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ON NEW IDENTITIES FOR 3 BY 3 MATRICES.

WOO LEE

ABSTRACT. In this paper we show that the polynomial of degree 9 called generalized algebraicity is a polynomial identity for 3×3 matrices. ([5])

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1. Introduction.

Let K be a field of characteristic 0, and $M_n(K)$ be the ring of $n \times n$ matrices over K. $K < X >= K < x_1,\ x_2, \ldots >$ denotes a free associative algebra over K with unity 1 of countable rank. We may use other variables $x,\ y,\ z,\ \ldots$ for notational simplicity.

Definition 1.1. A ring R is a polynomial identity ring (or R satisfies a polynomial identity) if there is a nonzero polynomial $f(x_1,\ x_2, \ldots, x_m) \in K < X >$ such that $f(r_1,\ r_2, \ldots, r_m) = 0$ for all $r_1,\ r_2, \ldots, r_m \in R$.

Example 1.2. (1) Any commutative ring satisfies $[x, y] = xy - yx$.
(2) The ring of strictly upper triangular $n \times n$ matrices satisfies $x_1 x_2 \cdots x_n$.
(3) $M_2(K)$ satisfies $[[x, y]^2, z]$.
(4) The free algebra $K < X >$ does not satisfy a polynomial identity.

Theorem 1.3. ([4]). *A primitive ring satisfying a polynomial identity is isomorphic to $M_n(D)$, where D is a division algebra finite dimensional over its center. Equivalently, it is central simple and finite dimensional over its center, of dimension n^2, for some n.*

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Definition 1.4. The standard polynomial of degree n is the polynomial

$$S_n(x_1,\ldots,x_n)=\sum \{\,\mathrm{sign}(\pi)\ x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)}\mid \pi\in \mathrm{Sym}(n)\}$$

where $\mathrm{Sym}(n)$ is the symmetric group on n letters.

In 1950, S. A. Amitsur and J. Levitzki proved the following theorem.

Theorem 1.5. ([1]). *$M_n(K)$ satisfies S_{2n}.*

$M_n(K)$ satisfies no polynomial identity of degree less than $2n$. Also it is easy to show that if $M_n(K)$ satisfies a polynomial identity of degree $2n$, then it is a scalar multiple of S_{2n}.

Now we list some properties of S_{2n}.

Lemma 1.6. (1) *S_n is multilinear.*

(2) *S_n is alternating, i.e., $S_n(\ldots,x,\ldots,y,\ldots)=-S_n(\ldots,y,\ldots,x,\ldots)$.*

(3) *$S_{n+1}(x_1,\ldots,x_{n+1})=\sum(-1)^{i+1}\,x_i\,S_n(x_1,\ldots,\hat{x}_i,\ldots,x_{n+1})$. Thus if a ring satisfies S_n, then it satisfies S_{n+1}.*

(4) *If R is an L-algebra where L is a field, and $a_1,a_2,\ldots,a_n\in A$ are linearly dependent over L, then $S_n(a_1,\ldots,a_n)=0$.*

Proof. (1).

$$\begin{aligned}
&S_n(x_1,\ldots,x_{i-1},x_i+y_i,x_{i+1},\ldots,x_n)\\
&=\sum\ \mathrm{sign}(\pi)\,x_{\pi(1)}\cdots(x_{\pi(i)}+y_{\pi(i)})\cdots x_{\pi(n)}\\
&=\sum\ \mathrm{sign}(\pi)\,\{x_{\pi(1)}\cdots x_{\pi(i)}\cdots x_{\pi(n)}+x_{\pi(1)}\cdots y_{\pi(i)}\cdots x_{\pi(n)}\}\\
&=\sum\ \mathrm{sign}(\pi)\,x_{\pi(1)}\cdots x_{\pi(i)}\cdots x_{\pi(n)}+\sum\ \mathrm{sign}(\pi)\,x_{\pi(1)}\cdots y_{\pi(i)}\cdots x_{\pi(n)}\\
&=S_n(x_1,\ldots,x_i,\ldots,x_n)+S_n(x_1,\ldots,y_i,\ldots,x_n).
\end{aligned}$$

(2). Since $\mathrm{sign}(\pi\cdot(ij))=(-1)\ \mathrm{sign}(\pi)=\mathrm{sign}((ij)\cdot\pi)$,

$$\begin{aligned}
S_n(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)&=\sum_{\pi\in\mathrm{Sym}(n)}\ \mathrm{sign}(\pi)\,x_{\pi(1)}\cdots x_{\pi(n)}\\
&=\sum\ -\mathrm{sign}(\pi(ij))\,x_{\pi(ij)(1)}\cdots x_{\pi(ij)(n)}\\
&=-\sum\ -\mathrm{sign}(\sigma)\,x_{\sigma(1)}\cdots x_{\sigma(n)}\\
&=-S_n(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n).
\end{aligned}$$

Therefore S_n is alternating.

(3). The conclusion follows from the simple fact,

$$
\begin{aligned}
S_{n+1}(x_1, \ldots, \ldots, x_{n+1}) &= \sum_{\tau \in \mathrm{Sym}(n+1)} \operatorname{sign}(\tau)\, x_{\tau(1)} \cdots x_{\tau(n+1)} \\
&= \sum_{i=1}^{n+1} (-1)^{i+1}\, x_i\, S_n(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}).
\end{aligned}
$$

(4). If $a_1, a_2, \ldots, a_n \in A$ are linearly independent, then there exist $l_1, l_2, \ldots, l_n \in L$ such that $l_1 a_1 + l_2 a_2 + \cdots + l_n a_n = 0$ for some nonzero l_j, $1 \leq j \leq n$. Thus there exists

$$a_k = l'_1 a_1 + \cdots + l'_{k-1} a_{k-1} + l'_{k+1} a_{k+1} + \cdots + l'_n a_n.$$

Then

$$
\begin{aligned}
&S_n(a_1, \ldots, a_n) \\
&= S_n(a_1, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_n) \\
&= S_n(a_1, \ldots, a_{k-1}, l'_1 a_1 + \cdots + l'_{k-1} a_{k-1} + l'_{k+1} a_{k+1} + \cdots \\
&\quad + l'_n a_n, a_{k+1}, \ldots, a_n) \\
&= \sum_{i \neq k} S_n(a_1, \ldots, a_{k-1}, l'_i a_i, a_{k+1}, \ldots, a_n).
\end{aligned}
$$

But $S_n(a_1, \ldots, a_{k-1}, l'_i a_i, a_{k+1}, \ldots, a_n) = 0$ by (2). Hence $S_n(a_1, \ldots, a_n) = 0$. □

2. Identities for 3 by 3 matrices.

Let's recall that a K-algebra A is algebraic over K if each element of A is algebraic over K, and algebraic of bounded degree $\leq d$ if each element of A is algebraic over K of degree $\leq d$.

Theorem 2.1. *Let A be an algebra over a field K.*

(1) *If $[A : K] = n < \infty$, then A satisfies $S_{n+1}(x_1, \ldots, x_{n+1})$.*
(2) *If A is algebraic over K of degree d, then A satisfies*

$$S_{d+1}(y, xy, x^2 y, \ldots, x^d y) \quad \text{and} \quad S_d([x, y], [x^2, y], \ldots, [x^d, y],),$$

where $[x, y] = xy - yx$ is the commutator of x and y.

Proof. (1). Since any $(n + 1)$-many elements in an algebra A such that $[A : K] = n < \infty$, are linearly dependent. Thus $S_{n+1}(x_1, \ldots, x_{n+1})$ is an identity by Lemma 1.6 (4)

(2). The conclusion follows by (1). □

Let's define a polynomial

$$P_n(x, y_1, \dots, y_n) = \sum_{\rho \in \mathrm{Sym}\{0,1,\dots,n\}} \mathrm{sign}(\rho)\, x^{\rho(0)} y_1 x^{\rho(1)} y_2 \cdots x^{\rho(n-1)} y_n x^{\rho(n)}.$$

Now we are ready to prove our main theorem.

Theorem 2.2. *The polynomial P_3 is an identity of $M_3(k)$.*

Proof. Since the cardinality of Sym$\{0, 1, 2, 3\}$ is 24, the polynomial P_3 has 24 different monomials.

$$\begin{aligned}
P_3(x, y_1, y_2, y_3) &= x^0y_1x^1y_2x^2y_3x^3 - x^0y_1x^1y_2x^3y_3x^2 - x^0y_1x^2y_2x^1y_3x^3 \\
&+x^0y_1x^2y_2x^3y_3x^1 + x^0y_1x^3y_2x^1y_3x^2 - x^0y_1x^3y_2x^2y_3x^1 - x^1y_1x^0y_2x^2y_3x^3 \\
&+x^1y_1x^0y_2x^3y_3x^2 + x^1y_1x^2y_2x^0y_3x^3 - x^1y_1x^2y_2x^3y_3x^0 - x^1y_1x^3y_2x^0y_3x^2 \\
&+x^1y_1x^3y_2x^2y_3x^0 + x^2y_1x^0y_2x^1y_3x^3 - x^2y_1x^0y_2x^3y_3x^1 - x^2y_1x^1y_2x^0y_3x^3 \\
&+x^2y_1x^1y_2x^3y_3x^0 + x^2y_1x^3y_2x^0y_3x^1 - x^2y_1x^3y_2x^1y_3x^0 - x^3y_1x^0y_2x^1y_3x^2 \\
&+x^3y_1x^0y_2x^2y_3x^1 + x^3y_1x^1y_2x^0y_3x^2 - x^3y_1x^1y_2x^2y_3x^0 - x^3y_1x^2y_2x^0y_3x^1 \\
&+x^3y_1x^2y_2x^1y_3x^0 \\
&= y_1x^1y_2x^2y_3x^3 - y_1x^1y_2x^3y_3x^2 - y_1x^2y_2x^1y_3x^3 + y_1x^2y_2x^3y_3x^1 \\
&+y_1x^3y_2x^1y_3x^2 - y_1x^3y_2x^2y_3x^1 - x^1y_1y_2x^2y_3x^3 + x^1y_1y_2x^3y_3x^2 \\
&+x^1y_1x^2y_2y_3x^3 - x^1y_1x^2y_2x^3y_3 - x^1y_1x^3y_2y_3x^2 + x^1y_1x^3y_2x^2y_3 \\
&+x^2y_1y_2x^1y_3x^3 - x^2y_1y_2x^3y_3x^1 - x^2y_1x^1y_2y_3x^3 + x^2y_1x^1y_2x^3y_3 \\
&+x^2y_1x^3y_2y_3x^1 - x^2y_1x^3y_2x^1y_3 - x^3y_1y_2x^1y_3x^2 + x^3y_1y_2x^2y_3x^1 \\
&+x^3y_1x^1y_2y_3x^2 - x^3y_1x^1y_2x^2y_3 - x^3y_1x^2y_2y_3x^1 + x^3y_1x^2y_2x^1y_3
\end{aligned}$$

Now we only need to verify that P_3 vanishes when x, y_1, y_2, y_3 are matrix units. So let

$$x = ae_{m,n}, y_1 = be_{i_1,j_1}, y_2 = ce_{i_2,j_2}, y_3 = de_{i_3,j_3}$$

where $e_{i,j}$ is a 3×3 matrix with one nonzero entry, a 1 in the (i, j)-position. If $m \neq n$, then $x^2 = x^3 = 0$. Thus we need to consider the case $m = n$, in which

$$\begin{aligned}
x^{\pi(0)}y_1x^{\pi(1)}y_2x^{\pi(2)}y_3x^{\pi(3)} &= x^{\pi(0)+\pi(1)+\pi(02)+\pi(3)}y_1y_2y_3 \\
&= x^6y_1y_2y_3 = a^6bcde_{m,m}.
\end{aligned}$$

But there are the same number of positive and negative monomials, which are cancelled to 0. Hence P_3 is a polynomial identity on $M_3(K)$ by a result of Bergman([2]). □

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Woo Lee received his B.A. from Sogang Univ., his M.A. from Univ. of Maine and PH.D. at Pennsylvania State Univ. under the supervision of Edward Formanek. Since 1998, he has been at Gwangju University. His research interests include noncommutative rings, braid groups and representations.

Department of Telecommunications, Gwangju University, Gwangju 503-703 S. Korea.
E-mail: woolee@gwangju.ac.kr