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FURTHER GENERALIZATION OF OSTROWSKI'S INEQUALITY AND APPLICATIONS IN NUMERICAL INTEGRATION AND FOR SPECIAL MEANS

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ABSTRACT. In this paper, by introducing a parameter, we establish a new Ostrowski's integral inequality which generalizes the result of [5]. Finally, we apply the new Ostrowski's inequality to numerical integration and special means.

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1. Introduction

In 1938, A. Ostrowski proved the following interesting integral inequality [6]:

Theorem 1. *Let $f : [a,b] \to R$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f' : (a,b) \to R$ is bounded in (a,b), that is, $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(x)| < \infty$. Then for any $x \in [a,b]$, we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty. \tag{1}$$

The inequality is sharp in the sense that the constant 1/4 cannot be replaced by a smaller one.

In [5], Dragomir and Sofo gave a generalization of Ostrowski's integral inequality for mappings whose second derivative belong to $L^\infty[a,b]$.

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Theorem 2. *Let $g:[a,b]\to R$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $g''\in L^{\infty}[a,b]$. Then we have the inequality:*

$$\left|\int_a^b g(t)\,dt-\frac{b-a}{2}\left[g(x)+\frac{g(a)+g(b)}{2}\right]+\frac{b-a}{2}\left(x-\frac{a+b}{2}\right)g'(x)\right|$$
$$\le\|g''\|_\infty\left[\frac{1}{3}\left|x-\frac{a+b}{2}\right|^3+\frac{(b-a)^3}{48}\right] \tag{2}$$

for all $a\le x\le b$.

In this paper, we will derive a new Ostrowski's integral inequality with a parameter $h\in[0,1]$ for twice differentiable functions, which will not only provide a generalization of [5], but also give some other interesting Ostrowski integral inequalities as special cases and showing that the case $h=1-\frac{\sqrt{2}}{2}$ is optimal. Applications in numerical integration and for special means are also given.

2. Main results

Our main results read as follows.

Theorem 3. *Let $g:[a,b]\to R$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $g''\in L^{\infty}[a,b]$. Then we have the inequality*

$$\left|\int_a^b g(t)\,dt-\frac{b-a}{2}\left[(1-h)\,g(x)+(1+h)\,\frac{g(a)+g(b)}{2}\right]\right.$$
$$\left.+\frac{b-a}{2}(1-h)\left(x-\frac{a+b}{2}\right)g'(x)-\frac{h}{8}(b-a)^2\left[g'(a)-g'(b)\right]\right|$$
$$\le\|g''\|_\infty\left[\frac{1}{3}\left|x-\frac{a+b}{2}\right|^3+\frac{(b-a)^3}{48}(1-3h+6h^2-2h^3)\right] \tag{3}$$

for all $a+h((b-a)/2)\le x\le b-h((b-a)/2)$ and $h\in[0,1]$.

Proof. Let us start with the following integral equality

$$(1-h)\,f(x)=\frac{1}{b-a}\left(\int_a^b f(t)\,dt+\int_a^b p(x,t)\,f'(t)\,dt\right)-\frac{h}{2}\left[f(a)+f(b)\right],$$

where $x\in[a,b]$, provided f is absolutely continuous on $[a,b]$, and the kernel $p:[a,b]^2\to R$ is given by

$$p(x,t)=\begin{cases} t-\left[a+\frac{h}{2}(b-a)\right], & t\in[a,x]\\ t-\left[b-\frac{h}{2}(b-a)\right], & t\in(x,b]\end{cases}. \tag{4}$$

Now choose $f(x) = \left(x - \frac{a+b}{2}\right) g'(x)$, to get

$$(1-h)\left(x - \frac{a+b}{2}\right) g'(x) = \frac{1}{b-a}\left(\int_a^b \left(t - \frac{a+b}{2}\right) g'(t)\,dt + \int_a^b p(x,t)\left[g'(t) + \left(t - \frac{a+b}{2}\right) g''(t)\right] dt\right) + \frac{h}{4}(b-a)\left[g'(a) - g'(b)\right]. \quad (5)$$

Integrating by parts we have

$$\int_a^b \left(t - \frac{a+b}{2}\right) g'(t)\,dt = \frac{b-a}{2}\left[g(a) + g(b)\right] - \int_a^b g(t)\,dt. \quad (6)$$

Also upon using (4), we have that

$$\begin{aligned}
&\int_a^b p(x,t)\left[g'(t) + \left(t - \frac{a+b}{2}\right) g''(t)\right] dt \\
&= \int_a^b p(x,t)\, g'(t)\,dt + \int_a^b p(x,t)\left(t - \frac{a+b}{2}\right) g''(t)\,dt \\
&= \frac{h}{2}(b-a)\left[g(a) + g(b)\right] + (1-h)(b-a)\,g(x) - \int_a^b g(t)\,dt \\
&\qquad + \int_a^b p(x,t)\left(t - \frac{a+b}{2}\right) g''(t)\,dt. \quad (7)
\end{aligned}$$

Now by (5), (6) and (7) we deduce that

$$\begin{aligned}
\int_a^b g(t)\,dt =& \frac{b-a}{2}\left[(1-h)\,g(x) + (1+h)\frac{g(a)+g(b)}{2}\right] \\
& - \frac{b-a}{2}(1-h)\left(x - \frac{a+b}{2}\right) g'(x) + \frac{h}{8}(b-a)^2\left[g'(a) - g'(b)\right] \\
& + \frac{1}{2}\int_a^b p(x,t)\left(t - \frac{a+b}{2}\right) g''(t)\,dt, \quad (8)
\end{aligned}$$

from where we get the identity

$$\begin{aligned}
&\left|\int_a^b g(t)\,dt - \frac{b-a}{2}\left[(1-h)\,g(x) + (1+h)\frac{g(a)+g(b)}{2}\right]\right. \\
&\left. + \frac{b-a}{2}(1-h)\left(x - \frac{a+b}{2}\right) g'(x) + \frac{h}{8}(b-a)^2\left[g'(a) - g'(b)\right]\right| \\
&\le \frac{1}{2}\int_a^b |p(x,t)|\left|t - \frac{a+b}{2}\right| |g''(t)|\,dt. \quad (9)
\end{aligned}$$

Obviously, we have

$$\int_a^b |p(x,t)|\left|t - \frac{a+b}{2}\right| |g''(t)|\,dt \le \|g''\|_\infty \int_a^b |p(x,t)|\left|t - \frac{a+b}{2}\right| dt,$$

where $\|g''\|_\infty = \sup_{t\in(a,b)} |g''(t)| < \infty$. Let

$$I = \int_a^b |p(x,t)| \left|t - \frac{a+b}{2}\right| dt = \int_a^x \left|t - \left[a + \frac{h}{2}(b-a)\right]\right| \left|t - \frac{a+b}{2}\right| dt$$
$$+ \int_x^b \left|t - \left[b - \frac{h}{2}(b-a)\right]\right| \left|t - \frac{a+b}{2}\right| dt.$$
$$:= I_1 + I_2. \tag{10}$$

We have two cases:

(1) For $x \in \left[a, \frac{a+b}{2}\right]$ we obtain

$$I_1 = \int_a^{a+\frac{h}{2}(b-a)} \left(a + h\frac{b-a}{2} - t\right)\left(\frac{a+b}{2} - t\right) dt$$
$$+ \int_{a+\frac{h}{2}(b-a)}^{x} \left[t - \left(a + h\frac{b-a}{2}\right)\right]\left(\frac{a+b}{2} - t\right) dt,$$

$$I_2 = \int_x^{\frac{a+b}{2}} \left(b - h\frac{b-a}{2} - t\right)\left(\frac{a+b}{2} - t\right) dt$$
$$+ \int_{\frac{a+b}{2}}^{b-h\frac{b-a}{2}} \left(b - h\frac{b-a}{2} - t\right)\left(t - \frac{a+b}{2}\right) dt$$
$$+ \int_{b-h\frac{b-a}{2}}^{b} \left[t - \left(b - h\frac{b-a}{2}\right)\right]\left(t - \frac{a+b}{2}\right) dt.$$

We have

$$I = I_1 + I_2 = \frac{2}{3}\left(\frac{a+b}{2} - x\right)^3 + \frac{(b-a)^3}{24}(1 - 3h + 6h^2 - 2h^3).$$

(2) For $x \in \left(\frac{a+b}{2}, b\right]$ we obtain

$$I_1 = \int_a^{a+\frac{b-a}{2}h} \left(a + h\frac{b-a}{2} - t\right)\left(\frac{a+b}{2} - t\right) dt$$
$$+ \int_{a+\frac{b-a}{2}h}^{\frac{a+b}{2}} \left[t - \left(a + h\frac{b-a}{2}\right)\right]\left(\frac{a+b}{2} - t\right) dt$$
$$+ \int_{\frac{a+b}{2}}^{x} \left[t - \left(a + h\frac{b-a}{2}\right)\right]\left(t - \frac{a+b}{2}\right) dt,$$

$$I_2 = \int_x^{b-\frac{b-a}{2}h} \left(b - h\frac{b-a}{2} - t\right)\left(t - \frac{a+b}{2}\right) dt$$
$$+ \int_{b-\frac{b-a}{2}h}^{b} \left[t - \left(b - h\frac{b-a}{2}\right)\right]\left(t - \frac{a+b}{2}\right) dt.$$

$$I = I_1 + I_2 = \frac{2}{3}\left(x - \frac{a+b}{2}\right)^3 + \frac{(b-a)^3}{24}(1 - 3h + 6h^2 - 2h^3).$$

By referring to (10), we obtain the result (3) of Theorem 3. □

Corollary 1. *Let $g : [a,b] \to R$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $g'' \in L^\infty[a,b]$. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b g(t)\,dt - \frac{1}{2}\left[(1-h)\,g\left(\frac{a+b}{2}\right) + (1+h)\,\frac{g(a)+g(b)}{2}\right] - \frac{h}{8}(b-a)\left[g'(a) - g'(b)\right]\right|$$
$$\leq \|g''\|_\infty \frac{(b-a)^2}{48}(1 - 3h + 6h^2 - 2h^3) \tag{11}$$

for all $h \in [0,1]$.

Proof. Choose $x = \frac{a+b}{2}$ in (3). □

Corollary 2. *Let $g : [a,b] \to R$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $g'' \in L^\infty[a,b]$. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b g(t)\,dt - \frac{1}{2}\left[g\left(\frac{a+b}{2}\right) + \frac{g(a)+g(b)}{2}\right]\right| \leq \|g''\|_\infty \frac{(b-a)^2}{48}.$$

Proof. Choose $h = 0$ in (11). □

Remark 1. Set $h = 0$ in (3) we can get (2). Therefore, Theorem 3 is a generalization of Theorem 2.

Corollary 3. *Let $g : [a,b] \to R$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $g'' \in L^\infty[a,b]$. Then we have the inequality*

$$\left|\int_a^b g(t)\,dt - \frac{b-a}{2}\left[g(a) + g(b)\right] - \frac{(b-a)^2}{8}\left[g'(a) - g'(b)\right]\right| \leq \|g''\|_\infty \frac{(b-a)^3}{24}.$$

Proof. Choose $h = 1$ in (11). □

Remark 2. For the right hand of (3),

$$P(x,h) := \|g''\|_\infty \left[\frac{1}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48}(1 - 3h + 6h^2 - 2h^3)\right].$$

When x be fixed, we can prove when $h = 1 - \frac{\sqrt{2}}{2}$, $P(x,h)$ obtains its minimum. Therefore, the following Corollary is optimal in the current situation.

Corollary 4. *Let $g : [a, b] \to R$ be a mapping whose first derivative is absolutely continuous on $[a, b]$ and assume that the second derivative $g'' \in L^\infty [a, b]$. Then we have the inequality*

$$\left| \int_a^b g(t)\,dt - \frac{b-a}{2}\left[\frac{\sqrt{2}}{2} g(x) + \left(2 - \frac{\sqrt{2}}{2}\right)\frac{g(a)+g(b)}{2}\right] \right.$$
$$\left. + \frac{\sqrt{2}}{4}(b-a)\left(x - \frac{a+b}{2}\right) g'(x) - \left(1 - \frac{\sqrt{2}}{2}\right)\frac{(b-a)^2}{8}\left[g'(a) - g'(b)\right] \right|$$
$$\le \|g''\|_\infty \left[\frac{1}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{2-\sqrt{2}}{48}(b-a)^3\right]$$

for all $\frac{2+\sqrt{2}}{4}a + \frac{2-\sqrt{2}}{4}b \le x \le \frac{2-\sqrt{2}}{4}a + \frac{2+\sqrt{2}}{4}b.$

3. Application in numerical integration

The following approximation of the integral $\int_a^b g(t)dt$ holds.

Theorem 4. *Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of the interval $[a, b]$, $d_i = x_{i+1} - x_i$, $\xi_i \in [x_i, x_{i+1}]$, $i = 0, 1, \cdots, n-1$. Then*

$$\int_a^b g(t)\,dt = S(g, I_n, \xi, h) + R(g, I_n, \xi, h),$$

where

$$S(g, I_n, \xi, h) = \frac{1}{2}\sum_{i=0}^{n-1}\left[(1-h)\,g(\xi_i) + (1+h)\frac{g(x_i)+g(x_{i+1})}{2}\right] d_i$$
$$-\frac{1}{2}\sum_{i=0}^{n-1}(1-h)\,d_i\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right) g'(\xi_i) + \frac{1}{8}\sum_{i=0}^{n-1} h d_i^2\left[g'(x_i) - g'(x_{i+1})\right]$$

and $|R(g, I_n, \xi, h)| \le \|g''\|_\infty \Big[\frac{1}{3}\sum_{i=0}^{n-1}\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|^3$
$$+\frac{1}{48}(1 - 3h + 6h^2 - 2h^3)\sum_{i=0}^{n-1} d_i^3\Big].$$

Proof. Applying (3) on the interval $\xi_i \in [x_i, x_{i+1}]$, $i = 0, \cdots, n-1$, we have

$$\left| \int_{x_i}^{x_{i+1}} g(t)\,dt - \frac{d_i}{2}\left[(1-h)\,g(\xi_i) + (1+h)\frac{g(x_i)+g(x_{i+1})}{2}\right] \right.$$
$$\left. + \frac{d_i}{2}(1-h)\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right) g'(\xi_i) - \frac{h}{8} d_i^2\left[g'(x_i) - g'(x_{i+1})\right] \right|$$

$$\leq \|g''\|_\infty \left[\frac{1}{3}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|^3 + \frac{1}{48}(1 - 3h + 6h^2 - 2h^3)d_i^3\right]$$

and therefore

$$\begin{aligned}
&\left|\int_a^b g(t)\,dt - S(g, I_n, \xi, h)\right| \\
&\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} g(t)\,dt - \frac{d_i}{2}\left[(1-h)\,g(\xi_i) + (1+h)\,\frac{g(x_i) + g(x_{i+1})}{2}\right]\right. \\
&\quad \left. + \frac{d_i}{2}(1-h)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) g'(\xi_i) - \frac{h}{8} d_i^2 [g'(x_i) - g'(x_{i+1})\right| \\
&\leq \|g''\|_\infty \sum_{i=0}^{n-1}\left[\frac{1}{3}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|^3 + \frac{1}{48}(1 - 3h + 6h^2 - 2h^3)d_i^3\right],
\end{aligned}$$

that is,

$$|R(g, I_n, \xi)| \leq \|g''\|_\infty \sum_{i=0}^{n-1}\left[\frac{1}{3}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|^3 + \frac{1}{48}(1 - 3h + 6h^2 - 2h^3)d_i^3\right].$$

□

Remark 3. Set $\xi_i = \dfrac{x_i + x_{i+1}}{2}$, $(i = 0, \cdots, n-1)$. Then

$$\begin{aligned}
S(g, I_n, h) =& \frac{1}{2}\sum_{i=0}^{n=1}\left[(1-h)\,g\left(\frac{x_i + x_{i+1}}{2}\right) + (1+h)\,\frac{g(x_i) + g(x_{i+1})}{2}\right] d_i \\
&+ \frac{h}{8}\sum_{i=0}^{n-1} d_i^2\,[g'(x_i) - g'(x_{i+1})],
\end{aligned}$$

where

$$|R(g, I_n, h)| \leq \|g''\|_\infty \frac{1}{48}(1 - 3h + 6h^2 - 2h^3)\sum_{i=0}^{n-1} d_i^3.$$

Remark 4. When $h = 0$, $S(g, I_n)$ may be thought of as the arithmetic mean of the Midpoint and the Trapezoidal quadrature rules. where

$$|R(g, I_n)| \leq \|g''\|_\infty \frac{1}{48}\sum_{i=0}^{n-1} d_i^3.$$

4. Applications for some special means

Let us recall the following means:

(1) The Arithmetic mean: $A(a, b) = \dfrac{a + b}{2}, \quad a, b > 0.$

(2) The Geometric mean: $G(a, b) = \sqrt{ab}, \quad a, b > 0.$

(3) The Logarithmic mean:

$$L(a,b)=\begin{cases} a, & a=b,\\ \dfrac{b-a}{\ln b-\ln a}, & a\neq b,\end{cases}\quad a,b>0.$$

(4) The Identic mean:

$$I\left(a,b\right)=\begin{cases} a, & a=b,\\ \dfrac{1}{e}\left(\dfrac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}}, & a\neq b.\end{cases}$$

(5) The harmonic mean:

$$H\left(a,b\right)=\frac{2}{\dfrac{1}{a}+\dfrac{1}{b}},\quad a,b>0.$$

The following inequality is well-known in the literature:

$$H(a,b)\leq G(a,b)\leq L(a,b)\leq I(a,b)\leq A(a,b).$$

The inequality (3) may be rewritten as

$$\begin{aligned}&\Bigg|-\frac{1-h}{2}\left(x-A\left(a,b\right)\right)g'\left(x\right)+\frac{1}{2}\left[\left(1-h\right)g\left(x\right)+\left(1+h\right)\frac{g\left(a\right)+g\left(b\right)}{2}\right]\\&-\frac{1}{b-a}\int_{a}^{b}g\left(t\right)dt+\frac{h}{8}\left(b-a\right)\left[g'\left(a\right)-g'\left(b\right)\right]\Bigg|\\&\leq\left\|g''\right\|_{\infty}\left[\frac{1}{3\left(b-a\right)}\left|x-A(a,b)\right|^{3}+\frac{\left(b-a\right)^{2}}{48}(1-3h+6h^{2}-2h^{3})\right].\end{aligned}\tag{12}$$

We may now apply (12) to deduce some inequalities for special means given above, by the use of some particular mappings as follows.

1. Consider $g\left(x\right)=\ln x, x\in\left[a,b\right]\subset\left(0,\infty\right)$. Then

$$\frac{1}{b-a}\int_{a}^{b}g\left(t\right)dt=\ln I\left(a,b\right),\quad\frac{g\left(a\right)+g\left(b\right)}{2}=\ln G\left(a,b\right)$$

and

$$\left\|g''\right\|_{\infty}=\sup_{t\in(a,b)}\left|g''\left(t\right)\right|=\frac{1}{a^{2}}.$$

From (12) we have that

$$\begin{aligned}&\Bigg|\left(h-1\right)\left(1-\frac{A\left(a,b\right)}{x}\right)+\left(1-h\right)\ln x+\left(1+h\right)\ln G\left(a,b\right)\\&-2\ln I\left(a,b\right)+\frac{h}{4}\frac{\left(b-a\right)^{2}}{ab}\Bigg|\end{aligned}$$

$$\leq \frac{2}{a^2}\left[\frac{1}{3(b-a)}|x-A(a,b)|^3+\frac{(b-a)^2}{48}(1-3h+6h^2-2h^3)\right]$$

for all $a+h((b-a)/2) \leq x \leq b-h((b-a)/2)$ and $h \in [0,1]$.

When $x=\frac{a+b}{2}=A(a,b)$, we have

$$\left|(1-h)\ln A(a,b)+(1+h)\ln G(a,b)-2\ln I(a,b)+\frac{h}{4}\frac{(b-a)^2}{ab}\right|$$
$$\leq \frac{2}{a^2}\left[\frac{(b-a)^2}{48}(1-3h+6h^2-2h^3)\right]$$

for all $h \in [0,1]$.

2. Consider $g(x)=\frac{1}{x}, x \in [a,b] \subset (0,\infty)$, then

$$\frac{1}{b-a}\int_a^b g(t)\,dt = L^{-1}(a,b), \quad \frac{g(a)+g(b)}{2}=H^{-1}(a,b)$$

and

$$\|g''\|_\infty = \sup_{t\in(a,b)} |g''(t)| = \frac{2}{a^3}.$$

From (12) we have that

$$\left|\frac{1-h}{x}\left(1-\frac{A(a,b)}{2x}\right)+\frac{1+h}{2}H^{-1}(a,b)-L^{-1}(a,b)\right.$$
$$\left.+\frac{h}{8}(b-a)\left[g'(a)-g'(b)\right]\right|$$
$$\leq \frac{2}{a^3}\left[\frac{1}{3(b-a)}|x-A(a,b)|^3+\frac{(b-a)^2}{48}(1-3h+6h^2-2h^3)\right]$$

for all $a+h((b-a)/2) \leq x \leq b-h((b-a)/2)$ and $h \in [0,1]$.

Choosing $x=\frac{a+b}{2}=A(a,b)$, we have

$$\left|\frac{1-h}{2A(a,b)}+\frac{1+h}{2}H^{-1}(a,b)-L^{-1}(a,b)+\frac{h}{8}(b-a)\left[g'(a)-g'(b)\right]\right|$$
$$\leq \frac{2}{a^3}\left[\frac{(b-a)^2}{48}(1-3h+6h^2-2h^3)\right]$$

for all $h \in [0,1]$.

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