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# MULTIPLE SCALE ANALYSIS OF A DELAYED PREDATOR PREY MODEL WITHIN RANDOM ENVIRONMENT

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ABSTRACT. We consider a delayed predator prey model. The local stability and Hopf bifurcation results are stated taking the time delay as a control parameter. We apply multiple scale analysis to analyze the effects of additive white noises near the Hopf bifurcation point at the positive interior equilibrium state. The governing equations for the amplitude of oscillations on a slow time scale are derived. We identify the process of amplitude of oscillations and derive its transient properties. We show that oscillations, which would decay in the deterministic system whenever time delay lies below its critical value, persists for long time under the validity of multiple scale analysis..

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## 1. Introduction

Predator-prey systems are very important in the models of multi-species population dynamics and have been studied by many authors (see [2, 6, 11, 12, 20, 22, 23, 28] and references cited therein). Two dimensional deterministic predator-prey models have only two basic patterns: either approach to an equilibrium point or to a limit cycle. It is a well known fact that past history as well as current environmental conditions have the ability to influence population dynamics and such interactions has motivated the introduction of time delays in population growth models. In most of the natural systems, population of one species does not respond instantaneously to changes in the environment or the interactions with other species of populations within their habitats. In recent years a large number of models involving time delays have been developed and analyzed using various mathematical techniques. It is commonly observed that

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the time delays have destabilizing effect on the dynamical behaviour and often time delays are responsible for the population oscillations within deterministic environment. Time delays of one type or another have been incorporated into predator-prey models by many researchers [1, 3, 4, 5, 7, 8, 9, 16, 19, 24] and they have observed various dynamical behaviours exhibited by the delay differential equation (DDE) model systems. Solutions to DDE's exhibit many interesting properties, among them, existence of periodic solutions gives us opportunity to explain the observed oscillatory behavior induced by a delay in time variable.

Major parts of the work in this direction are based on deterministic models of differential equations. The deterministic approach has some limitations as it is always difficult to predict the future dynamics of the model system accurately knowing its state at an earlier time. The deterministic approach is based upon the hypothesis that in case of large populations, stochastic deviations (or effect of random environmental fluctuation) are small enough to be ignored. A stochastic model provides a more realistic picture of an ecological system compared to its deterministic counterpart. A major obstacle in the stochastic modelling of an ecosystem is the lack of mathematical machinery available to analyze non-linear multi-dimensional stochastic models.

In the presence of noise, DDEs are referred to as stochastic delay differential equation (SDDE) and SDDE models have been studied by various researchers in last few years (see [10, 15, 17, 18, 21, 26] and references cited therein). Well-known deterministic DDE models are the starting point of SDDE models which include demographic or environmental stochasticity by adding noise term either in the parameters involved with the model system or directly to the growth equations. The resulting stochastic models involve non-linear stochastic differential equations with delay parameters whose solutions pose great difficulties. In particular, analytical tools and methods for such nonlinear equations are not well developed. In most of the cases, different linearization techniques of non-linear stochastic differential equations give rise to a set of deterministic moment equations and the magnitude of second and higher order moments give us the information about the stochastic stability of the model system under consideration.

In this paper we consider a well-known delayed predator-prey model within fluctuating environment, SDDE model is developed from DDE model by adding white noise terms in growth equations of both prey and predator population. We derive the threshold level of the delay parameter for which system undergoes a Hopf bifurcation and small amplitude periodic solution exists around the coexisting equilibrium point. We apply multiple scale analysis as developed in [14] to study the effects of additive white noises on the dynamical behavior of the model system near the positive interior equilibrium point at Hopf frequency. We assume the form of the periodic solution resulting from Hopf bifurcation around the interior equilibrium point in such a way that the amplitude of oscillations evolves stochastically on a slow time scale. We derive the governing equations for the amplitude of oscillations on a slow time scale for the SDDE model and show that these governing equations are also coupled SDDEs in terms of small delay parameter. The dynamics are sensitive to noise when the noise intensities are small and balanced with the proximity of perturbation. In this case the oscillatory behaviour sustains for a long time near the positive interior equilibrium state. We identify the process of amplitude of oscillations and show that the process is a stationary Gaussian process whenever the time delay lies below its critical value. Stochastic fluctuations govern the dynamics of the SDDE model system whenever the noise intensities are very large compared to the proximity of perturbation. The condition for the validity of multi-scale analysis with fixed but very small values of noise intensities provides a parametric regime where the oscillatory behavior persists for a long time.

The paper is organized as follows: In the next section we present the mathematical model describing delayed predator prey interaction and there after briefly discuss the local stability and Hopf bifurcation results by considering time delay as bifurcation parameter. In section 3 we describe the standard multiple scale analysis to obtain the periodic solution of DDE model system [13]. Section 4 consists of the applicability of multiple scale analysis in the presence of additive white noises near the positive interior equilibrium state at Hopf frequency and we establish the fact that the amplitude of oscillations evolves stochastically on a slow time scale. We made an attempt to identify the nature of stochastic oscillation for the amplitudes of oscillatory solution in section 5. The basic outcomes of multiple scale analysis are mentioned in the concluding section.

## 2. The delayed model : Stability results

Classical two-dimensional predator-prey dynamics is governed by the following system of nonlinear ordinary differential equations

$$\frac{dN_1}{dt} = R(N_1)N_1 - F(N_1, N_2)N_2 \tag{2.1a}$$

$$\frac{dN_2}{dt} = eF(N_1, N_2)N_2 - \delta(N_2)N_2 \tag{2.1b}$$

subject to the initial conditions $N_1(0) = N_{10} > 0$, $N_2(0) = N_{20} > 0$. Here $N_1$ and $N_2$ denote the populations of prey and predator species at any instant of time '$t$'. $R(N_1)$ is the density dependent specific growth rate of prey in the absence of predator, $\delta(N_2)$ is the per capita decline rate of predator in the absence of prey. The predator consumes the prey with functional response $F(N_1, N_2)$ and '$e$' denotes the conversion rate of prey biomass into predator biomass and satisfies the restriction '$0 < e < 1$'. In this paper we have considered Holling-type III functional response, which depends on the density of prey population only. The Holling type III function has proved to be relatively successful in describing the feeding process of predator [20]. In case of Holling type III functional response, the maximum uptake rate of predator and half saturation prey density will be denoted by '$\rho$' and '$\theta$' respectively. In the present paper we will consider the logistic form of growth function for prey in the absence of predator in the following way

$$R(N_1) = r\left(1 - \frac{N_1}{k}\right) \tag{2.2}$$

where '$r$' is the intrinsic growth rate and '$k$' is the environmental carrying capacity. We assume that the per capita decline rate of predator in the absence of prey is density independent. Thus the dynamics of the model system are governed by the following system of nonlinear ordinary differential equations

$$\frac{dN_1}{dt} = rN_1\left(1 - \frac{N_1}{k}\right) - \frac{\rho N_1^2 N_2}{\theta + N_1^2} \tag{2.3a}$$

$$\frac{dN_2}{dt} = \frac{e\rho N_1^2 N_2}{\theta + N_1^2} - mN_2 \tag{2.3b}$$

All the parameters $r$, $k$, $\rho$, $\theta$, $e$, and $m$ involved in the model system are positive. In order to reduce the number of parameters, model system (2.3) can be non-dimensionalized in the following way

$$\frac{dx}{dt} = x(\alpha - x) - \frac{\beta x^2 y}{1 + x^2} \tag{2.4a}$$

$$\frac{dy}{dt} = \frac{\beta_1 x^2 y}{1 + x^2} - \gamma y \tag{2.4b}$$

where $\alpha$, $\beta$, $\beta_1$ and $\gamma$ are dimensionless parameters and '$0 < \beta_1 < \beta$' (for details see [25]). In the above model system it has been assumed that the conversion of prey biomass into the predator biomass is instantaneous but in reality this does not happen. Now we assume that the contribution of consumed prey population to the growth of predator population is not instantaneous rather it is mediated by some discrete time lag $\tau$, which is known as 'gestation delay' [18]. Under this assumption we write the system of equations (2.4) as

$$\frac{dx}{dt} = x(\alpha - x) - \frac{\beta x^2 y}{1 + x^2} \tag{2.5a}$$

$$\frac{dy}{dt} = \frac{\beta_1 x^2(t-\tau) y}{1 + x^2(t-\tau)} - \gamma y \tag{2.5b}$$

where '$\tau$' denotes dimensionless gestation time delay. The delayed model system (2.5) is subjected to the positive initial conditions $x(\theta) = \phi(\theta) > 0$, $\theta \in [-\tau, 0]$, where $\phi \in C([-\tau, 0], R_+)$, and $y(0) = y_0 \geq 0$. Under this initial condition the model system (2.5) satisfies the existence and uniqueness criterion [9]. At this position we would like to mention that such a type of delayed prey-predator model with gestation delay has been studied by several authors ([9, 16] and references cited therein) and hence without going into the details of calculations here, we will only mention the basic dynamical results for the convenience of the readers. In absence of time delay (*i.e.*, $\tau = 0$) non-negative equilibrium points for the model system (2.5) are given by (i) $E_0(0,0)$ (trivial equilibrium), (ii)

$E_1(\alpha, 0)$ (axial equilibrium) and (iii) $E_*(x^*, y^*)$ (positive interior equilibrium) where

$$x^* = \sqrt{\frac{\gamma}{\beta_1 - \gamma}}, \qquad y^* = \frac{\beta_1(\alpha - x^*)}{\beta x^*(\beta_1 - \gamma)} \tag{2.6}$$

The existence of most interesting equilibrium state $E_*$, where both prey and predator population coexist demands the following restrictions

$$\beta_1 > \gamma \quad \textit{and} \qquad 0 < x^* < \alpha \tag{2.7}$$

Simple calculation shows that in the absence of time delay, $E_0$ is always a saddle point which is unstable along the positive direction of $x$-axis. The equilibrium state $E_1$ is locally asymptotically stable whenever $E_*$ does not exist, but the existence of $E_*$ implies that $E_1$ is a saddle point which is unstable along the positive direction of $y$-axis. The positive interior equilibrium $E_*$ will be locally asymptotically stable if the following condition holds

$$\alpha < \alpha^* = \frac{2x^*\gamma}{(2\gamma - \beta_1)} \quad \textit{with} \quad \gamma < \beta_1 < 2\gamma \tag{2.8}$$

It can be easily shown that in this case $E_*$ is also a global attractor [25]. As $\alpha$ passes through its critical value $\alpha^*$, the non-degeneracy and transversality conditions for Hopf bifurcation are satisfied. The existence of Hopf bifurcation implies that the model system (2.5) exhibits small amplitude periodic solution around $E_*$. We explain this phenomena using the following lemma

**Lemma 1.** *In the absence of time delay the model system (2.5) exhibits Hopf bifurcation around $E_*$ whenever $\alpha$ crosses through* $\alpha = \alpha^* = \dfrac{2x^*\gamma}{2\gamma - \beta_1}(\alpha > 0)$.

In the presence of gestation time delay '$\tau$' we don't have any change in the position of equilibrium states and hence $E_0$, $E_1$ and $E_*$ are also the equilibrium points for the delayed system (2.5) (for details, see [16]). To study the local stability of the interior equilibrium $E_*(x^*, y^*)$ we linearize the model system (2.5) around $E_*(x^*, y^*)$ and this results in the following system of equations

$$\frac{dx}{dt} = ax + by \tag{2.9a}$$

$$\frac{dy}{dt} = cx(t - \tau) \tag{2.9b}$$

where $a = \alpha - \dfrac{2x^*\gamma}{(2\gamma - \beta_1)}$, $b = -\dfrac{\beta(x^*)^2}{1 + (x^*)^2} < 0$ and $c = \dfrac{2\beta_1 x^* y^*}{(1 + (x^*)^2)^2} > 0$ . The characteristic equation corresponding to (2.9) is given by

$$G(\lambda, \tau) = \lambda^2 - a\lambda - bce^{-\lambda\tau} = 0 \tag{2.10}$$

The equilibrium $E_*$ is locally asymptotically stable if

$$\sup_{-\tau \le \theta \le 0} \Big[ |\phi(\theta) - x^*| + |y_0 - y^*| \Big] < \delta$$

which implies

$$\lim_{t\to\infty}(x(t),y(t)) = (x^*,y^*)$$

where $(x(t),y(t))$ is any solution of (2.5) subjected to the earlier initial conditions. In the presence of time delay '$\tau$', stability of $E_*$ carries two notions : one is absolute stability and the other corresponds to conditional stability. In case of absolute stability, $E_*$ is asymptotically stable for all $\tau \geq 0$, but for conditional stability, $E_*$ is asymptotically stable for '$\tau$' in some finite interval. The first one corresponds to the case that the real parts of characteristic roots are negative for all $\tau \geq 0$ and the second one shows an existence of critical time delay $\tau_0$ (smallest delay) such that for $0 \leq \tau < \tau_0$, the real parts of characteristic roots are negative and for $\tau > \tau_0$ there exists at least one root of (2.10) whose real part is positive. We recall that for $\tau = 0$, $E_*$ is locally asymptotically stable whenever (2.8) holds. We now investigate the nature of the roots of (2.10) which depends on time delay '$\tau$'. Let for some value of '$\tau$', $\lambda = i\omega$, ($\omega > 0$ and real) is a root of the characteristic equation (2.10). Then separating real imaginary parts in $G(i\omega,\tau) = 0$ we get

$$\omega^2 + bc\cos\omega\tau = 0, \tag{2.11}$$

$$a\omega - bc\sin\omega\tau = 0. \tag{2.12}$$

The above two equations can be combined as

$$\omega^4 + a^2\omega^2 - b^2c^2 = 0 \tag{2.13}$$

The roots of the biquadratic equation (2.13) are given by

$$\omega_\pm^2 = \frac{1}{2}\left(-a^2 \pm \sqrt{a^4+4b^2c^2}\right). \tag{2.14}$$

From (2.14) it follows that we have a unique positive root $\omega_0$ of (2.13), given by

$$\omega_0 = \sqrt{\frac{1}{2}\left(-a^2 + \sqrt{a^4+4b^2c^2}\right)} \tag{2.15}$$

From equations (2.11) and (2.12) we have

$$\sin(\omega_0\tau) = \frac{a\omega_0}{bc} > 0, \tag{2.16}$$

$$\cos(\omega_0\tau) = -\frac{\omega_0^2}{bc} > 0. \tag{2.17}$$

Thus $\tau_k = \frac{1}{\omega_0}[\arcsin\left(\frac{a\omega_0}{bc}\right) + 2k\pi]$, $k = 0,1,2,....$ and $0 < \arcsin\left(\frac{a\omega_0}{bc}\right) < \frac{\pi}{2}$. Consequently the smallest delay for which there is a purely imaginary root is given by

$$\tau_0 = \frac{1}{\omega_0}\arcsin\left(\frac{a\omega_0}{bc}\right). \tag{2.18}$$

These results lead us to the following lemma.

**Lemma 2.** *Suppose that the following conditions hold*

(*i*) $\beta_1 > \gamma$, $0 < x^* < \alpha$, *(ii)* $a < 0$ *with* $\gamma < \beta_1 < 2\gamma$. *Then for* $\tau = \tau_k = \frac{1}{\omega_0}\Big[\arcsin\Big(\frac{a\omega_0}{bc}\Big) + 2k\pi\Big]$, $k = 0, 1, 2, ....$ *the characteristic equation (2.10) have a unique pair of imaginary roots* $\pm i\omega_0$*, where* $\omega_0$ *is given by (2.15).*

We will now study how the real parts of the roots of (2.10) changes as '$\tau$' varies in a small neighbourhood of $\tau_k$. Let $\lambda = u(\tau) + i\omega(\tau)$ be a root of the equation (2.10). Substituting $\lambda = u(\tau) + i\omega(\tau)$ in (2.10) and then separating real and imaginary parts we get

$$H_1(u, \omega, \tau) = u^2 - au - \omega^2 - bce^{-u\tau}\cos(\omega\tau) = 0,$$

$$H_2(u, \omega, \tau) = 2u\omega - a\omega + bce^{-u\tau}\sin(\omega\tau) = 0.$$

Now it follows that $H_1(0, \omega_0, \tau_k) = H_2(0, \omega_0, \tau_k) = 0$. Also we have $|J|_{(0,\omega_0,\tau_k)} > 0$, where $J = \begin{pmatrix} \frac{\partial H_1}{\partial u} & \frac{\partial H_1}{\partial \omega} \\ \frac{\partial H_2}{\partial u} & \frac{\partial H_2}{\partial \omega} \end{pmatrix}$. Hence by implicit function theorem, $H_1(u, \omega, \tau) = 0 = H_2(u, \omega, \tau)$ defines $u$, $\omega$ as a function of $\tau$ in a neighborhood of $(0, \omega_0, \tau_k)$ such that $u(\tau_k) = 0$, $\omega_{\tau_k} = \omega_0$ and $\frac{du}{d\tau}|_{\tau=\tau_k,\omega=\omega_0} > 0$. We now state the following theorem regarding Hopf-bifurcation

**Theorem 1.** *For the model system (2.5) suppose the following conditions are satisfied*

*(i)* $0 < x^* < \alpha$, $\beta_1 > \gamma$

*(ii)* $a < 0$ *with* $\gamma < \beta_1 < 2\gamma$. *Then* $E_*$ *is asymptotically stable whenever* $0 \leq \tau < \tau_0$ *and unstable whenever* $\tau > \tau_0$. *The model system (2.5) undergoes a Hopf bifurcation at* $E_*$ *for* $\tau = \tau_k$.

## 3. Standard multiple scale analysis

In the setting of Hopf bifurcation, a multiple scale approximation explicitly employs the natural frequency ($\omega = \omega_0$) of the oscillation associated with the Hopf bifurcation [13]. Now $\omega_0$ is related to $\tau_0$ and is defined in lemma-2 with $k = 0$. For characterization of the behaviour of the solutions near the bifurcation point, we assume the form of the solution as

$$x(t, T) = A(T)\cos\omega t + B(T)\sin\omega t, \tag{3.1a}$$

$$y(t, T) = A_1(T)\cos\omega t + B_1(T)\sin\omega t \tag{3.1b}$$

where $T = \epsilon^2 t$, $\tau = \tau_0 + \epsilon^2\tau_p$ , $0 < \epsilon \ll 1$ and $\tau_p$ is a real quantity measuring the deviation from bifurcation point. Here $\epsilon^2$ is the parameter measuring the proximity to the bifurcation. $A(T)$ and $B(T)$ are functions of a slow time $T$ and are treated as constant with respect to the fast oscillation with frequency $\omega$ on $t$ time scale. $A_1(T)$ and $B_1(T)$ are given by

$$A_1(T) = \frac{1}{b}\left[\omega B(T) - aA(T)\right], \tag{3.2a}$$

$$B_1(T) = \frac{1}{b}\left[\omega A(T) + aB(T)\right]. \tag{3.2b}$$

The method treats $x$ and $y$ as a functions of two independent times $t$ and $T$ with a perturbation expansion $x(t,T) \sim x_0(t,T) + \epsilon x_1(t,T) + ....$, $y(t,T) = y_0(t,T)+\epsilon y_1(t,T)+.....$ The time derivatives $x_t$, $y_t$ are replaced by $x_t+\epsilon^2 x_T$ and $y_t+\epsilon^2 y_T$. Proceeding with the perturbation expansion; the equations for higher order contributions $x_j$, $y_j$ for $j > 0$ are subjected to the solvability conditions, which give governing equations for $A(T)$ and $B(T)$. These solvability conditions are often in the form of conditions of orthogonality to the oscillatory modes $\cos\omega t$ and $\sin\omega t$. The benefit of analyzing the governing equations is that they allow us to continue the computation over the long time scale.

In our earlier section we have observed that the model system (2.5) undergoes a Hopf bifurcation at $E_*$ for $\tau = \tau_0$ (the smallest delay). This implies that a small amplitude periodic orbit emerges for the model system (2.5) around $E_*$ for $\tau = \tau_0$ (the conditions of the theorem 1 must be satisfied). For $\tau < \tau_0$, oscillations decay over time and for $\tau > \tau_0$ the oscillation grows exponentially. The method of multiple scales can be applied for the full nonlinear system (2.5) with stochastic perturbations, but in that case the envelope equations for the amplitude of oscillations will be highly nonlinear and this is because of the nonlinearity involved in (2.5). Because of this nonlinearity and also the feedback process, it becomes very difficult to analyze the behavior of such governing equations. The paucity of available techniques gives us the opportunity to apply the method of multiple scales to the stochastic model system obtained from the deterministic model system (2.5) in the vicinity of the critical delay parameter $\tau = \tau_0$.

## 4. Multiple scale analysis of stochastic delay differential equations

The main assumption that leads us to extend the deterministic model system to a stochastic one is that all prey-predator type interactions take place in open environment and hence environmental fluctuation always have an effect on the evolutionary behaviour of prey and predator populations. There are several ways in which the effect of fluctuating environment can be incorporated in a deterministic model system to construct its stochastic counterpart. In the present study, we assume that randomly fluctuating driving forces have effect on the growth of prey and predator population at any instant of time '$t$', so that the deterministic model system (2.5) results in following SDDE model with 'additive noise' terms

$$\frac{dx}{dt} = x(\alpha - x) - \frac{\beta x^2 y}{1+x^2} + \sigma_1\xi_1(t), \tag{4.1a}$$

$$\frac{dy}{dt} = \frac{\beta_1 x^2(t-\tau)y}{1+x^2(t-\tau)} - \gamma y + \sigma_2\xi_2(t) \tag{4.1b}$$

where $\xi_1(t)$ and $\xi_2(t)$ are two independent Gaussian white noise characterized by $\langle\xi_1(t)\rangle = 0 = \langle\xi_2(t)\rangle$ and $\langle\xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$, $i,j = 1,2$. Here $\delta$ is the Dirac-delta function, $\delta_{ij}$ is the Kronecker delta and $\langle.\rangle$ stands for the ensemble average due to the effect of fluctuating environment. Linearizing SDDE model system (4.1) around $E_*$ with help of the transformation $x = x^* + x'$, $y = y^* + y'$ and then dropping $'$ we get the following system of equations

$$\frac{dx}{dt} = ax + by + \sigma_1 \xi_1(t), \tag{4.2a}$$

$$\frac{dy}{dt} = cx(t-\tau) + \sigma_2 \xi_2(t) \tag{4.2b}$$

where the expressions for $a$, $b$ and $c$ are same as in section 2. At this position we like to mention that, due to stochastic perturbations $\xi_i(t)$, $i = 1, 2$ the variables $x(t)$ and $y(t)$ have zero mean values. Formally a white noise process is the derivative of the Wiener process. The sample trajectories of a Wiener process are continuous but nowhere differentiable and have infinite variations on any finite interval [18]. The parameters $\sigma_1$ and $\sigma_2$ involved with our model system are intensities of the two independent Gaussian white noises $\xi_1(t)$ and $\xi_2(t)$. The above system is a system of two coupled linear stochastic delay differential equations and can be regarded as delay differential equations (2.9) which are acted on by two independent additive white noises.

To know the effect of noise near the bifurcation point over a long time, we seek a periodic solution of which the amplitude varies stochastically on a slow time scale $T = \epsilon^2 t$. We have outlined the standard method of multiple scales in the earlier section and assume the solution of (4.2) near the bifurcation point of the form as in (3.1), but in this case $A(T)$ and $B(T)$ evolves stochastically. We proceed in a similar fashion as in [14] and write the governing equations for the amplitude of oscillations as follows

$$\begin{pmatrix} dA \\ dB \end{pmatrix} = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} dT + \begin{pmatrix} dW_A(T) \\ dW_B(T) \end{pmatrix}. \tag{4.3}$$

Here $\psi_A$ and $\psi_B$ are called drift coefficients and $(dW_A(T)/dT = \xi_A(T)$, $dW_B(T)/dT = \xi_B(T))$ are two independent white noises. Our task is to determine the drift coefficients $(\psi_A, \psi_B)$ and a relation between the white noises $(\xi_A(T), \xi_B(T))$ with the white noises $(\xi_1(t), \xi_2(t))$. We write the model system (4.2) as

$$d\begin{pmatrix} x \\ y \end{pmatrix} = M_1 \begin{pmatrix} x \\ y \end{pmatrix} dt + M_2 \begin{pmatrix} x(t-\tau) \\ y(t-\tau) \end{pmatrix} dt + M_3 \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} \tag{4.4}$$

where $M_1 = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ and $M_3 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$. By using the well known properties of Brownian motions [27], we may write the noise term in (4.3) for our convenience on slow time scale $T$ as

$$M_3 \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} = \frac{M_3}{\epsilon} \begin{pmatrix} \cos\omega t \; dW_{11}(T) + \sin\omega t \; dW_{12}(T) \\ \cos\omega t \; dW_{21}(T) + \sin\omega t \; dW_{22}(T) \end{pmatrix} \tag{4.5}$$

where $W_{ij}$'s $(i, j = 1, 2)$ are independent standard Brownian motions on slow time scale $T$.

With the help of Ito's formula and (3.1), we get from the equations (4.4) the following set of coupled equations

$$\begin{pmatrix} \dfrac{\partial x}{\partial t} \\ \\ \dfrac{\partial y}{\partial t} \end{pmatrix} dt + \begin{pmatrix} \dfrac{\partial x}{\partial A} & \dfrac{\partial x}{\partial B} \\ \\ \dfrac{\partial y}{\partial A} & \dfrac{\partial y}{\partial B} \end{pmatrix} \begin{pmatrix} \dfrac{dA}{dt} \\ \\ \dfrac{dB}{dt} \end{pmatrix} dt =$$

$$\begin{pmatrix} a(A(T)\cos\omega t + B(T)\sin\omega t) - A(T)(a\cos\omega t \\ +\omega\sin\omega t) + B(T)(\omega\cos\omega t - a\sin\omega t) \\ cA(T-\epsilon^2\tau)\cos\omega(t-\tau) + cB(T-\epsilon^2\tau)\sin\omega(t-\tau) \end{pmatrix} dt$$

$$+\frac{M_3}{\epsilon}\begin{pmatrix} \cos\omega t\ dW_{11}(T) + \sin\omega t\ dW_{12}(T) \\ \cos\omega t\ dW_{21}(T) + \sin\omega t\ dW_{22}(T) \end{pmatrix}. \tag{4.6}$$

We write

$$A(T-\epsilon^2\tau) = A(T) + \epsilon^2\left(\frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2}\right), \tag{4.7a}$$

$$B(T-\epsilon^2\tau) = B(T) + \epsilon^2\left(\frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2}\right) \tag{4.7b}$$

and since $\epsilon$ is small we treat $\left(\dfrac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2}\right)$ and $\left(\dfrac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2}\right)$ as O(1). Also we have

$$\cos\omega(t-\tau) = \cos\omega t\cos\omega\tau_0 - \epsilon^2\omega\tau_p\cos\omega t\sin\omega\tau_0 \\ + \sin\omega t\sin\omega\tau_0 + \epsilon^2\omega\tau_p\sin\omega t\cos\omega\tau_0 + O(\epsilon^4), \tag{4.8a}$$

$$\sin\omega(t-\tau) = \sin\omega t\cos\omega\tau_0 - \epsilon^2\omega\tau_p\sin\omega t\sin\omega\tau_0 \\ - \cos\omega t\sin\omega\tau_0 - \epsilon^2\omega\tau_p\cos\omega t\cos\omega\tau_0 + O(\epsilon^4). \tag{4.8b}$$

With the help of (4.7), (4.8), (2.15) and (2.16) we note that $O(1)$ terms get cancelled in (4.6). We neglect $O(\epsilon^4)$ terms to obtain the governing equations for the amplitude of oscillations and on comparison we derive from (4.6) the following equations

$$\psi_A(t)\cos\omega t + \psi_B(t)\sin\omega t = 0, \tag{4.9a}$$

$$-(a\cos\omega t + \omega\sin\omega t)\psi_A + (\omega\cos\omega t - a\sin\omega t)\psi_B = \\ bc\omega\tau_p(\sin\omega t\cos\omega\tau_0 - \cos\omega t\sin\omega\tau_0)A(T) - bc\omega\tau_p(\sin\omega t\sin\omega\tau_0 + \\ \cos\omega t\cos\omega\tau_0)B(T) + bc\left(\frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2}\right)(\sin\omega t\sin\omega\tau_0 + \cos\omega t\cos\omega\tau_0) \\ +bc\left(\frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2}\right)(\sin\omega t\cos\omega\tau_0 - \cos\omega t\sin\omega\tau_0), \tag{4.9b}$$

$$\cos\omega t\ dW_A(T) + \sin\omega t\ dW_B(T) = \frac{\sigma_1}{\epsilon}(\cos\omega t\ dW_{11}(T) + \sin\omega t\ dW_{12}(T)) \tag{4.10a}$$

$$-(a\cos\omega t + \omega\sin\omega t)\ dW_A(T) + (\omega\cos\omega t - a\sin\omega t)\ dW_B(T) =$$

$$\frac{b\sigma_2}{\epsilon}\left(\cos\omega t\; dW_{21}(T) + \sin\omega t\; dW_{22}(T)\right). \tag{4.10b}$$

We now use the method of multiple scales, we project the equations for $A'(T)$ and $B'(T)$ onto $\cos\omega t$ and $\sin\omega t$, while treating functions of $T$ as independent of $t$ (orthogonality condition). Thus the governing equations for the amplitude of oscillations on slow time scale $T$ are

$$\begin{pmatrix} dA \\ dB \end{pmatrix} = P\begin{pmatrix} A \\ B \end{pmatrix} dT + Q\begin{pmatrix} A(T-\epsilon^2\tau) \\ B(T-\epsilon^2\tau) \end{pmatrix} dT + R\begin{pmatrix} dW_{11}(T) \\ dW_{12}(T) \end{pmatrix} + R_1\begin{pmatrix} dW_{21}(T) \\ dW_{22}(T) \end{pmatrix} \tag{4.11}$$

where $P$, $Q$, $R$ and $R_1$ are $(2\times 2)$ matrices given by

$$P = \begin{pmatrix} \omega^2\tau_p + \frac{a}{\epsilon^2} & a\omega\tau_p - \frac{\omega}{\epsilon^2} \\ -a\omega\tau_p + \frac{\omega}{\epsilon^2} & \omega^2\tau_p + \frac{a}{\epsilon^2} \end{pmatrix}, \quad Q = \begin{pmatrix} -\frac{a}{\epsilon^2} & \frac{\omega}{\epsilon^2} \\ -\frac{\omega}{\epsilon^2} & -\frac{a}{\epsilon^2} \end{pmatrix}$$

$$R = \frac{\sigma_1}{2\epsilon}\begin{pmatrix} 1 & -\frac{a}{\omega} \\ \frac{a}{\omega} & 1 \end{pmatrix}, \quad R_1 = \frac{\sigma_2}{2\epsilon}\begin{pmatrix} 0 & -\frac{b}{\omega} \\ \frac{b}{\omega} & 0 \end{pmatrix}.$$

It is clear from (4.11) that the governing equations for the amplitude of oscillations are linear stochastic delay differential equations with small delay $\epsilon^2\tau = \overline{\tau}$. In absence of noise (4.11) results in the following system of differential equations

$$\begin{pmatrix} dA \\ dB \end{pmatrix} = P\begin{pmatrix} A \\ B \end{pmatrix} dT + Q\begin{pmatrix} A(T-\epsilon^2\tau) \\ B(T-\epsilon^2\tau) \end{pmatrix} dT. \tag{4.12}$$

The characteristic equation corresponding to (4.12) is

$$\left(\lambda - \omega^2\tau_p - \frac{a}{\epsilon^2}\left(1 - e^{-\lambda\epsilon^2\tau}\right)\right)^2 + \left(a\omega\tau_p - \frac{\omega}{\epsilon^2}\left(1 - e^{-\lambda\epsilon^2\tau}\right)\right)^2 = 0. \tag{4.13}$$

Since we have $0 < \epsilon \ll 1$, the roots of (4.13) are

$$\lambda = \frac{1}{(\omega^2\tau_0^2 + (1 - a\tau_0)^2)}\left\{\omega^2\tau_p \pm i\omega\tau_p(\omega^2\tau_0 - a + a^2\tau_0)\right\} + O(\epsilon^2). \tag{4.14}$$

The roots given in (4.14) will have negative real parts if $\tau_p < 0$ which implies $\tau < \tau_0$.

## 5. Transient properties of the process $(A(T), B(T))^T$

In this section we discuss the transient properties of the process $(A(T), B(T))^T$ whose governing equations on slow time scale are given by (4.11). There are two ways to characterize the transient properties of the process $(A(T), B(T))^T$. In the first case one can derive the Fokker Planck equation corresponding to the SDDE (4.11). But the usual Fokker Planck equation approach can not be applied directly to a stochastic delay differential equation and we need a small delay expansion [10]. Only then it is possible to characterize the stationary properties of the process $(A(T), B(T))^T$. In the other case, one can directly derive the moment equations of the process $(A(T), B(T))^T$ from (4.11) and can characterize the stationary properties of the process $(A(T), B(T))^T$. Here we will derive the moment equations of the process $(A(T), B(T))^T$. The behaviour of the first and second moments of the process $(A(T), B(T))^T$ are sufficient for the determination of transient properties of the process $(A(T), B(T))^T$ and we need not derive equations for higher moments.

Let $m_A(T) = E(A(T))$ and $m_B(T) = E(B(T))$. Then from (4.11) we have the following set of equations for the first moments :

$$\begin{pmatrix} \dfrac{dm_A}{dT} \\ \dfrac{dm_B}{dT} \end{pmatrix} = P \begin{pmatrix} m_A \\ m_B \end{pmatrix} + Q \begin{pmatrix} m_A(T-\bar{\tau}) \\ m_B(T-\bar{\tau}) \end{pmatrix} \tag{5.1}$$

From (4.13), it follows that whenever $\tau < \tau_0$, $m_A(T)$ and $m_B(T)$ approach zero as $T$ increases. To derive the equations for the second moments we use Ito differential rule and the notations $k_A(T,S) = E\{A(T)A(S)\}$, $k_B(T,S) = E\{B(T)B(S)\}$ and $k_{AB}(T,S) = E\{A(T)B(S)\}$. We then have the following set of equations for the second moments :

$$\frac{dk_A(T,T)}{dT} = 2\theta_1 k_A(T,T) + 2\theta_2 k_{AB}(T,T) + 2\theta_3 k_{AB}(T,T-\bar{\tau}) - 2\theta_4 k_A(T,T-\bar{\tau}) + \Delta^2 \tag{5.2}$$

$$\frac{dk_B(T,T)}{dT} = 2\theta_1 k_B(T,T) - 2\theta_2 k_{AB}(T,T) - 2\theta_3 k_{AB}(T,T-\bar{\tau}) - 2\theta_4 k_B(T,T-\bar{\tau}) + \Delta^2 \tag{5.3}$$

$$\begin{aligned} \frac{dk_{AB}(T,T)}{dT} &= -\theta_2 k_A(T,T) + 2\theta_1 k_{AB}(T,T) + \theta_2 k_B(T,T) - \theta_3 k_A(T,T-\bar{\tau}) \\ &\quad -2\theta_4 k_{AB}(T,T-\bar{\tau}) + \theta_3 k_B(T,T-\bar{\tau}) \end{aligned} \tag{5.4}$$

where $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ and $\Delta^2$ are given by

$$\theta_1 = \omega^2\tau_p + \frac{a}{\epsilon^2}, \quad \theta_2 = a\omega\tau_p - \frac{\omega}{\epsilon^2},$$

$$\theta_3 = \frac{\omega}{\epsilon^2}, \quad \theta_4 = \frac{a}{\epsilon^2}, \quad \Delta^2 = \frac{1}{4\epsilon^2}\left(\sigma_1^2(a^2+\omega^2) + \sigma_2^2 b^2\right).$$

The equilibrium state is given by

$$k_A^* = k_B^* = -\frac{\Delta^2}{2\omega^2\tau_p}, \quad k_{AB}^* = 0. \tag{5.5}$$

The characteristic equation corresponding to the linearized version of the system of equations for the second moments around the steady state given by (5.5) is equivalent to (4.13). Thus whenever $\tau < \tau_0$, the second moments approaches to the equilibrium state given by (5.5). This shows that whenever $\tau < \tau_0$, the process $(A(T), B(T))^T$ approaches its stationary realizations at a rate on slow time scale $T$. The result (5.5) suggests that whenever $\tau < \tau_0$, $A(T)$ and $B(T)$

are two independent stationary Gaussian processes each with mean zero and variance $-\frac{\Delta^2}{2\omega^2\tau_p}$.

## 6. Conclusion

To analyze the effects of additive white noise in a delayed predator prey model we apply multiple scale analysis as developed by [14] near the Hopf bifurcation point around the positive interior equilibrium state $E_*$. In this method we derive the governing equations for the amplitude of oscillations on a slow time scale $T$. This analysis helps us to discuss the behaviour of stochastic and deterministic effects seperately for the model system near $E_*$. In the absence of noise when time delay lies below its critical value, the oscillations decay exponentially but after stochastic perturbations the combined effect of noise and delay sustain these oscillations. The periodic solution near $E_*$ at Hopf frequency serves as a carrier whose amplitude evolves stochastically. The amplitude of oscillations are identified as a stationary Gaussian process whenever time delay lies below its critical value. The variance of the steady state of the amplitude of oscillations depends on the noise intensities as well as also on the perturbing delay parameter. The variance increases if we increase the noise intensities and if we approach the perturbing delay parameter towards the zero value. This shows that the validity of the method of multiple scale analysis that we adopted here depends on two conditions : $0 < \epsilon \ll 1$ and $-\frac{\Delta^2}{2\omega^2\tau_p} \ll 1$ where $\Delta^2 = \frac{1}{4\epsilon^2}\left(\sigma_1^2(a^2+\omega^2) + \sigma_2^2 b^2\right)$. The first condition is trivial and for the second we need '$\sigma_i$', $i = 1, 2$ must be small, of the order of $\epsilon$ and the value of '$\Delta$' should not exceed the product of the square root of the deviation of delay from its critical value and the frequency of oscillations. The method does not hold whenever the noise intensities are large and the stochastic fluctuations govern the dynamics. Now if we fix the noise intensities '$\sigma_i = O(\epsilon)$', then $-\frac{\Delta^2}{2\omega^2\tau_p} \ll 1$ defines a parametric regime in *ab*-parameter space where the oscillatory behavior persists for a long time around the coexisting equilibrium point. Finally we would like to mention that the method we have followed here can be applied to other two dimensional SDDE models of ecological systems.

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