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DOMINATION IN GRAPHS WITH MINIMUM DEGREE SIX

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ABSTRACT. A set D of vertices of a graph $G = (V(G), E(G))$ is called a dominating set if every vertex of $V(G) - D$ is adjacent to at least one element of D. The domination number of G, denoted by $\gamma(G)$, is the size of its smallest dominating set. Haynes et al.[5] present a conjecture: For any graph G with $\delta(G) \geq k$, $\gamma(G) \leq \frac{k}{3k-1}n$. When $k \neq 6$, the conjecture was proved in [7], [8], [10], [12] and [13] respectively. In this paper we prove that every graph G on n vertices with $\delta(G) \geq 6$ has a dominating set of order at most $\frac{6}{17}n$. Thus the conjecture was completely proved.

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1. Introduction

The graphs considered here are finite, undirected, and simple. The set of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The minimum degree of graph G is denoted by $\delta(G)$. A set D of vertices of a graph G is called a dominating set if every vertex of $V(G) - D$ is adjacent to at least one element of D. The domination number of G, denoted by $\gamma(G)$, is the size of its smallest dominating set. It has been proved [4] that the decision problem corresponding to the domination number for arbitrary graphs is NP-complete. Thus, the exploration of lower and upper bounds for the domination number as sharp as possible is of great significance. In fact, many results on upper bounds on the domination number in terms of some basic parameters such as the numbers of vertices and edges, the minimum and maximum degree and so on, have been obtained. For a survey, we refer the reader to [5]. Haynes et al.[5] present a conjecture:

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Conjecture 1. *For any graph G with $\delta(G) \geq k$, $\gamma(G) \leq \dfrac{k}{3k-1}n$.*

For $\delta(G) \geq 1, 3, 4, 5$, Conjecture 1 was partially proved in [8], [10], [12] and [13] respectively. For $\delta(G) \geq 2$, McCraig and Shepherd [7] proved that $\gamma(G) \leq \dfrac{2n}{5}$ except for seven graphs. For $\delta(G) \geq 7$, Caro and Roditty (see [1], [2]) gave the following better bound. For any graph G,

$$\gamma(G) \leq n\left[1 - \delta\left(\frac{1}{\delta+1}\right)^{1+\frac{1}{\delta}}\right].$$

Thus, the question remains open only for graphs G with $\delta(G) \geq 6$. In this paper one shall prove the question for $\delta(G) \geq 6$. The following theorem and above results will complete the proof of Conjecture 1.

Theorem 1. *Let G be a graph of order n with $\delta(G) \geq 6$. Then*

$$\gamma(G) \leq \frac{6}{17}n.$$

The proof of Theorem 1 will be completed by choosing a dominating set D of G based on the so-called vertex disjoint paths cover, which was introduced by Reed in [10]. In this paper, for $x, y \in V(G)$, xy denotes the edge with ends x and y. If $xy \in E(G)$, we say that y is a neighbor of x or y is adjacent to x, and the set of neighbors of x is denote by $N(x)$, $d(x) = |N(x)|$ is called the degree of x. A subgraph H is said to be induced by U if $V(H) = U$ and $xy \in E(H)$ if and only if $xy \in E(G)$, $x, y \in U$. The number of vertices of the graph G is denoted by $|V(G)|$.

A vertex disjoint paths cover of G, or simply called a $vdp-cover$, is a set of vertex disjoint paths $P_1, \cdots, P_k$ such that $V(G) = V(P_1) \cup \cdots \cup V(P_k)$. A path P is called a $0-, 1-$ or $2-path$ if $|P|$ is congruent to 0, 1 or 2 mod 3, respectively. For a $vdp-cover$ S of G, let S_i $(i = 0, 1, 2)$ be the set of i-paths in S. If $P = P'xP''$, where P' is an i-path and P'' is a j-path (x is on neither P' nor P''), then we say x is an $(i,j)-vertex$ of P. Let $P \in S$ and x be an endvertex of P. We say that x is an out-endvertex if it has a neighbor which is not on P. If P is a 2-path, we say that x is a $(2,2)$-endvertex if it is not an out-endvertex and is adjacent to some $(2,2)$-vertex of P.

2. Choose a dominating set

We assume that G is a graph with order n and $\delta(G) \geq 6$. For convenience, we assume that G is connected. We first choose a $vdp-cover$ S of G such that

(1) $2|S_1| + |S_2|$ is minimized.

(2) Subject to (1), $|S_2|$ is minimized.

(3) Subject to (2), $\sum_{P_i \in S_0} |P_i|$ is minimized.

(4) Subject to (3), $\sum_{P_i \in S_1} |P_i|$ is minimized.

Let x be an out-endvertex of $P_i \in S_1 \cup S_2$ and y be a neighbor of x on some path P_j distinct from P_i. Let $P_j = P_j' y P_j''$, then we have the following assertion (for the proof, see [10], Observation 1-3).

Assertion 1. *P_j is not a 1-path. If P_j is a 0-path, then both P_j' and P_j'' are 1-paths; if P_j is a 2-path, then both P_j' and P_j'' are 2-paths.*

Having chosen the minimal *vdp – cover* $S = \{P_1, \cdots, P_k\}$, we rearrange the paths of S to obtain a new *vdp – cover* $S' = \{P_1', \cdots, P_k'\}$ such that:

(i)$P_i'(1 \leq i \leq k)$ is a Hamiltonian path in $G[P_i]$;

(ii) subject to (i), the number of out-endvertices in S' is maximized;

(iii) subject to (ii), the number of $(2,2)$-vertices in S' is maximized.

Clearly, S' is still minimal with respect to the above conditions. For convenience, we still denote the new *vdp – cover* of G by S.

If a 1-path P in S has at least one out-endvertex, then we choose an out-endvertex x of P and a vertex $y \notin P$ which is adjacent to x, we say that y is the *acceptor* for P. If a 2-path P in S has two out-endvertices, then for each of the two out-endvertices, we choose a vertex of $V(G) - V(P)$ which is adjacent to it and designate it as the acceptor corresponding to the out-endvertex. If a 2-path P in S which has precisely one out-endvertex x and $|P| \leq 14$, we choose a vertex $y \notin P$ which is adjacent to x and designate y as the acceptor for P. We call a path in S *accepting* if it contains an acceptor. Now we specify a set $A \subseteq S$ of 2-paths. Initially, let A be the set of accepting 2-paths. While there is any out-endvertex x of a path in A for which we have not chosen an acceptor, we choose a neighbor of this endvertex in $V(G) - V(P)$ and designate it as an acceptor for x. If this new acceptor is on a previously non-accepting 2-path P', then we add P' to A. We continue this process until there is an acceptor for every out-endvertex of the paths in A. In addition, for any $(2,2)$-endvertex x of any path P in A, we choose a $(2,2)$-vertex y of P which is adjacent to x and designate y as an *inacceptor* for x.

For any accepting 2-path P, we partition $P = P_1P_2P_3$ such that P_1 and P_3 are maximal 1-paths which contain neither acceptors nor inacceptors. We say that P_1 and P_3 are *tips* of P and P_2 is its *central path.* By the maximality of P_1, P_3, and by Assertion 1, if $x \in P_2$ is adjacent to an endvertex of P_2, then it is an acceptor or an inacceptor. Before the description of choosing the dominating set, one present the following assertion.

Assertion 2. *Let $P \in S$ be a 2-path with at most one out-endvertex. If $|P| \leq 14$, then all vertices of $V(P)$ except for the possible out-endvertex can be dominated by $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices.*

One will prove Assertion 2 in section 3. Now one choose a dominating set D of G in the following manner:

Step 1: For each 0-path P, we put every $(1,1)$-vertex of P in D.

Step 2: For each accepting 2-path P, we put into D every $(2,2)$-vertex of P which is in the central path of P .

Step 3: For each 1-path P with at least one out-endvertex, we choose $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices of P which dominate all of the vertices of P except for the endvertex of P which is adjacent to the acceptor of P. We put these vertices in D. For each non-accepting 2-path P with two out-endvertices, we choose $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices of P to dominate its interior vertices. We put these vertices in D. For each non-accepting 2-path P which has precisely one out-endvertex x and $|P| \leq 14$, By assertion 2, we can choose $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices of P which dominate all of the vertices of P except for the endvertex x of P which is adjacent to the acceptor of P. We put these vertices in D.

Step 4: For each 1-path P with no out-endvertex, we choose a subset of $V(P)$ which dominate $V(P)$ and put it in D. If possible, we choose a set of $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices; otherwise we choose a set of $\left\lfloor \frac{|P|}{3} \right\rfloor + 1$ vertices. For each non-accepting 2-path P with at most one out-endvertex and $|P| \geq 11$, we choose a subset of $V(P)$ which dominate $V(P)$ and put it in D. If possible, we choose a set of $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices, otherwise we choose a set of $\left\lceil \frac{|P|}{3} \right\rceil$ vertices.

Step 5: For each tip P_1 of an accepting 2-path P, if the common endvertex x of P_1 and P is adjacent to a vertex chosen in step 1 or 2, we choose $\left\lfloor \frac{|P_1|}{3} \right\rfloor$ of vertices of P_1 which dominate the remaining vertices of P_1 and put them in D. If x is not adjacent to a vertex chosen in step 1 or 2, we choose a set which dominates P_1 and put it in D. If possible, we choose $\left\lfloor \frac{|P_1|}{3} \right\rfloor$ vertices, otherwise we choose $\left\lfloor \frac{|P_1|}{3} \right\rfloor + 1$ vertices.

It is easy to see that D is a dominating set of G (see [10], Observation 5-8). To calculate the size of D, we define the following sets.

(i) O_1: the set of 1-paths P which either have at least one out-endvertex or contain a dominating set of size $\left\lfloor \frac{|P|}{3} \right\rfloor$.

(ii) O_2: the set of non-accepting 2-paths P which have two out-endvertices or contain a dominating set with size $\left\lfloor \frac{|P|}{3} \right\rfloor$ that dominates all the vertices of P, and all non-accepting 2-paths which have precisely one out-endvertex and $|P| \leq 14$.

(iii) I_1: the set of 1-paths not in O_1.

(iv) I_2: the set of non-accepting 2-paths not in O_2.

(v) E: the tip T of an accepting 2-path P is in E if and only if the corresponding endvertex of P is neither an out-endvertex nor a $(2,2)$-endvertex and we cannot dominate T by using $\left\lfloor \frac{|T|}{3} \right\rfloor$ vertices.

(vi) W: the set of $(2,2)$-endvertices of accepting 2-paths for which we have chosen an inacceptor.

Then, the size of D can be calculated easily as

$$|D| = \sum_{P \in O_1} \frac{|P|-1}{3} + \sum_{P \in O_2} \frac{|P|-2}{3} + \sum_{P \in I_1} \frac{|P|+2}{3} + \sum_{P \in I_2} \frac{|P|+1}{3} + \sum_{P \in S_0} \frac{|P|}{3} + \sum_{P \in A} \frac{|P|-2}{3} + |E|.$$

Equivalently,

$$|D| = \frac{n}{3} - \frac{1}{3}|O_1| - \frac{2}{3}|O_2| + \frac{2}{3}|I_1| + \frac{1}{3}|I_2| - \frac{2}{3}|A| + |E|.$$

Note that each accepting 2-path corresponds to an endvertex of some path in $O_1 \cup O_2$ or to an endvertex of an accepting 2-path of A which is not in $E \cup W$. Thus, we have $|A| \le |O_1| + 2|O_2| + 2|A| - |E| - |W|$, so $|E| \le |O_1| + 2|O_2| + |A| - |W|$. Also, $|E| \le 2|A| - |W|$. Thus,

$$|D| \le \frac{n}{3} + \frac{2}{3}|I_1| + \frac{1}{3}|I_2| + \frac{|E|}{2} - \frac{|W|}{2}.$$

To any element T of E there corresponds an accepting 2-path P_T such that T is a tip of P_T. Now we define a set E', $E' \subseteq E$ by saying that each $T \in E$ is in E' if the endvertex of P_T not in T is not an element of W.

Clearly, $|E'| \ge |E| - |W|$, and so

$$|D| \le \frac{n}{3} + \frac{2}{3}|I_1| + \frac{1}{3}|I_2| + \frac{1}{2}|E'| \quad (*)$$

3. Proof of Theorem 1

The proof will be completed by a sequence of two lemmas and four assertions. The following three observations are straightforward to verify.

Observation 1. *Let $P = x_1x_2 \cdots x_{3k+1}$ $(k \ge 1)$ be a path. If x_1 is adjacent to a vertex x_{3i} for some $1 \le i \le k$, then $V(P)$ can be dominated by k vertices.*

Observation 2. *Let C be a circle with $3k+1$ $(k \ge 1)$ vertices and let $L = x_1x_2x_3$ be a path such that $V(C) \cap V(L) = \emptyset$. If x_2 has a neighbor in C, then $V(C) \cup V(L)$ can be dominated by $k+1$ vertices.*

Observation 3. *Let $P = x_1x_2\cdots x_{3k-1}$ $(k \geq 1)$ be a path and let $x \notin P$. If x is adjacent to one vertex of $\bigcup_{i=1}^{k}\{x_{3i-2}, x_{3i-1}\}$, then $V(P) \cup \{x\}$ can be dominated by k vertices.*

From the above three observations, one shall prove Lemma 1 and Lemma 2.

Lemma 1. *Let $C = x_1x_2\cdots x_{3k+1}x_1 (2 \leq k \leq 7)$ be a circle of G, H be a subgraph of G induced by $V(C)$. For $v \in V(C)$, if in H there is a Hamiltonian path between v and x_{3k+1}, we have $N(v) \subseteq V(C)$, then H can be dominated by k vertices.*

Proof. If $k = 2$, the conclusion is obvious. Here we only prove the case for $k = 7$, the cases for $3 \leq k \leq 6$ can be proved by similar reasoning and omitted.

When $k = 7$, then $C = x_1x_2\cdots x_{22}x_1$. Let $C^+ = x_1x_2\cdots x_{22}$, for $1 \leq i < j \leq 22$, let $x_iC^+x_j$ (or $x_jC^-x_i$) denotes the path between x_i and x_j of C^+ (both x_i and x_j are contained). We prove by contradiction, assume H can not be dominated by k vertices. We first check the neighbors of x_1, then deduce a contradiction.

Obviously, there are Hamiltonian paths between x_1 and x_{22}, x_{21} and x_{22} in H, so we have $N(x_1) \subseteq V(C)$, $N(x_{21}) \subseteq V(C)$, by Observation 1, x_1 is not adjacent to $x_{3i}(1 \leq i \leq 7)$, by symmetry, x_{21} is not adjacent to $x_{3i+1}(0 \leq i \leq 6)$. In the following, we check the neighbors of x_1.

Case 1. x_1 is not adjacent to x_{10}.

We prove by contradiction. Assume x_1 is adjacent to x_{10}, now we check the neighbors of x_{21}, then deduce a contradiction.

For $x_{20}x_{21}x_{22}$ and the circle $x_1C^+x_{10}x_1$, by Observation 2, x_{21} is not adjacent to the circle. So,

$$N(x_{21}) - \{x_{20}, x_{22}\} \subseteq \{x_{11}, x_{12}, x_{14}, x_{15}, x_{17}, x_{18}\}.$$

Since $d(x_{21}) \geq 6$, then x_{21} must be adjacent to both x_{11} and x_{12}, or both x_{14} and x_{15}, or both x_{17} and x_{18}.

Firstly, if x_{21} is adjacent to both x_{11} and x_{12}, then there is a Hamiltonian path $x_{13}C^+x_{21}x_{12}C^-x_1x_{22}$, so $N(x_{13}) \subseteq V(C)$, for $x_{12}x_{13}x_{14}$ and the circle $x_1C^+x_{11}x_{21}$
$x_{22}x_1$, by Observation 2, x_{13} is not adjacent to the circle. By Observation 1, x_{13} is not adjacent to x_{15} or x_{18}. So $N(x_{13}) \subseteq \{x_{16}, x_{17}, x_{19}, x_{20}\}$. Since $d(x_{13}) \geq 6$, then x_{13} must adjacent to all vertices of $\{x_{16}, x_{17}, x_{19}, x_{20}\}$. Now there is a Hamiltonian path

$$x_{15}x_{14}x_{13}x_{16}C^+x_{21}x_{12}C^-x_1x_{22},$$

so, $N(x_{15}) \subseteq V(C)$. For $x_{14}x_{15}x_{16}$ and the circle $x_1C^+x_{13}x_{17}C^+x_{22}x_1$, by Observation 2, x_{15} is not adjacent to the circle, this means $d(x_{15}) \leq 5$, a contradiction. So x_{21} is at most adjacent to one of $\{x_{11}, x_{12}\}$.

Secondly, by similar reasoning, x_{21} is at most adjacent to one of $\{x_{14}, x_{15}\}$, or at most one of $\{x_{17}, x_{18}\}$. This means $d(x_{21}) \leq 5$, a contradiction. So x_1 is not adjacent to x_{10}. This proves Case 1.

By similar reasoning, x_1 is not adjacent to the vertex of $\{x_{13}, x_{16}, x_{19}\}$.

Case 2. x_1 is not adjacent to x_{11}.

We prove by contradiction. Assume x_1 is adjacent to x_{11}, now x_1 dominates x_2, x_{22} and x_{11}, for x_{21} and the path $x_3C^+x_{10}$, by Observation 3, we have

$$N(x_{21}) - \{x_{20}, x_{22}\} \subseteq \{x_2, x_5, x_8, x_{11}, x_{12}, x_{14}, x_{15}, x_{17}, x_{18}\}.$$

Case 2.1. x_{21} is not adjacent to x_2.

Assume x_{21} is adjacent to x_2, now x_{21} dominates x_{20}, x_{22} and x_2, for x_1 and the path $x_3C^+x_{19}$, by Observation 3, we have $N(x_1) \subseteq \{x_5, x_8, x_{11}, x_{14}, x_{17}, x_{20}\}$.

Firstly, if x_1 is adjacent to x_5, there is a Hamiltonian path $x_3C^+x_{21}x_2x_1x_{22}$, so $N(x_3) \subseteq V(C)$. For $x_2x_3x_4$ and the circle $x_5C^+x_{22}x_1x_5$, by Observation 2, x_3 has no neighbor in $x_5C^+x_{22}x_1x_5$, a contradiction to $d(x_3) \geq 6$. So x_1 is not adjacent to x_5, by similar reasoning, x_1 is not adjacent to x_8. Then x_1 must be adjacent to all the vertices of $\{x_{11}, x_{14}, x_{17}, x_{20}\}$.

Secondly, x_1 is adjacent to all the vertices of $\{x_{11}, x_{14}, x_{17}, x_{20}\}$.

Now there is a Hamiltonian path $x_{19}C^-x_1x_{20}x_{21}x_{22}$, so $N(x_{19}) \subseteq V(C)$. For $x_{18}x_{19}x_{20}$ and the circle $x_2C^+x_{17}x_1x_{22}x_{21}x_2$, by Observation 2, x_{19} has no neighbor in the circle, a contradiction to $d(x_{19}) \geq 6$, so x_1 is not adjacent to all the vertices of $\{x_{11}, x_{14}, x_{17}, x_{20}\}$. This means $d(x_1) \leq 6$, a contradiction. This proves Case 2.1. by similar reasoning, x_{21} is not adjacent to x_5 or x_8. So

$$N(x_{21}) - \{x_{20}, x_{22}\} \subseteq \{x_{11}, x_{12}, x_{14}, x_{15}, x_{17}, x_{18}\}.$$

Case 2.2. x_{21} is at most adjacent to one of $\{x_{11}, x_{12}\}$.

Assume x_{21} is adjacent to both x_{11} and x_{12}, there is a Hamiltonian path $(x_{13}C^+x_{21}$
$x_{12}C^-x_1x_{22}$, so $N(x_{13}) \subseteq V(C)$. For $x_{12}x_{13}x_{14}$ and the circle $x_1C^+x_{11}x_{21}x_{22}x_1$, by Observation 2, x_{13} has no neighbor in the circle, and by Observation 1, x_{13} must be adjacent to all the vertices of $\{x_{16}, x_{17}, x_{19}, x_{20}\}$, then there is a Hamiltonian path

$$x_{15}x_{14}x_{13}x_{16}C^+x_{21}x_{12}C^-x_1x_{22},$$

so $N(x_{15}) \subseteq V(C)$. For $x_{14}x_{15}x_{16}$ and the circle $x_1C^+x_{13}x_{17}C^+x_{22}x_1$, by Observation 2, x_{15} has no neighbor in the circle, a contradiction to $d(x_{15}) \geq 6$. This proves Case 2.2. By similar reasoning, x_{21} is at most adjacent to one of $\{x_{14}, x_{15}\}$ or at most one of $\{x_{17}, x_{18}\}$.

From the Case 2.1 and Case 2.2, we have $d(x_{21}) \leq 5$, a contradiction. So x_1 is not adjacent to x_{11}. This proves Case 2. By similar reasoning as Case 2, x_1 is not adjacent to the vertex of $\{x_{14}, x_{17}, x_{20}\}$.

Case 3. x_1 is at most adjacent to one of $\{x_4, x_5\}$.

Assume x_1 is adjacent to both x_4 and x_5, then there is a Hamiltonian path $x_3x_2x_1x_4C^+x_{22}$, so $N(x_3) \subseteq V(C)$. For $x_2x_3x_4$ and the circle $x_5C^+x_{22}x_1x_5$, by Observation 2, x_3 has no neighbor in the circle, a contradiction to $d(x_3) \geq 6$. This proves Case 3. By similar reasoning, x_1 is at most adjacent to one of $\{x_7, x_8\}$.

From the above three cases we have $d(x_1) \leq 5$, a contradiction. This proves Lemma 1. □

Lemma 2. *Let $C = x_1x_2\cdots x_{3k+2}x_1$ $(2 \leq k \leq 7)$ be a circle of G, H be a subgraph of G induced by $V(C)$. For $v \in V(C)$, if in H there is a Hamiltonian path between v and x_{3k+2}, we have $N(v) \subseteq V(C)$, then $V(C) - \{x_{3k+2}\}$ can be dominated by k vertices.*

Proof. When $k = 2$, the conclusion is obvious. We only prove the conclusion for $k = 7$. The proof of other cases is similar and thus omitted.

When $k = 7$, $C = x_1x_2\cdots x_{23}x_1$. Similarly, let $C^+ = x_1x_2\cdots x_{23}$. For $1 \leq i < j \leq 23$, $x_iC^+x_j$ or $x_jC^-x_i$ denotes the path between x_i and x_j of C^+(x_i and x_j are contained).

We prove by contradiction. Assume $V(C) - \{x_{3k+2}\}$ can not be dominated by k vertices. First we check the neighbors of x_1, then deduce a contradiction.

Since there are Hamiltonian paths between x_1 and x_{23}, x_{22} and x_{23}, we have $N(x_1) \subseteq V(C)$ and $N(x_{22}) \subseteq V(C)$. Noted that x_1 and x_{22} are symmetrical about x_{23} on C, so the properties about the neighbors of x_1 are the same as the neighbors of x_{22}. By Observation 1, x_1 is not adjacent to $x_{3k}(1 \leq k \leq 7)$, and symmetrically, x_{22} is not adjacent to $x_{3k-1}(1 \leq k \leq 7)$.

Case 1. x_1 is not adjacent to x_{22}.

Assume x_1 is adjacent to x_{22}, now in H there are Hamiltonian paths between x_2 and x_{23}, x_{21} and x_{23}, this is similar to the Lemma 1, so by the similar proof of Case 1-2 of Lemma 1 we get

$$N(x_1) - \{x_2, x_{23}, x_{22}\} \subseteq \{x_4, x_5, x_7, x_8\},$$

similar to the proof of Case 3 of Lemma 1, we get x_1 is at most adjacent to one of $\{x_4, x_5\}$, or at most adjacent to one of $\{x_7, x_8\}$. This means $d(x_1) \leq 5$, a contradiction.

By similar reasoning as the proof of Lemma l we have x_1 is not adjacent to the vertex of $\{x_{13}, x_{16}, x_{19}\}$. By symmetry, x_{22} is not adjacent to the vertex of $\{x_1, x_4, x_7, x_{10}\}$.

Case 2. x_1 is not adjacent to x_{20}.

Assume x_1 is adjacent to x_{20}, then x_{20} dominates $\{x_1, x_{19}, x_{21}\}$, for x_{22} and the path $x_2C^+x_{18}$, by Observation 3,

$$N(x_{22}) - \{x_{21}, x_{23}\} \subseteq \{x_{13}, x_{16}, x_{19}\}.$$

This is a contradiction to $d_{22} \geq 6$. This proves Case 2.

By the same reason as Case 2, x_1 is not adjacent to x_{14} or x_{17}. By symmetry,

$$N(x_{22}) - \{x_{21}, x_{23}\} \subseteq \{x_{12}, x_{13}, x_{15}, x_{16}, x_{18}, x_{19}\}.$$

Case 3. x_1 is at most adjacent to one of $\{x_{10}, x_{11}\}$.

Assume x_1 is adjacent to both x_{10} and x_{11}, now we check the neighbors of x_{22}.

Case 3.1. x_{22} is at most adjacent to one of $\{x_{12}, x_{13}\}$.

If x_{22} is adjacent to both x_{12} and x_{13}, then there is a Hamiltonian path $x_{21}C^-x_{13}x_{22}x_{12}C^-x_1x_{23}$, so $N(x_{21}) \subseteq V(C)$. For $x_{20}x_{21}x_{22}$ and the circle $x_1C^+x_{10}x_1$, by Observation 2, x_{21} is not adjacent to the circle. And by Observation 1, we have

$$N(x_{21}) - \{x_{20}, x_{22}\} \subseteq \{x_{12}, x_{14}, x_{15}, x_{17}, x_{18}, x_{23}\}.$$

Now we check the neighbors of x_{21}.

Case 3.1.1. x_{21} is at most adjacent to one vertex of $\{x_{12}, x_{23}\}$.

Assume x_{21} is adjacent to both x_{12} and x_{23}, there is a Hamiltonian path $x_{14}C^+x_{22}$
$x_{13}C^-x_1x_{23}$, for $x_{13}x_{14}x_{15}$ and the circle $x_1C^+x_{10}x_1$, by Observation 2, x_{14} is not adjacent to the circle. By Observation 1,

$$N(x_{14}) - \{x_{13}, x_{15}\} \subseteq \{x_{12}, x_{17}, x_{18}, x_{20}, x_{21}, x_{23}\}.$$

Firstly, if x_{14} is adjacent to x_{12}, then there are 7 vertices $\{x_2, x_5, x_8, x_{11}, x_{12}, x_{16}, x_{19}\}$ dominate $V(C)-\{x_{23}\}$, this is contrary to the supposition that $V(C)-\{x_{23}\}$ can not be dominated by 7 vertices. So x_{14} is not adjacent to x_{12}.

Secondly, if x_{14} is adjacent to x_{20}, for $x_{13}x_{22}x_{21}$ and the circle $x_{14}C^+x_{20}x_{14}$, by Observation 2, x_{22} is not adjacent to the circle, then

$$N(x_{22}) - \{x_{21}, x_{23}\} \subseteq \{x_{12}, x_{13}\},$$

a contradiction, so x_{14} is not adjacent to x_{20}.

Similarly, x_{14} is not adjacent to x_{17}.

This means $d(x_{14}) \leq 5$, a contradiction. So x_{21} is at most adjacent to one vertex of $\{x_{12}, x_{23}\}$. This proves Case 3.1.1.

Case 3.1.2. x_{21} is at most adjacent to one vertex of $\{x_{17}, x_{18}\}$.

Assume x_{21} is adjacent to both x_{17} and x_{18}, then there is a Hamiltonian path

$$x_{19}x_{20}x_{21}x_{18}C^-x_{13}x_{22}x_{12}C^-x_1x_{23},$$

for $x_{18}x_{19}x_{20}$ and the circle $x_1C^+x_{10}x_1$ or the circle $x_{13}C^+x_{17}x_{21}x_{22}x_{13}$, by Observation 2, x_{19} is not adjacent to the circles, this means $d(x_{19}) \leq 5$, a contradiction to $d(x_{19}) \geq 6$. So x_{21} is at most adjacent to one vertex of $\{x_{17}, x_{18}\}$.

Similarly, x_{21} is at most adjacent to one vertex of $\{x_{14}, x_{15}\}$.

From the two cases 3.1.1-3.1.2, we get $d(x_{21}) \leq 5$, a contradiction. So x_{22} is at most adjacent to one of $\{x_{12}, x_{13}\}$. This proves Case 3.1.

By similar reasoning, we have x_{22} is at most adjacent to one vertex of $\{x_{15}, x_{16}\}$, or one of $\{x_{18}, x_{19}\}$. This means $d(x_{22}) \leq 5$, a contradiction. So x_1 is at most adjacent to one of $\{x_{10}, x_{11}\}$. This proves Case 3.

By similar reasoning as Case 3, x_1 is at most adjacent to one of $\{x_4, x_5\}$ or one of $\{x_7, x_8\}$.

From Case 1-3, we get $d(x_1) \leq 5$, a contradiction. This proves Lemma 2. □

A lasso L is defined as a graph formed by identifying any vertex in a circle C with an endvertex of a path P. The other endvertex of the path P is called the end of L and the common vertex of C and P is called the connecting vertex of L. Especially, a cycle can be regarded as a lasso.

Assertion 3. *Let $P \in S$ be a 2-path with at most one out-endvertex. If $|P| \leq 14$, then all vertices of $V(P)$ except for the possible out-endvertex can be dominated by $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices.*

Proof. If $|V(G)| = 14$, then the conclusion is immediate. We may assume that $|V(G)| > 14$. Let $P = x_1x_2\cdots x_{3m+2}$ $(2 \leq m \leq 4)$ be a 2-path in S with at most one out-endvertex. Let H be a subgraph of G induced by $V(P)$. Since $\delta \geq 6$, when $|P| = 8, 11$, the conclusion is obvious, so in the following we prove only $|P| = 14$, i.e., $P = x_1x_2\cdots x_{14}$.

Case 1. P has no out-endvertex.

As G is connected, there is at least one edge between $V(P)$ and $V(G)-V(P)$. If there is a Hamiltonian circle of H, then each vertex of H is an out-endvertex of some Hamiltonian path, a contradiction. So there has no Hamiltonian circle in H.

Now we choose a lasso L in H such that the number of vertices on the circle of the lasso is maximum. For convenience, we label the vertices of L along a Hamiltonian path on L from the end of L as $x_{14}, x_{13}, \cdots, x_2, x_1$. Since x_1 and x_{14} are not out-endvertices, so x_1 and x_{14} are only adjacent to the vertices of P. As there has no Hamiltonian circle in H, x_1 is not adjacent to x_{14}, and $d(x_1) \geq 6$, let u be the connecting vertex of the lasso, we have

$$u \in \{x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},$$

by the labeling, x_1 is adjacent to u.

In the following, we prove only $u = x_{13}$ because the proof of the other can be done in a similar way.

Case 1.1. When $u = x_{13}$.

We prove by contradiction, assume $V(P)$ can not be dominated by 4 vertices. We check the neighbors of x_1.

Case 1.1.1. x_1 is not adjacent to x_6.

Firstly, if x_1 is adjacent to x_6, we check the neighbors of x_{14}. Now x_5 is an endvertex of a Hamiltonian path $x_5C^-x_1x_6C^+x_{13}x_{14}$, so by the choice of the circle of L, x_{14} is not adjacent to x_5.

Secondly, as x_1 dominates $\{x_2, x_6, x_{13}\}$, if x_{14} is adjacent to one of $\{x_4, x_8, x_{11}\}$, then $\{x_1, x_4, x_8, x_{11}\}$ dominate $V(P)$, this is contrary to the supposition that $V(P)$ can not be dominated by 4 vertices.

Finally, since x_6 dominate $\{x_1, x_5, x_7\}$, if x_{14} is adjacent to one vertex of $\{x_3, x_6, x_9, x_{12}\}$, then $\{x_3, x_6, x_9, x_{12}\}$ dominate $V(P)$, a contradiction.

So $N(x_{14}) \subseteq \{x_2, x_7, x_{10}, x_{13}\}$, this means $d(x_{14}) \leq 4$, a contradiction to $d(x_{14}) \geq 6$. So x_1 is not adjacent to x_6. By similar reasoning, x_1 is not adjacent to the vertex of $\{x_3, x_9, x_{12}\}$. So x_1 is adjacent to at least four vertices of $\{x_4, x_5, x_7, x_8, x_{10}, x_{11}\}$.

Case 1.1.2. x_1 is at most adjacent to one of $\{x_4, x_5\}$.

Now there is a Hamiltonian path $x_2C^+x_4x_1x_5C^+x_{13}x_{14}$, so $N(x_2) \subseteq V(P)$, and by the choice of the circle of L, we have x_2 is not adjacent to x_{14}. Since x_4 dominates $\{x_1, x_3, x_5\}$, for x_2 and the path $x_6C^+x_{13}$, by Observation 3, we have

$$N(x_2) - \{x_1, x_3\} \subseteq \{x_5, x_8, x_{11}\},$$

this means $d(x_2) \leq 5$, a contradiction.

By similar reasoning, x_1 is at most adjacent to one of $\{x_7, x_8\}$, or one of $\{x_{10}, x_{11}\}$.

From Cases 1.1.1-1.1.2, we get $d(x_1) \leq 5$, a contradiction. This completes Case 1.

Case 2. P has precisely one out-endvertex.

When P has precisely one out-endvertex, assume x_{14} is an out-endvertex of P, as x_1 is not an out-endvertex, x_1 is only adjacent to the vertices of P.

Similarly as Case 1, we choose a lasso L in H such that the vertices on the circle of the lasso is maximum, let C be the circle of the lasso, and x_{14} be an out-endvertex of the lasso. For convenience, we label the vertices of L along a Hamiltonian path on L from the end of L as $x_{14}, \cdots, x_1$. Let u be the connecting vertex. By the labeling, x_1 is adjacent to u.

We prove by contradiction. By proposition 1, x_1 is not adjacent to $x_{3i}(1 \leq i \leq 4)$, so $u = x_{3k+1}$ or x_{3k+2} $(2 \leq k \leq 4)$. Assume $x_l \in V(C)$, Noted that if there is a path from x_l to u that all the vertices of C are on the path, then there is a Hamiltonian path from x_l to x_{14}, thus $N(x_l) \subseteq V(C)$, i.e, the vertex of C satisfy the conditions of Lemma 1 or Lemma 2, so by Lemma 1 or Lemma 2, we have the conclusion. □

Assertion 4. *Let $T \in E'$ be a tip of a 2-path P in A. If $|T| \leq 22$, then T can be dominated by $\left\lfloor \frac{|T|}{3} \right\rfloor$ vertices.*

Proof. We prove by contradiction, assume T can not be dominated by $\left\lfloor \frac{|T|}{3} \right\rfloor$ vertices, then deduce a contradiction.

Let $T = a_0 \cdots a_k \in E'$ be a tip of 2-path P, $C = c_0 \cdots c_l$ be a central path of P. Assume c_0 is adjacent to a_k on the path P, by definition, c_1 is an acceptor or inacceptor. As $T \in E'$, there is not $(2,2)$-endvertex in P, so c_1 is an acceptor. We first present a Claim proved by Reed (for the proof, see [10] p285, Fact 11).

Claim 1. a_0 is only adjacent to the vertex of $V(T) \cup \{c_0\}$.

If $a'_0 \cdots a'_k$ is a Hamiltonian path on $V(T)$ such that a'_k is adjacent to c_0, then by the choice of S, a'_0 also is only adjacent to the vertex of $T \cup \{c_0\}$.

As T is 1−path, then $|T| = 3m + 1 (0 \leq m \leq 6)$. Let H be a subgraph of G induced by $V(T) \cup \{c_0\}$. As a_0 is only adjacent to the vertex of $V(T) \cup \{c_0\}$, there is a lasso in H with one endvertex c_0. Now we choose a lasso L in H such that the number of vertex on the circle of the lasso is maximum, and c_0 is an endvertex of the lasso (perhaps there is a Hamiltonian circle of H). We label the vertices of L along a Hamiltonian path on L from the end of L as $c_0 x_{3m+1} \cdots x_1$, $(1 \leq m \leq 6)$. Let u be the connecting vertex, by the labeling, x_1 is adjacent to u, by Observation 1, x_1 is not adjacent to $x_{3k} (1 \leq k \leq m \leq 6)$. Since $d(x_1) \geq 6$ and x_1 is not an out-endvertex, then $u = x_{3k+1}$ or $u = x_{3k+2}$, where $2 \leq k \leq m \leq 6$. Designate the circle of L as C, similarly we can deduce that the vertices of C satisfy the conditions of Lemma 1 or Lemma 2, so by Lemma 1 or Lemma 2, we have the conclusion that $V(T)$ can be dominated by $\left\lfloor \frac{|T|}{3} \right\rfloor$ vertices, a contradiction. □

Assertion 5. *Let $P \in S$ be a 1-path with no out-endvertex. If $|P| \leq 31$, then P can be dominated by $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices.*

Proof. Here we only prove $|P| = 31$, the other cases can be proved by similar reasoning and omitted. Let H be a subgraph of G induced by $V(P)$.

Case 1. When $|V(G)| = 31$ and there has a Hamiltonian circle in G.

If $|V(G)| = 31$ and there has a Hamiltonian circle in G, $G = H$, by Lemma 1, G can be dominated by 10 vertices, as $10 \leq \frac{6n}{17} = \frac{6 \times 31}{17}$, This satisfies Theorem 1.

Case 2. When $|V(G)| \geq 31$ and there has no Hamiltonian circle in H.

If $|V(G)| > 31$ and there has a Hamiltonian circle in H, since P has no out endvertex, this is contrary to that G is connected. So in the following, we always assume $|V(G)| \geq 31$ and there has no Hamiltonian circle in H. We prove by contradiction, assume P can not be dominated by $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices.

Now we choose a lasso L in H such that the number of vertices on the circle of the lasso is maximum, x_{31} is an endvertex of the lasso. For convenience, we label the vertices of L along a Hamiltonian path on L from the end of L as $x_{31}x_{30}\cdots x_1$. Let u be the connecting vertex, by the labeling, x_1 is adjacent to u.

We prove by contradiction, assume P can not be dominated by $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices.

By Observation 1, x_1 is not adjacent to $x_{3i}, (1 \leq i \leq 10)$, by the choice of the circle of L, x_1 is not adjacent to x_{31}, since $d(x_1) \geq 6$ and x_1 is not an out-endvertex, then $u = x_{3k+1}$ or $x_{3k+2}, (2 \leq k < 10)$. Let C be the circle of the lasso. When $k \leq 7$, the vertices of C satisfy the conditions of Lemma 1 or Lemma 2, so by the Lemma 1 or Lemma 2, we have $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices dominate $V(P)$, a contradiction. Thus, assume $k \geq 8$, for convenience, we denote $C^+ = x_1x_2\cdots x_{31}$. For $1 \leq i < j \leq 31$, let $x_iC^+x_j$ (or $x_jC^-x_i$) denote the path between x_i and x_j of C^+(both x_i and x_j are contained). Here we only prove $k = 8$, i.e., $u = x_{25}$, the other cases can be similarly proved and omitted.

When $u = x_{3k+1} = x_{25}$, by the choice of the lasso, x_1 is not adjacent to the vertex of $\{x_{26}, x_{27}, \cdots, x_{31}\}$ (otherwise there is a longer circle). So

$$N(x_1) - \{x_2, x_{25}\} \subseteq \{x_4, x_5, x_7, x_8, x_{10}, x_{11}, x_{13}, x_{14}, x_{16}, x_{17}, x_{19}, x_{20}, x_{22}, x_{23}\}.$$

Now we check the neighbors of x_{31}. Since x_{31} is not an out-endvertex, x_{31} is only adjacent to the vertices of P. By the choice of the lasso, x_{31} is not adjacent to the vertex of $\{x_1, x_2, \cdots, x_6\}$, by symmetry, x_{31} is not adjacent to the vertex of $\{x_{19}, x_{20}, \cdots, x_{24}\}$, by Observation 1, x_{31} is not adjacent to the vertex of $\{x_8, x_{11}, x_{14}, x_{17}, x_{26}, x_{29}\}$. Thus,

$$N(x_{31}) - \{x_{30}\} \subseteq \{x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{18}\} \cup \{x_{25}, x_{27}, x_{28}\}.$$

Case 2.1. x_{31} is not adjacent to x_7.

If x_{31} is adjacent to x_7, now we check the neighbors of x_1. By the choice of the lasso, x_1 is not adjacent to the vertex of $\{x_8, \cdots, x_{13}\}$.

Case 2.1.1. x_1 is not adjacent to x_{22}.

If x_1 is adjacent to x_{22}, since x_{24} is an endvertex of a Hamiltonian path of H, so $N(x_{24}) \subseteq V(P)$, for $x_{23}x_{24}x_{25}$ and the circle $x_1C^+x_{22}x_1$, by Observation 2, x_{24} is not adjacent to the circle, this means $d(x_{24}) \leq 5$, a contradiction. So x_1 is not adjacent to x_{22}, similarly, x_1 is not adjacent to x_{16} or x_{19}.

Case 2.1.2. x_1 is not adjacent to x_{23}.

Now we check the neighbors of x_{24}.

Firstly, by Observation 1, x_{24} is not adjacent to x_{27}, x_{30} and $x_{3i+1}, (0 \leq i \leq 7)$.

Secondly, by the choice of the circle of L, x_{24} is not adjacent to the vertex of $x_1C^+x_6$ and $\{x_{25}, \cdots x_{31}\}$,

Finally, since x_1 dominates $\{x_2, x_{23}, x_{25}\}$, for x_{24} and the path $x_3C^+x_{22}$, by Observation 3, we have x_{24} is not adjacent x_{3i} and x_{3i+1}, $0 \leq i \leq 7$.

So $N(x_{24}) \subseteq \{x_8, x_{11}, x_{14}, x_{17}, x_{20}\}$. If x_{24} is adjacent to x_{20}, then there is a Hamiltonian path $x_{22}C^-x_1x_{23}x_{24}x_{25}\cdots x_{31}$, so $N(x_{24}) \subseteq V(H)$, for $x_{21}x_{22}x_{23}$ and the circle $x_1C^+x_{20}x_{24}x_{25}x_1$, by Observation 2, x_{22} is not adjacent to the circle, this means $d(x_{22}) \leq 5$, a contradiction. So x_{24} is not adjacent to x_{20}, similarly, x_{24} is not adjacent to x_{17}, this means $d(x_{24}) \leq 5$, a contradiction. So x_1 is not adjacent to x_{23}. similarly, x_1 is not adjacent to x_{20} or x_{17}.

Combining with Case 2.1.1, since $d(x_1) \geq 6$, then x_1 must be adjacent to the four vertices of $\{x_4, x_5, x_7, x_{14}\}$.

Case 2.1.3. x_1 is at most adjacent to three vertices of $\{x_4, x_5, x_7, x_{14}\}$.

If x_1 is adjacent to all vertices of $\{x_4, x_5, x_7, x_{14}\}$. Now there is a Hamiltonian path $x_3x_2x_1x_4C^+x_{25}\cdots x_{31}$, so $N(x_3) \subseteq V(H)$. For $x_2x_3x_4$ and the circle $x_5C^+x_{25}x_1x_5$, by Observation 2, x_3 is not adjacent to the circle, this means $d(x_3) \leq 5$, a contradiction.

So x_{31} is not adjacent to x_7, This proves Case 2.1.

Similarly, x_{31} is not adjacent to x_{10} or x_{13}. By symmetry, x_{31} is not adjacent to the vertex of $\{x_{18}, x_{15}, x_{12}\}$.

So

$$N(x_{31}) - \{x_{30}\} \subseteq \{x_9, x_{16}, x_{25}, x_{27}, x_{28}\}.$$

Since $d(x_{31}) \geq 6$, x_{31} must be adjacent to all vertices of $\{x_9, x_{16}, x_{25}, x_{27}, x_{28}\}$.

Case 2.2. x_{31} is at most adjacent to one vertex of $\{x_{27}, x_{28}\}$.

Assume x_{31} is adjacent to both x_{27} and x_{28}. Now there is a Hamiltonian path $x_{29}x_{30}x_{31}x_{28}\cdots x_{25}C^-x_1$, so $N(x_{29}) \subseteq V(H)$. Since x_{27} dominates x_{26} and x_{31}, for $x_{28}x_{29}x_{30}$ and the circle C, by Observation 2, x_{29} is not adjacent to C, this means $d(x_{29}) \leq 5$, a contradiction.

From Cases 2.1-2.1, we get $d(x_{31}) \leq 5$, a contradiction. This proves Case 2. It completes Assertion 4. □

Now by using the three assertions we deduce Theorem 1. By Assertion 2 and Assertion 4, if $P \in I_2$, then $|P| \geq 17$. If $P \in I_1$, then $|P| \geq 34$. Hence

$$\sum_{P \in I_1} |P| \geq 34|I_1|;$$

$$\sum_{P \in I_2} |P| \geq 17|I_2|.$$

By Assertion 3,

$$\sum_{P \in A} |P| \geq 26|E'|.$$

So we have $n \geq \sum_{P \in I_1} |P| + \sum_{P \in I_2} |P| + \sum_{P \in A} |P| \geq 34|I_1| + 17|I_2| + 26|E'|$,

i.e., $\frac{n}{51} \geq \frac{2}{3}|I_1| + \frac{1}{3}|I_2| + \frac{26}{51}|E'|$. Combining with (*), we have $|D| \leq \frac{6}{17}n$. This completes Theorem 1.

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