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WEIGHTED POSSIBILISTIC VARIANCE AND MOMENTS OF FUZZY NUMBERS

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ABSTRACT. In this paper, a method to find the weighted possibilistic variance and moments about the mean value of fuzzy numbers via applying a difuzzification using minimizer of the weighted distance between two fuzzy numbers is introduced. In this way, we obtain the nearest weighted point with respect to a fuzzy number, this main result is a new and interesting alternative justification to define of weighted mean of a fuzzy number. Considering this point and the weighted distance quantity, we introduce the weighted possibilistic mean (WPM) value and the weighted possibilistic variance(WPV) of fuzzy numbers. This paper shows that WPM is the nearest weighted point to fuzzy number and the WPV of fuzzy number is preserved more properties of variance in probability theory so that it can simply introduce the possibilistic moments about the mean of fuzzy numbers without problem. The moments of fuzzy numbers play an important role to estimate of parameters, skewness, kurtosis in many of fuzzy times series models.

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1. Introduction

Dubois and Prade [3] defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. Carlsson and Fullér [2] defined an interval-valued mean of fuzzy numbers, viewing them as possibiliy distribution. They also introduced notions of crisp possibilistic mean value and crisp possibilistic variance of continous possibility distribution. These notions are partially consistent with the extension principle and with the well-known definitions of expectation and variance in probability theory. Recently, Fullér and Majlender [4] defined the weighted lower possibilistic and upper possibilistic mean values, the WPM value and variance of fuzzy numbers similar to the same definitions as in [2].

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In this paper we introduce a method to obtain the WPM value, variance and moments of fuzzy numbers. We show that they are suitable, applicable and play an important role in fuzzy data analysis. In addition, for the WPV of fuzzy numbers, many properties of variance in probability theory are preserved and it has real concept of a variance.

The following properties are necessary for defining of the possibilistic variance of a fuzzy number:

1. The concept of the variance of fuzzy numbers must be an accurate measure of the spread or dispersion about the mean value.
2. The possibilistic variance value of fuzzy number must be minimum about the mean value with respect to any other crisp point of support function. As we know many relations including some inequalities in probability theory are similarly held in possibility theory. In other words, similar to the probability contexts, the variance inequality in possibility theory should be held.
3. In the physical interpretation of the variance, it gives the moment of inertia of the mass distributed about the center of mass, also the variance gives information about the spread of variables around the mean value. It is a very important factor to finding out the fluctuations in the observed values.
4. The high-order possibilistic moments (about the mean) of fuzzy numbers are obtained by the extension of the definition of the possibilistic variance (if existing).

The above mentioned properties lead us to a proper definition of the possibilistic variance of fuzzy numbers. The rest of this paper is organized as follows:

In section 2, we briefly state some defines of fuzzy numbers. In section 3, we introduce a new method to obtaining the nearest weighted point of a fuzzy number which is defined as a WPM value of fuzzy numbers. The main result of this section is a new and interesting alternative justification to the definition of the weighted mean of a fuzzy number so that this result is not mentioned in [2,4]. In section 4, we introduce the WPV of fuzzy numbers via a new path such that it has all mentioned properties for a variance. In section 5, the possibilistic moments about the mean of fuzzy numbers are proposed, it is a very important topic for the fuzzy community.

2. Preliminary

A fuzzy number A is a fuzzy set of the real line R with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by F. A $\gamma-level$ set of a fuzzy number u is defined by $[A]^\gamma = \{t \in R | A(t) \geq \gamma\}$ if $\gamma > 0$ and $[A]^\gamma = cl\{t \in R | A(t) > 0\}$ (the closer of the support of A) if $\gamma = 0$. It is well known that if A is a fuzzy number then $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ is a compact subset of R for all $\gamma \in [0,1]$. If A is a symmetric fuzzy number, we denote it as

$$[A]^\gamma = [a_1(\gamma), a_2(\gamma)] = [\lambda - s(\gamma), \lambda + s(\gamma)],$$

where λ is a real number and $s(\gamma) = \lambda - a_1(\gamma)$. Therefore for any symmetric fuzzy number $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ we have $\frac{a_1(\gamma)+a_2(\gamma)}{2} = \lambda$.

For arbitrary fuzzy numbers $A, B \in F$ the quantity

$$d(A,B) = \left[\int_0^1 (a_1(\gamma) - b_1(\gamma))^2 d\gamma + \int_0^1 (a_2(\gamma) - b_2(\gamma))^2 d\gamma\right]^{\frac{1}{2}}, \qquad (1)$$

is the distance between A and B. The function $d(A,B)$ is a metric in F and is a particular member of the family of distance $\delta_{p,q}$ defined as follows:

$$\delta_{p,q} = \left[\int_0^1 (1-q)|a_1(\gamma) - b_1(\gamma)|^p d\gamma + \int_0^1 q|a_2(\gamma) - b_2(\gamma)|^p d\gamma\right]^{\frac{1}{p}}, \qquad (2)$$

where, $1 \le p \le \infty$ and $0 \le q \le 1$, (see [1]).

We now define a possibilistic distance quantity of fuzzy number A similar to (2) as follows:

$$\begin{aligned} d(A, C(A)) \;\; = \;\; & \left[\int_0^1 \mathrm{Pos}(A \le a_1(\gamma))(a_1(\gamma) - C(A))^2 d\gamma\right. \\ & \left. + \int_0^1 \mathrm{Pos}(A \ge a_2(\gamma))(a_2(\gamma) - C(A))^2 d\gamma\right]^{\frac{1}{2}}, \end{aligned}$$

$$d(A, C(A)) = \left[\int_0^1 \gamma(a_1(\gamma) - C(A))^2 d\gamma + \int_0^1 \gamma(a_2(\gamma) - C(A))^2 d\gamma\right]^{\frac{1}{2}} \qquad (3)$$

where $C(A)$ is a real number from support function of fuzzy number, and here Pos denotes possibility, i.e.

$$\begin{aligned} \mathrm{Pos}(A \le a_1(\gamma)) &= \Pi((-\infty, a_1(\gamma)]) = \sup\nolimits_{x \le a_1(\gamma)} A(x), \\ \mathrm{Pos}(A \ge a_2(\gamma)) &= \Pi([a_2(\gamma), -\infty)) = \sup\nolimits_{x \ge a_2(\gamma)} A(x). \end{aligned}$$

3. A weighted point of a fuzzy number

In this section, we will propose a defuzzification which is called the nearest weighted point of a fuzzy number. Interval $EI(A)$ of a fuzzy number A introduced independently by Dubois and Prade [3] and Heilpern [6]. It is defined by

$$EI(A) = \left[\int_0^1 a_1(\gamma), \int_0^1 a_2(\gamma)\right]. \qquad (4)$$

The middle point of interval $EI(A)$ is as the follows:

$$\bar{EI}(A) = 1/2 \int_0^1 (a_1(\gamma) + a_2(\gamma)) d\gamma. \qquad (5)$$

Also Grzegorzewski [5] show that the interval $EI(A)$ is the nearest interval to the fuzzy number A. Carlsson and Fullér [2] introduced the interval-valued possibilistic mean of fuzzy number A as the interval $M(A) = [M_*(A), M^*(A)]$, where

$$M_*(A) = 2\int_0^1 \gamma a_1(\gamma)d\gamma \quad \text{and} \quad M^*(A) = 2\int_0^1 \gamma a_2(\gamma)d\gamma, \tag{6}$$

hence, they have considered the mean of fuzzy number A, as follows:

$$\bar{M}(A) = \frac{M_*(A) + M^*(A)}{2} = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma. \tag{7}$$

Now we will try to find a weighting crisp point $C_f(A)$ which is the nearest to $A \in F$ with respect to a weighted distance quantity. We can do this, since each crisp number is a fuzzy number, too. Therefore, regarding the definitions (2, 3), we define a defuzzification as a $f-$weighted distance quantity similar to the distance between two fuzzy numbers $A, B \in F$ [1, 5].

Definition 1. Let $A \in F$ and $C_f(A) \in R$ is a crisp point of support function. A $f-$weighted distance quantity is define by

$$d_f(A, C_f(A)) = \frac{1}{2}\left[\int_0^1 \gamma(a_1(\gamma) - C_f(A))^2 d\gamma + \int_0^1 \gamma(a_2(\gamma) - C_f(A))^2 d\gamma\right]^{\frac{1}{2}},$$

where $f : [0,1] \to R$ is said to be a weighting function if f is non-negative, monotone increasing and satisfies the following normalization condition

$$\int_0^1 f(\alpha)d\alpha = 1.$$

By minimizing $d_f^2(A, C_f(A))$ with respect to crisp point $C_f(A)$ from support function, we get

$$C_f(A) = \int_0^1 f(\gamma)\frac{a_1(\gamma) + a_2(\gamma)}{2}d\gamma. \tag{8}$$

Note that $C_f(A)$ exists and it is unique (with respect to the above distance); in addition, it is the nearest point of fuzzy number A.

We define $C_f(A)$ as the $f-$WPM value of $A \in F$ that is denoted by $\bar{M}_f(A)$, i.e.

$$\bar{M}_f(A) = \int_0^1 f(\gamma)\frac{a_1(\gamma) + a_2(\gamma)}{2}d\gamma, \tag{9}$$

therefore we have the following theorem:

Theorem 1. *Let f be a weighting and let A be a fuzzy number and $C_f(A)$ a crisp point. Then, the function $d_f(A, C_f(A))$ with respect to $C_f(A)$ is minimum value if $C_f(A) = \bar{M}_f(A)$ and $\bar{M}_f(A)$ is unique.*

Proof. Proof is similar to proof of equation 18 in [5]. □

This theorem shows that $\bar{M}_f(A)$ is the nearest $f-$weighted point of $A \in F$ which is unique. In addition, $\bar{M}_f(A)$ is a WPM value of a fuzzy number (see

[4]). If $f(\gamma) = 1$, then $\bar{M}_f(A)$ is the same as the the middle point of the interval-valued mean of Dubois, Prade and Heilpren [3, 6] and if $f(\gamma) = 2\gamma$ then $\bar{M}_f(A)$ is the same as the possibilistic mean value of Carlsson and Full [2].

Should be noted that we have showed f−WPM is the nearest weighted point to fuzzy number A and belongs to support function. And theorem 1 is the main result of this section that is a new and intersting alternative justification to the definition of the f−weighted mean of a fuzzy number so that it is not mentioned in [2, 4].

4. Weighted possibilistic variance (WPV)

In this section, we introduce a notion of the variance of fuzzy numbers that satisfies in all desired properties mentioned in the introduction of paper.

By use definition 1, we introduce the f−WPV of $A \in F$ by

$$\mathrm{Var}'_f(A) = \frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma) - \bar{M}_f(A))^2 + (a_2(\gamma) - \bar{M}_f(A))^2]d\gamma. \tag{10}$$

The above expression is a weighted possibilistic quantity, because we can rewrite it as:

$$\begin{aligned}\mathrm{Var}'_f(A) &= \frac{1}{2}\int_0^1 f(\mathrm{Pos}[A \le a_1(\gamma)])(\min[A]^\gamma - \bar{M}_f(A))^2 d\gamma + \\ &\quad \frac{1}{2}\int_0^1 f(\mathrm{Pos}[A \ge a_2(\gamma)])(\max[A]^\gamma - \bar{M}_f(A))^2 d\gamma.\end{aligned}$$

Note that $\bar{M}_f(A)$ and $\mathrm{Var}'_f(A)$ have all the properties of the possibilistic mean value and variance stated in [2, 4]; furthermore, $\mathrm{Var}'_f(A)$ has preserved some other properties of variance in probability theory, this means that for any other crisp point $C \in R$, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma) - \bar{M}_f(A))^2 + (a_2(\gamma) - \bar{M}_f(A))^2]d\gamma \\ \le\ &\frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma) - C)^2 + (a_2(\gamma) - C)^2]d\gamma.\end{aligned} \tag{11}$$

Equation (10) is the natural generalization of the possibilistic variance that introduced in [2], remark 4.2. But here the f−WPV of $A \in F$ is introduced to help definition 1. Furthermore we add the theorem of the variance inequality as follows:

Theorem 2. *Let f be a weighting function and $A \in F$. Then for any $C \in R$*

$$\mathrm{Var}'_f(A) \le \frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma) - C)^2 + (a_2(\gamma) - C)^2]d\gamma. \tag{12}$$

Proof. For proof of theorem, it is sufficient that

$$\frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma) - C)^2 + (a_2(\gamma) - C)^2]d\gamma - \mathrm{Var}'_f(A) \ge 0, \tag{13}$$

holds. We can write:

$$
\begin{aligned}
& \frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma)-C)^2+(a_2(\gamma)-C)^2]d\gamma \\
= & \frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma)-\bar{M}_f(A) \\
& +\bar{M}_f(A)-C)^2+(a_2(\gamma)-\bar{M}_f(A)+\bar{M}_f(A)-C)^2]d\gamma \\
= & \frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma)-\bar{M}_f(A))^2+(a_2(\gamma)-\bar{M}_f(A))^2]d\gamma+ \\
& \frac{1}{2}\int_0^1 f(\gamma)[(\bar{M}_f(A)-C)^2+(\bar{M}_f(A)-C)^2]d\gamma+2(\bar{M}_f(A)-C))\times \\
& \left[\int_0^1 f(\gamma)[(a_1(\gamma)-\bar{M}_f(A))+(a_2(\gamma)-\bar{M}_f(A))]d\gamma\right],
\end{aligned}
$$

the last sentence of the above expression is zero; therefore,

$$\frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma)-C)^2+(a_2(\gamma)-C)^2]d\gamma=\mathrm{Var}_f^{'}(A)+(\bar{M}_f(A)-C)^2.$$

Consequently,

$$\frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma)-C)^2+(a_2(\gamma)-C)^2]d\gamma-\mathrm{Var}_f^{'}(A)=(\bar{M}_f(A)-C))^2\geq 0.$$

This completes the proof of theorem. □

Theorem 2 shows that the second moment about $\bar{M}_f(A)$(variance) is less than or equal to the second moment about crisp point $C \in R$.

Carlsson and Fullér [2] have introduced the possibilistic variance of fuzzy number A as follows:

$$
\begin{aligned}
\mathrm{Var}(A) &= \int_0^1 \mathrm{Pos}(A \leq a_1(\gamma))\Big[\frac{a_1(\gamma)+a_2(\gamma)}{2}-a_1(\gamma)\Big]^2 d\gamma+ \\
& \quad \int_0^1 \mathrm{Pos}(A \geq a_2(\gamma))\Big[\frac{a_1(\gamma)+a_2(\gamma)}{2}-a_2(\gamma)\Big]^2 d\gamma \\
&= \int_0^1 \left(\gamma\Big[\frac{a_1(\gamma)+a_2(\gamma)}{2}-a_1(\gamma)\Big]^2+\gamma\Big[\frac{a_1(\gamma)+a_2(\gamma)}{2}-a_2(\gamma)\Big]^2\right)d\gamma,
\end{aligned}
$$

so,

$$\mathrm{Var}(A)=\frac{1}{2}\int_0^1 \gamma(a_2(\gamma)-a_1(\gamma))^2 d\gamma. \tag{14}$$

It seems that this definition does not have the real concept of the variance, because that stated properties in introduction, for a variance are not satisfy, e.g. it does not have physical interpretation of variance. The variance inequality is not stateable. In addition, in the next section we show that this definition of variance of fuzzy numbers is not generable to moments of fuzzy numbers.

Also Fullér and Majlender [4] introduced the $f-$ weighted possibilistic variance as:

$$\mathrm{Var}_f(A) = \frac{1}{4}\int_0^1 f(\gamma)(a_2(\gamma) - a_1(\gamma))^2 d\gamma. \tag{15}$$

This definition is generalization of (14).

4.1. Comparison of variances. In this subsection, we state several theorem to compare two variances of a fuzzy number, $\mathrm{Var}_f(A)$ and $\mathrm{Var}'_f(A)$.

Theorem 3. *Let A and B be two fuzzy numbers of type LR with strictly decreasing and continuous shape functions. If $A \subseteq B$ then*

$$\mathrm{Var}'_f(A) \leq \mathrm{Var}'_f(B).$$

Proof. The proof is similar to proof of theorem 3.3 in [8]. □

This theorem shows that subsethood does entail smaller variance.

Theorem 4. *Let $A \in F$. Then*

$$\mathrm{Var}_f(A) \leq \mathrm{Var}'_f(A).$$

Proof. See the proof of theorem 3.4 in [8]. □

Theorem 5. *Let A be a symmetric fuzzy number with $[A]^\gamma = [\lambda - s(\gamma), \lambda + s(\gamma)]$ for all $\gamma \in [0, 1]$. then*

$$\mathrm{Var}_f(A) = \mathrm{Var}'_f(A).$$

Proof. According to the definitions of $\bar{M}_f(A), \mathrm{Var}_f(A)$ and $\mathrm{Var}'_f(A)$, we easily get

$$\bar{M}_f(A) = \int_0^1 f(\gamma)\frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma = \int_0^1 f(\gamma)\frac{\lambda - s(\gamma) + \lambda + s(\gamma)}{2} d\gamma = \lambda.$$

$$\begin{aligned}\mathrm{Var}_f(A) &= \frac{1}{4}\int_0^1 f(\gamma)[a_2(\gamma) - a_1(\gamma)]^2 d\gamma \\ &= \frac{1}{4}\int_0^1 f(\gamma)[(\lambda + s(\gamma)) - (\lambda - s(\gamma))]^2 d\gamma \\ &= \int_0^1 f(\gamma)[s(\gamma)]^2 d\gamma.\end{aligned}$$

Similarly,

$$\mathrm{Var}'_f(A) = \frac{1}{2}\int_0^1 f(\gamma)[(\lambda - s(\gamma) - \lambda)^2 + (\lambda + s(\gamma) - \lambda)^2] d\gamma = \int_0^1 f(\gamma)[s(\gamma)]^2 d\gamma.$$

Observe that $\mathrm{Var}_f(A) = \mathrm{Var}'_f(A)$, which ends the proof. □

5. Weighted possibilistic moments

In this section, the weighted possibilistic moments about the mean for every $A \in F$ are suggested. In a similar way with the definition of possibilistic variance of fuzzy number A we state the following definition.

Definition 2. Let f be a weighting function and let A be a fuzzy number. Then, we define the $f-$weighted possibilistic moments of order r about the mean value of A as:

$$\mu_r^{'}(A) = \frac{1}{2}\int_0^1 f(\gamma)[(a_1(\gamma) - \bar{M}_f(A))^r + (a_2(\gamma) - \bar{M}_f(A))^r]d\gamma, r = 1, 2, 3, ... \quad (16)$$

clearly, $\mathrm{Var}_f^{'}(A) = \mu_2^{'}(A)$.

The first moment about the mean is zero and the second moment is the variance. Third and fourth moments are also used for computing statistical quantities known as skewness and kurtosis, in other words in a similar way we can define the possibilistic skewness and kourtosis in possibility theory such as those definitions in the probability theory.

If $f(\gamma) = 2\gamma$, then

$$\mu_r^{'}(A) = \int_0^1 \gamma[(a_1(\gamma) - \bar{M}_f(A))^r + (a_2(\gamma) - \bar{M}_f(A))^r]d\gamma, r = 1, 2, 3, ... \quad (17)$$

the above expression is the possibilistic moments of r order about the mean value of a fuzzy number.

There are higher order possibilistic moments that are commonly used to the description of fuzzy numbers. The most common and useful are the second and third possibilistic moments, the second possibilistic moments is called variance, which is the square deviation of the fuzzy number from its mean value, the variance is always positive and it is a measure of dispersion or spread of the fuzzy number. The third possibilistic moments is called the possibilistic skewness and it is a measure of symmetric of fuzzy number. A symmetrical fuzzy number has a possibilistic skewness of zero. If the skewness be positive, then fuzzy number has a longer tail to the right, and if the possibilistic skewness be negative, then fuzzy number has a longer tail to the left. Also the forth possibilistic moments is use for compute the possibilistic kurtosis of fuzzy numbers. This indices can apples to recognize many of fuzzy times series models (see [7]), especially, those are use for the estimation of parameters in the times series models . However, the possibilistic moments play an important role in many of fuzzy times series process.

Theorem 6. Let $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ *be a fuzzy number and* λ_1, λ_2 *be real numbers. Then,*

$$\mu_r^{'}(\lambda_1 A + \lambda_2) = \lambda_1^r \mu_r^{'}(A).$$

Proof. Suppose $\lambda_1 > 0$. Then, we have $[\lambda_1 A+\lambda_2]^\gamma = [\lambda_1 a_1(\gamma)+\lambda_2, \lambda_1 a_2(\gamma)+\lambda_2]$. Therefore,

$$\begin{aligned}\overline{M}_f(\lambda_1 A+\lambda_2) &= \int_0^1 f(\gamma)\frac{\lambda_1 a_1(\gamma)+\lambda_2+\lambda_1 a_2(\gamma)+\lambda_2}{2}d\gamma \\ &= \lambda_1\int_0^1 f(\gamma)\frac{a_1(\gamma)+a_2(\gamma)}{2}d\gamma+\lambda_2\int_0^1 f(\gamma)d\gamma \\ &= \lambda_1\bar{M}_f(A)+\lambda_2.\end{aligned}$$

$$\begin{aligned}&\mu_r'(\lambda_1 A+\lambda_2) \\ =\ & \frac{1}{2}\int_0^1 f(\gamma)[(\lambda_1 a_1(\gamma)+\lambda_2-\bar{M}_f(\lambda_1 A+\lambda_2))^r+ \\ & (\lambda_1 a_2(\gamma)+\lambda_2-\overline{M}_f(\lambda_1 A+\lambda_2))^r]d\gamma \\ =\ & \frac{1}{2}\int_0^1 f(\gamma)[(\lambda_1 a_1(\gamma)-\lambda_1\bar{M}_f(A))^r+(\lambda_1 a_2(\gamma)-\lambda_1\bar{M}_f(A))^r]d\gamma \\ =\ & \lambda_1^r\mu_r''(A).\end{aligned}$$

Similarly, when $\lambda_1 < 0$, theorem can be easily proved. □

Theorem 6 show that the possibilistic moments of fuzzy numbers with respect to locally parameter λ_2 are invariant.

Theorem 7. *Let $[A]^\gamma = [\lambda-s(\gamma), \lambda+s(\gamma)]$ be a symmetric fuzzy number. Then,*

$$\mu_r'(A) = \frac{(1+(-1)^r)}{2}\int_0^1 f(\gamma)[s(\gamma)]^r d\gamma,\ r\in\{1,2,3,...\}. \tag{18}$$

Proof. Suppose A be a symmetric fuzzy numbers, then

$$\begin{aligned}\mu_r'(A) &= \frac{1}{2}\int_0^1 f(\gamma)[(\lambda-s(\gamma)-\lambda)^r+(\lambda+s(\gamma)-\lambda)^r]d\gamma \\ &= \frac{1}{2}\int_0^1 f(\gamma)[(-s(\gamma))^r+(s(\gamma))^r]d\gamma,\end{aligned}$$

so,

$$\mu_r'(A) = \frac{(1+(-1)^r)}{2}\int_0^1 f(\gamma)[s(\gamma)]^r d\gamma. \tag{19}$$

That completes the proof of Theorem 7. □

Corollary 1. *If A be a symmetric fuzzy number. Then the moments of odd orders are zero.*

Corollary 2. *If A be a symmetric fuzzy number. Then the moments of even orders are:*

$$\mu_r'(A) = \int_0^1 f(\gamma)[s(\gamma)]^r d\gamma. \tag{20}$$

Example 1. Let $f(\gamma) = 2\gamma$. Consider fuzzy number A with the following membership function.

$$A(x) = \begin{cases} 1 - \left(\frac{x-2}{2}\right)^2, & 0 \le x \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

We have $[A]^\gamma = [a_1(\gamma), a_2(\gamma)] = [2 - 2\sqrt{1-\gamma}, 2 + 2\sqrt{1-\gamma}]$, since for every $\gamma \in [0,1]$, $\frac{a_1(\gamma)+a_2(\gamma)}{2} = 2 = \lambda$; therefore, A is a symmetric fuzzy number and $\bar{M}_f(A) = 2$ (see Fig.1).

$$\mu_r^{'}(A) = \frac{(1+(-1)^r)}{2} \int_0^1 2\gamma(2\sqrt{1-\gamma})^r d\gamma = \frac{2^{r+2}(1+(-1)^r)}{(r+4)(r+2)}, \ r = 1,2,3,...$$

$$\text{Var}_f^{'}(A) = \int_0^1 2\gamma(2\sqrt{1-\gamma})^2 d\gamma = \frac{4}{3}.$$

Observe that for all $r \in \{1,3,5,...\}$, $\mu_r^{'}(A) = 0$, and this shows that A is a symmetric fuzzy number about its mean value.

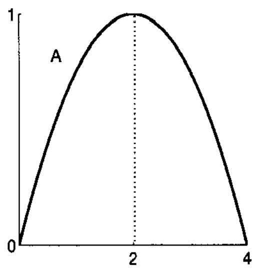

Fig.1.

We will now generalize the equation (14) to possibilistic moments of fuzzy numbers. In a similar way with the definition of the possibilistic variance of Carlsson and Fullér, the r_{th} possibilistic moment of $A \in F$ is defined as below:

$$\begin{aligned} \mu_r(A) &= \int_0^1 \text{Pos}(A \le a_1(\gamma))[\frac{a_1(\gamma)+a_2(\gamma)}{2} - a_1(\gamma)]^r d\gamma + \\ &\int_0^1 \text{Pos}(A \ge a_2(\gamma))\left[\frac{a_1(\gamma)+a_2(\gamma)}{2} - a_2(\gamma)\right]^r d\gamma, \ r = 1,2,3,... \end{aligned}$$

so,

$$\mu_r(A) = \frac{(1+(-1)^r)}{2^r} \int_0^1 \gamma(a_2(\gamma) - a_1(\gamma))^r d\gamma, \ r = 1,2,3,... \quad (21)$$

obviously, $\text{Var}(A) = \mu_2(A)$ and $\mu_r(A) = 0$ for any $r \in \{1,3,5,...\}$ which shows for any $A \in F$, A is symmetric.

5.1. Comparison of moments. In this subsection two types of possibilistic moments of fuzzy numbers are compared. We have introduced the $f-$ weighted possibilistic variance ($\mathrm{Var}'_f(A)$) by a weighted distance quantity, so that, it has all important properties of the notion of a variance and according to the extension principle our notion for the possibilistic variance and moments of fuzzy numbers is consistent with the physical interpretation of the variance and well-known definition of variance in probability theory. In addition to the possibilistic moments ($\mu'_r(A)$ of $A \in F$ which is introduced in section 5, are preserved the important properties of the moments about the mean in probability theory, e.g. the second possibilistic moment about the mean is the possibilistic variance and the third possibilistic moment is a measure of the lopsidedness of the fuzzy number.

Theorem 8. *Let* $[A]^\gamma = [\lambda - s(\gamma), \lambda + s(\gamma)]$ *be a symmetric fuzzy number and* $f(\gamma) = 2\gamma$. *Then*

$$\mu_r(A) = \mu'_r(A), \;\; r \in \{1, 2, 3, ...\}.$$

Proof. Suppose A be a symmetric fuzzy number, we easily get

$$\mu_r(A) = \frac{(1 + (-1)^r)}{2^r} \int_0^1 \gamma[2s(\gamma)]^r d\gamma,$$

$$\mu_r(A) = (1 + (-1)^r) \int_0^1 \gamma[s(\gamma)]^r d\gamma. \tag{22}$$

From comparing relations (19) and (22) result is yield . □

Corollary 3. *If A be a symmetric fuzzy number, then for all* $r \in \{1, 3, 5, ...\}$

$$\mu_r(A) = \mu'_r(A) = 0.$$

Corollary 4. *If A be an asymmetric fuzzy number, then for all* $r \in \{1, 3, 5, ...\}$

$$\mu_r(A) = 0.$$

This corollary shows that the possibilistic variance defined by Carlsson and Fullér is suitable for symmetric fuzzy numbers. In other words, we can not define the possibilistic moments of fuzzy numbers by extending of the definition of the possibilistic variance. Therefore, we have the following main results:

1. For symmetric fuzzy numbers

$$\mu_r(A) = \mu'_r(A), \; r \in \{1, 2, 3, ...\} \quad \text{and}$$

$$\mu_r(A) = \mu'_r(A) = 0, \; r \in \{1, 3, 5, ...\}.$$

2. For asymmetric fuzzy numbers

$$\mu_r(A) = 0, \; r \in \{1, 3, 5, ...\},$$

and this is a contradictory to physical interpretation of moments, in addition to, this shows that the possibilistic moments of $A \in F$ ($\mu_r(A)$) are only suitable for symmetric fuzzy numbers. In other words, in a similar manner with definition of the variance of Carlsson and Fullér [2] we can not define the possibilistic moments of asymmetric fuzzy numbers. More important that $\mu_3(A) = 0$, this imply that the possibilistic skewness of asymmetric fuzzy numbers are zero, that it is quite unreasonable.

3. For all asymmetric fuzzy numbers $\mu_r^{'}(A) \neq 0$, $r \in \{1,3,5,...\}$. And this shows that possibilistic moments of A $(\mu_r^{'}(A))$ are suitable for all fuzzy numbers. Especially, the second possibilistic moment that is called as the possibilistic variance.

We think that $\mu_r^{'}(A)$ is a suitable definition and well-define of the possibilistic moments about the mean value of a fuzzy number, so that it the important properties of the central moments in probability theory are preserved and it has physical interpretation, completely. The possibilistic moments of fuzzy numbers are more generally of the variance and the variance is a special case of moments. Therefore, if the possibilistic moments of fuzzy number be well-define; then, the second possibilistic moment (variance) is suitable and well-define.

4. $\mathrm{Var}^{'}(A) = \mu_2^{'}(A)$ is the same as alternative variance of Carlsson and Fullér (remark 4.2). This means that our definition of the possibilistic variance of fuzzy numbers is the same as the definition of $\mathrm{Var}^{'}(A)$ which is introduced by Carlsson and Fullér. Therefore, $\mathrm{Var}^{'}(A)$ has all the properties of the notion of a variance as mentioned in the introduction and our method is confirmed this variance.

Should be noted that in this paper we have stated several the new theorem about the properties of variance and moments of fuzzy numbers.

Example 2. Consider triangular fuzzy numbers $[B]^\gamma = [4\gamma, 5-\gamma]$ (Fig.2), $[C]^\gamma = [3+\frac{\gamma}{2}, 4-\frac{\gamma}{2}]$ (Fig.3) and $f(\gamma) = 2\gamma$. We get

$$\bar{M}_f(B) = 3.5, \mathrm{Var}^{'}(B) = 1.667, \mathrm{Var}(B) = 1.042,$$

the possibilistic moments of orders odd of $B \in F$ are zero in $\mu_r(B)$. This shows that B is a symmetric fuzzy number and it is unreasonable. But the possibilistic moments of orders odd of $B \in F$ are negative in $\mu_r^{'}(B)$. Therefore, fuzzy number B is skewed to the left (see Fig.2, Table 1).

For fuzzy number C since, $\frac{(c_1(\gamma)+(c_2(\gamma)}{2} = \frac{(3+\frac{\gamma}{2}+4-\frac{\gamma}{2})}{2} = 3.5 = \lambda$; hence, C is a symmetric fuzzy number (see Fig.3) and we have

$$\bar{M}_f(C) = 3.5,\ \mathrm{Var}(C) = \mathrm{Var}^{'}(C) = 2\int_0^1 \gamma\Big(3.5 - 3 - \frac{\gamma}{2}\Big)^2 d\gamma = 0.041672,$$

for all $r \in \{1,2,3,...\}$ $\mu_r(C) = \mu_r^{'}(C) = 0$ (see Table 1).

Table 1: Comparative possibilitic moments of fuzzy numbers B, C

r	1	2	3	4	5	6	7	...
$\mu_r(B)$	0	1.042	0	2.667	0	8.719	0	...
$\mu_r^{'}(B)$	0	1.667	-1.275	4.204	-9.170	25.589	-67.883	...
$\mu_r(C)$	0	0.041667	0	0.004167	0	0.000558	0	...
$\mu_r^{'}(C)$	0	0.041667	0	0.004167	0	0.000558	0	...

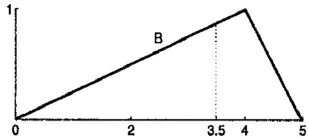

Fig.2.

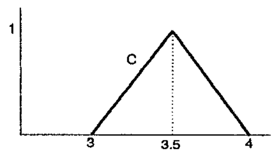

Fig.3.

Example 3. Consider fuzzy numbers $[D]^\gamma = [5-\sqrt{-2Ln\gamma}, 5+\sqrt{-2Ln\gamma}]$, $[E]^\gamma = [4-\gamma, 7-2\gamma]$ (Fig.4) and $f(\gamma) = 2\gamma$. We get $\bar{M}_f(D) = 5$, $\bar{M}_f(E) = 5.167$, $\mathrm{Var}_f(D) = 1$, $\mu_3(D) = 0, \mu_4(D) = 2$, $\mu_5(D) = 0$, $\mu_6(D) = 6$.

$$\mathrm{Var}_f(E) = 0.375,\ \mu_3(E) = 0,\ \mu_4(E) = 0.407,\ \mu_5(E) = 0,\ \mu_6(E) = 0.407.$$

$$\mathrm{Var}_f^{'}(D) = 1,\ \mu_3^{'}(D) = 0,\ \mu_4^{'}(D) = 2,\ \mu_5^{'}(D) = 0,\ \mu_6^{'}(D) = 6.$$

$$\mathrm{Var}_f^{'}(E) = 0.389,\ \mu_3^{'}(E) = 0.151,\ \mu_4^{'}(E) = 0.400,\ \mu_5^{'}(E) = 0,\ \mu_6^{'}(E) = 0.631,$$

observe that, $E \subseteq D$ and $\mathrm{Var}_f^{'}(E) = 0.389 \leq 1 = \mathrm{Var}_f^{'}(D)$.

The third moment of fuzzy number $D\,(\mu_3^{'}(D))$ is zero, therefore D is a symmetric fuzzy number, and the third moment of fuzzy number $E\,(\mu_3^{'}(E))$ equals 0.151, therefore fuzzy number E is skewed to right side (see Fig.4). But the third moment of fuzzy number $E\,(\mu_3(E))$ is zero, this means that E is a symmetric fuzzy number, and unreasonable.

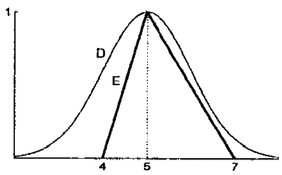

Fig.4. Fuzzy numbers D, E

6. Summary

In this paper, first the necessities and proper properties are mentioned for the notion of possibilistic variance of a fuzzy number. Then a WPM value is obtained via minimizing a weighted distance quantity, in fact in this work a distance path is selected, and then introduced a notion of the possibilistic variance of fuzzy numbers which is yielded from a weighted distance quantity. We have showed that the WPM value is nearest weighted point of a fuzzy number, the main result of section 3 is a new and interesting alternative justification to define of $f-$weighted mean of a fuzzy number. In addition a notion of the variance of fuzzy numbers is introduced such that according to the the extension principle our notion of the possibilistic variance of fuzzy numbers is consistent with physical interpretation of the variance and well-known definition of variance in probability theory. We also have presented the high-order possibilistic moments about the mean value of fuzzy numbers as the generalization of the possibilistic variance of fuzzy numbers that they play an important role in fuzzy data analysis, e.g. the possibilistic skewness and kurtosis of fuzzy numbers. Furthermore, we have concluded from the two definitions of possibilistic variance $(\mathrm{Var}(A), \mathrm{Var}^{'}(A))$ which are introduced by Carlsson and Fullér, $\mathrm{Var}^{'}(A)$ is suitable for all fuzzy number and our method is confirmed this matter, but $\mathrm{Var}(A)$ which has very well efficiency, it is more suitable for symmetric fuzzy numbers.

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