}}$ a.e. Moreover,* $\|P_{U\perp V}\| = \operatorname{ess-sup}_{x \in \mathbb{T}^d} \left\|P_{\hat{U}_{||x}\perp \hat{V}_{||x}}\right\|$.

Proof. The first part follows by slightly adapting the argument in the proof of Lemma 3.7 in [6]. The second part follows from the first part and Theorem 4.5 of [3] by noticing that $x \to P_{\hat{U}_{||x}\perp \hat{V}_{||x}}$ is the range operator of the clearly shift-preserving operator $P_{U\perp V}$. □

Theorem 1. *The oblique projection of $L^2(\mathbb{R}^d)$ on U along $V^\perp$ is well-defined if and only if $G(x)^{-1}$ exist and their norms are bounded above uniformly a.e. $x \in \mathbb{T}^d$.*

Proof. ($\Rightarrow$): Let $b = (b_1, b_2, \cdots, b_r)^T \in \mathbb{C}^r$ be arbitrary. Define

$$c(x) = (c_1(x), \cdots, c_r(x))^T = G_\Psi(x)^{-1}b, \tag{1}$$

$$\hat{g}_{||x} = \sum_{i=1}^{r} c_i(x)\hat{\psi}_{i||x}. \tag{2}$$

Note that $c(x)$ is a 1-periodic r-dimensional vector-valued function and $G_\Psi(x)^{-1} \leq (1/A)I_r$ a.e. Hence

$$\sum_{j=1}^{r} \int_{\mathbb{T}^d} |c_i(x)|^2\, dx = \int_{\mathbb{T}^d} \left\|G_\Psi(x)^{-1}b\right\|^2_{\mathbb{C}^r}\, dx \leq \frac{1}{A^2}\int_{\mathbb{T}^d} \|b\|^2_{\mathbb{C}^r}\, dx < \infty.$$

Hence each $c_i \in L^2(\mathbb{T}^d)$. Therefore (2) defines $g \in V$ by Proposition 1. Then there exist unique $u \in U, v^\perp \in V^\perp$ such that $g = u + v^\perp$. Since $u \in U$, there exist $a_i \in L^2(\mathbb{T}^d), 1 \leq i \leq r$ such that

$$\hat{u}_{||x} = \sum_{i=1}^{r} a_i(x)\hat{\phi}_{i||x}, \tag{3}$$

by Proposition 1. Since $g - u \in V^\perp$, $\hat{g}_{||x} - \hat{u}_{||x} \perp \hat{V}_{||x}$ a.e. (see the discussion in Introduction). Therefore, $\left\langle \hat{u}_{||x}\hat{\psi}_{l||x}\right\rangle = \left\langle \hat{g}_{||x}\hat{\psi}_{l||x}\right\rangle$ for each $l = 1, \cdots, r$. Now notice that

$$\left\langle \hat{u}_{||x}\hat{\psi}_{l||x}\right\rangle = \sum_{i=1}^{r} G(x)_{l,i}a_i(x),$$

and that

$$\begin{aligned}\langle \hat{g}_{||x}, \hat{\psi}_{l||x}\rangle &= \sum_{i=1}^{r}\sum_{j=1}^{r}(G_\Psi(x)^{-1})_{i,j}b_j G_\Psi(x)_{l,i} \\ &= \sum_{j=1}^{r} b_j \sum_{i=1}^{r}(G_\Psi(x)^{-1})^T_{j,i}G_\Psi(x)^T_{i,l} \\ &= \sum_{j=1}^{r} b_j\delta_{j,l} = b_l.\end{aligned}$$

Hence $G(x)a(x) = b$ a.e. This shows that $G(x)$ is invertible a.e.

It remains to show that $||G(x)^{-1}|| \leq M$ a.e. $x \in \mathbb{T}^d$ for some $M < \infty$. Let $||b||^2_{\mathbb{C}^r} = 1$. Then from what we have shown above we have $a(x) = G(x)^{-1}b$. Since $\{\hat{\phi}_{i||x} : 1 \leq i \leq r\}$ is a Riesz basis for $\hat{U}_{||x}$ with Riesz bounds A and B,

(3) implies that $||a(x)||^2_{\mathbb{C}^r} \leq (1/A)||\hat{u}_{||x}||^2_{\ell^2(\mathbb{Z}^d)}$. Moreover,

$$\left\|\hat{u}_{||x}\right\|^2_{\ell^2(\mathbb{Z}^d)} = \left\|P_{\hat{U}_{||x}\perp\hat{V}_{||x}}\hat{g}_{||x}\right\|^2_{\ell^2(\mathbb{Z}^d)} \leq \|P_{U\perp V}\|^2 \, ||\hat{g}_{||x}||^2_{\ell^2(\mathbb{Z}^d)}$$

by Lemma 1. Since $\left\{\hat{\psi}_{i||x} : 1 \leq i \leq r\right\}$ is a Riesz basis for $\hat{V}_{||x}$ with Riesz bounds A and B a.e., $\left\|\hat{g}_{||x}\right\|^2_{\ell^2(\mathbb{Z}^d)} \leq B\,\|c(x)\|^2_{\mathbb{C}^r}$ a.e. by (2). Finally, $\|c(x)\|^2_{\mathbb{C}^r} \leq (1/A)^2$ a.e. by (1). This shows that the norm of $G(x)^{-1}$ is bounded above uniformly a.e.

($\Leftarrow$): Let $f \in L^2(\mathbb{R}^d)$, $a_i(x) = \left\langle \hat{f}_{||x}, \hat{\psi}_{i||x}\right\rangle, x \in \mathbb{T}^d, 1 \leq i \leq r$, $a(x) = (a_1(x), \cdots, a_r(x))^T$, and let $\hat{g}_{||x} = \sum_{i=1}^{r} b_i(x)\hat{\phi}_{i||x}$, where $b(x) = G(x)^{-1}a(x)$. Now, let $P_{\hat{V}_{||x}}$ denote the orthogonal projection of $\ell^2(\mathbb{Z}^d)$ onto $\hat{V}_{||x}$. Then,

$$\begin{aligned}\sum_{j=1}^{r} |a_j(x)|^2 &= \sum_{j=1}^{r} |\langle \hat{f}_{||x}, \hat{\psi}_{j||x}\rangle|^2 \\ &= \sum_{j=1}^{r} |\langle P_{\hat{V}_{||x}}\hat{f}_{||x}, \hat{\psi}_{j||x}\rangle|^2 \leq B\left\|P_{\hat{V}_{||x}}\hat{f}_{||x}\right\|^2_{\ell^2(\mathbb{Z}^d)} \leq B\left\|\hat{f}_{||x}\right\|^2_{\ell^2(\mathbb{Z}^d)},\end{aligned}$$

which holds since, being a Riesz basis of $\hat{V}_{||x}$ with Riesz bounds A and B, $\left\{\hat{\psi}_{j||x}\right\}_{j=1}^{r}$ is a frame for $\hat{V}_{||x}$ with bounds A and B. It is now easy to see that $a_j \in L^2(\mathbb{T}^d)$ for each j. Since the operator norm $G(x)^{-1}$ is bounded above uniformly a.e., $b_j \in L^2(\mathbb{T}^d)$ for each j. Then $g \in U$ by Proposition 1. Since $f = g + (f - g)$, it is enough to show that $f - g \perp V$. This is tantamount to showing that $\left\langle \hat{f}_{||x}\hat{\psi}_{l||x}\right\rangle = \left\langle \hat{g}_{||x}\hat{\psi}_{l||x}\right\rangle$, $x \in \mathbb{T}^d, 1 \leq l \leq r$. Now,

$$\begin{aligned}\langle \hat{g}_{||x}, \hat{\psi}_{l||x}\rangle &= \sum_{i=1}^{r}\sum_{j=1}^{r} G(x)^{-1}_{i,j}a_j(x)G(x)_{l,i} \\ &= \sum_{j=1}^{r} a_j(x)\delta_{j,l} \\ &= a_l(x) = \left\langle \hat{f}_{||x}\hat{\psi}_{l||x}\right\rangle.\end{aligned}$$

For the uniqueness of this decomposition we argue as follows: Let $f \in L^2(\mathbb{R}^d)$. Suppose that $f = u + v^{\perp} \in U + V^{\perp}$. Then, for a.e. $x \in \mathbb{T}^d$, there exist unique $a_1(x), \cdots, a_r(x)$ such that $\hat{u}_{||x} = \sum_{i=1}^{r} a_i(x)\hat{\phi}_{i||x}$ since $\hat{u}_{||x} \in \hat{V}_{||x}$ and $\left\{\hat{\phi}_{i||x}\right\}_{i=1}^{r}$ is a Riesz basis of $\hat{V}_{||x}$. A calculation similar to the previous ones shows that $a(x) = G(x)^{-1}b(x)$, where $a(x) = (a_1(x), \cdots, a_r(x))^T, b(x) =$

$\left(\langle \hat{f}_{||x}, \hat{\psi}_{1||x}\rangle, \cdots, \langle \hat{f}_{||x}, \hat{\psi}_{r||x}\rangle\right)^T$. This shows that the decomposition is unique. □

We now give a formula for the angle between U and V (see [6, Equation (4.1)]).

Lemma 2. *If the assumptions of Theorem 1 are satisfied, then*

$$R(U,V) = R(V,U) = \operatorname*{ess-inf}_{x\in\mathbb{T}^d} \left\| G_\Phi(x)^{1/2} G(x)^{-1} G_\Psi(x)^{1/2} \right\|^{-1}.$$

Proof. We have $R(U,V) = \operatorname*{ess-inf}_{x\in\mathbb{T}^d} R\left(\hat{U}_{||x}, \hat{V}_{||x}\right)$ by Proposition 2.10 of [4]. Now let $a = (a_1, \cdots, a_r)^T \in \mathbb{C}^r \setminus \{0\}$. Then $\left\|\sum_{i=1}^r a_i \hat{\phi}_{i||x}\right\|^2 = \langle G_\Phi(x)a, a\rangle_{\mathbb{C}^r}$. Let $b = (b_1, \cdots, b_r)^T$ be such that $\sum_{i=1}^r b_i \hat{\psi}_{i||x}$ is the orthogonal projection of $\sum_{i=1}^r a_i \hat{\phi}_{i||x}$ onto $\hat{V}_{||x}$. Then, for each $l = 1, \cdots, r$, the following should be satisfied:

$$\left\langle \sum_{i=1}^r a_i \hat{\phi}_{i||x} - \sum_{i=1}^r b_i \hat{\psi}_{i||x} \hat{\psi}_{l||x} \right\rangle_{\mathbb{C}^r} = 0.$$

This leads us to:

$$G_\Psi(x) b = G(x) a.$$

Hence

$$b = G_\Psi(x)^{-1} G(x) a.$$

A direct calculation shows that

$$\left\|\sum_{i=1}^r b_i \hat{\psi}_{i||x}\right\|^2 = \left\langle G(x) a G_\Psi(x)^{-1} G(x) a\right\rangle_{\mathbb{C}^r}.$$

This shows that

$$R\left(\hat{U}_{||x}, \hat{V}_{||x}\right) = \inf_{a\in\mathbb{C}^r\setminus\{0\}} \left(\frac{\langle G(x)a, G_\Psi(x)^{-1}G(x)a\rangle_{\mathbb{C}^r}}{\langle G_\Phi(x)a, a\rangle_{\mathbb{C}^r}}\right)^{\frac{1}{2}}.$$

Notice that $G_\Phi(x)$ and $G_\Psi(x)$ are strictly positive-definite. In particular, they are hermitian. Recall that $G(x)$ is invertible. We now use a change of variables such that $a = G(x)^{-1}G_\Psi(x)^{1/2}b$. As a runs over $\mathbb{C}^r \setminus \{0\}$, so does b. We have

$$\begin{aligned}
&\inf_{a\in\mathbb{C}^r\setminus\{0\}} \left(\frac{\langle G(x)a, G_\Psi(x)^{-1}G(x)a\rangle_{\mathbb{C}^r}}{\langle G_\Phi(x)a, a\rangle_{\mathbb{C}^r}} \right)^{\frac{1}{2}} \\
&= \left(\sup_{a\in\mathbb{C}^r\setminus\{0\}} \frac{\langle G_\Phi(x)a, a\rangle_{\mathbb{C}^r}}{\langle G(x)a, G_\Psi(x)^{-1}G(x)a\rangle_{\mathbb{C}^r}} \right)^{-\frac{1}{2}} \\
&= \left(\sup_{b\in\mathbb{C}^r\setminus\{0\}} \frac{\langle G_\Phi(x)G(x)^{-1}G_\Psi(x)^{1/2}b, G(x)^{-1}G_\Psi(x)^{1/2}b\rangle_{\mathbb{C}^r}}{||b||^2_{\mathbb{C}^r}} \right)^{-\frac{1}{2}} \\
&= \left\| G_\Psi(x)^{1/2}G(x)^{*-1}G_\Phi(x)G(x)^{-1}G_\Psi(x)^{1/2} \right\|^{-\frac{1}{2}} \\
&= \left\| G_\Phi(x)^{1/2}G(x)^{-1}G_\Psi(x)^{1/2} \right\|^{-1}.
\end{aligned}$$

Hence

$$R(U, V) = \operatorname*{ess-inf}_{x\in\mathbb{T}^d} \left\| G_\Phi(x)^{1/2}G(x)^{-1}G_\Psi(x)^{1/2} \right\|^{-1}.$$

We have, by symmetry,

$$R(V, U) = \operatorname*{ess-inf}_{x\in\mathbb{T}^d} \left\| G_\Psi(x)^{1/2}G_{\Psi,\Phi}(x)^{-1}G_\Phi(x)^{1/2} \right\|^{-1}.$$

$R(U, V) = R(V, U)$ since $G_{\Psi,\Phi}(x) = G(x)^*$. □

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