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CONDENSED CRAMER RULE FOR COMPUTING A KIND OF RESTRICTED MATRIX EQUATION

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ABSTRACT. The problem of finding Cramer rule for solutions of some restricted linear equation $Ax = b$ has been widely discussed. Recently Wang and Qiao consider the following more general problem

$$AXB = D, \quad R(X) \subset T, \ N(X) \supset \tilde{S}.$$

They present the solution of above general restricted matrix equation by using generalized inverses and give an explicit expression for the elements of the solution matrix for the matrix equation. In this paper we re-consider the restricted matrix equation and give an equivalent matrix equation to it. Through the equivalent matrix equation, we derive condensed Cramer rule for above restricted matrix equation. As an application, condensed determinantal expressions for $A^{(2)}_{T,S}A$ and $AA^{(2)}_{T,S}$ are established. Based on above results, we present a method for computing the solution of a kind of restricted matrix equation.

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1. Introduction

Let $\mathbb{C}^n$ denote the n-dimensional complex vector space, $\mathbb{C}^{m\times n}$ denote the set of all complex $m\times n$ matrices, $\mathbb{C}^{m\times n}_r = \{X \in \mathbb{C}^{m\times n} : \operatorname{rank}(X) = r\}$, and $\dim(L)$ denote the dimension of a subspace L of $\mathbb{C}^n$. The symbols A^T, A^*, $R(A)$, $N(A)$, $\operatorname{rank}(A)$ and $\det(A)$ denote, respectively, the transpose, the conjugate transpose, the range, the null space, the rank and the determinant of A. For $A \in \mathbb{C}^{m\times n}$, $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$, let $A(i \to x)$ stand for the matrix obtained by replacing ith column of A by x, and $A(y^T \leftarrow j)$ denote the matrix obtained by replacing jth row of A by y^T.

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Let $A \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{n\times m}$ such that

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA.$$

Then X is called the Moore-Penrose inverse of A, and denoted by $X = A^\dagger$.

For $A \in \mathbb{C}^{n\times n}$, the smallest nonnegative integer k such that $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$ is called the index of A, and denoted by $k = \mathrm{Ind}(A)$.

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, and $X \in \mathbb{C}^{n\times n}$ such that

$$A^{k+1}X = A^k,\ XAX = X,\ AX = XA.$$

Then X is called the Drazin inverse of A, and denoted by $X = A_d$. In particular, when $\mathrm{Ind}(A) = 1$, the matrix X satisfying above matrix equations is called the group inverse of A, and denoted by $X = A_g$.

If A is nonsingular then, for any b, the solution of the nonsingular linear equation

$$Ax = b$$

is given by the classical Cramer rule (for an elegant proof see [3]). A number of authors have extended the Cramer rule to general A and b. Since 1982, the trick of Robinson has been used to derive a series of Cramer rules [4-8] for the minimum-norm solution $A^\dagger b$ of a consistent linear equation

$$Ax = b, \quad b \in R(A),$$

where $A^\dagger$ is the Moore-Penrose inverse of A; for the minimum-norm(M) and least-squares(N) solution $A^\dagger_{M,N}b$ of inconsistent linear equation

$$Ax = b,$$

where $A^\dagger_{M,N}$ is the weighted Moore-Penrose inverse of A, M and N are Hermitian positive definite matrices [1, 2]; for the Drazin inverse solution $A_d b$ of a class of singular equation

$$Ax = b, \quad x \in R(A^k),\ b \in R(A^k),$$

where $k = \mathrm{Ind}(A)$; for the unique solution $A^{(2)}_{T,S}b$ of a general restricted linear equation

$$Ax = b, \quad x \in T, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times n}_r$, $b \in AT$ and $T \cap N(A) = \{0\}$, $AT \oplus S = \mathbb{C}^m$.

Recently, Wang and Qiao consider the following more general problem [10]. Given $A \in \mathbb{C}^{m\times n}_r$, $B \in \mathbb{C}^{p\times q}_{\tilde{r}}$, $D \in \mathbb{C}^{m\times q}$, solve X in the restricted matrix equation

$$AXB = D, \quad R(X) \subset T,\ N(X) \supset \tilde{S} \tag{1.2}$$

for the predetermined subspaces $T \subset \mathbb{C}^n$ and $\tilde{S} \subset \mathbb{C}^p$.

If we define the range and null space of a pair of matrices A and B as sets of matrices:

$$R(A,B) = \{\, Y : Y = AXB \ \text{ for some } X \,\}$$

and

$$N(A,B) = \{\, X : AXB = 0 \,\},$$

then the restricted matrix equation (1.2) has a solution if $D \in R(A, B)$. Wang and Qiao [10] present the solution of above general restricted matrix equation with some conditions by using generalized inverses and give an explicit expression for the elements of the solution by general Cramer rule. Their basic idea is to construct a nonsingular bordered matrix from the original matrix by adjoining to it certain matrices.

In this paper, we will refine the earlier work in [10]. First, we present an equivalent matrix equation of (1.2). Second, we give a condensed Cramer rule for the solution of (1.2) and condensed determinantal expressions of $A^{(2)}_{T,S}A$ and $AA^{(2)}_{T,S}$. Finally, we present a method for computing the solution of matrix equation (1.2).

Lemma 1.1[1,2]. *Let $A \in \mathbb{C}^{m\times n}_r$, T be a subspace of $\mathbb{C}^n$ of dimension $t \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - t$. Then A has a $\{2\}$-inverse X such that $R(X) = T$ and $N(X) = S$ if and only if*

$$AT \oplus S = \mathbb{C}^m,$$

in which case X is unique and denoted by $A^{(2)}_{T,S}$.

Lemma 1.2[2]. (1) *Let $A \in \mathbb{C}^{m\times n}$. Then for the Moore-Penrose inverse $A^\dagger$, one has:*

(a) $A^\dagger = A^{(2)}_{R(A^*),N(A^*)}$.

(2) *Let A, $E \in \mathbb{C}^{n\times n}$. Then for the Drazin inverse A_d and the group inverse E_g, one has:*

(b) $A_d = A^{(2)}_{R(A^k),N(A^k)}$, *where $k = Ind(A)$;*

(c) $E_g = A^{(2)}_{R(E),N(E)}$, *where $Ind(E) = 1$;*

(d) $R(E_g) = R(E)$, $N(E_g) = N(E)$;

(e) $EE_g = E_gE = P_{R(E),N(E)}$, *where $P_{R(E),N(E)}$ is a projector;*

(f) $P_{R(E),N(E)} + P_{N(E),R(E)} = I$, *where I is an identity matrix.*

Lemma 1.3[11]. *Let $A \in \mathbb{C}^{m\times n}_r$, $T \subset \mathbb{C}^n$, $S \subset \mathbb{C}^m$, dim(T)=dim($S^\perp$)= $t \le r$. In addition, suppose $G \in \mathbb{C}^{n\times m}$ such that*

$$R(G) = T, \ N(G) = S.$$

If A has a $\{2\}$-inverse $A^{(2)}_{T,S}$, then $Ind(AG) = Ind(GA) = 1$. Furthermore, we have $A^{(2)}_{T,S} = G(AG)_g = (GA)_gG$.

2. Equivalent matrix equation

Now, we consider the solution of (1.2). Let $A \in \mathbb{C}^{m\times n}_r$, $B \in \mathbb{C}^{p\times q}_{\tilde{r}}$, $D \in \mathbb{C}^{m\times q}$, $T \subset \mathbb{C}^n$, $S \subset \mathbb{C}^m$, $\tilde{T} \subset \mathbb{C}^q$ and $\tilde{S} \subset \mathbb{C}^p$ satisfy

$$\dim(T) = \dim(S^\perp) = t \le r \text{ and } AT \oplus S = \mathbb{C}^m,$$

and

$$\dim(\tilde{T}) = \dim(\tilde{S}^{\perp}) = \tilde{t} \leq \tilde{r} \text{ and } B\tilde{T} \oplus \tilde{S} = \mathbb{C}^p.$$

From [10], we know that the restricted matrix equation (1.2) has unique solution.

Lemma 2.1[2,10]. *Given the matrices A, B, D, and the subspaces T, S, $\tilde{T}$, $\tilde{S}$ as above. If $D \in R(AG, \tilde{G}B)$, for some matrices $G \in \mathbb{C}^{n\times m}$ and $\tilde{G} \in \mathbb{C}^{q\times p}$ satisfying*

$$R(G) = T,\ N(G) = S,\ R(\tilde{G}) = \tilde{T},\ N(\tilde{G}) = \tilde{S},$$

then the matrix equation (1.2) has the unique solution

$$X = A^{(2)}_{T,S} D B^{(2)}_{\tilde{T},\tilde{S}}.$$

In the following we present some special cases of (1.2) which can be found in [2,9]. The Lemma 2.2 is from Lemma 2.1 by setting $G = A^*$ and $\tilde{G} = B^*$.

Lemma 2.2[2,10]. *Let $A \in \mathbb{C}^{m\times n}_r$, $B \in \mathbb{C}^{p\times q}_s$, $D \in \mathbb{C}^{m\times q}$. Let D satisfy*

$$D \in R(AA^*, B^*B).$$

Then the restricted matrix equation

$$AXB = D, \quad R(X) \subset R(A^*),\ N(X) \supset N(B^*)$$

has a unique solution $X = A^{\dagger} D B^{\dagger}$.

The Lemma 2.3 is from Lemma 2.1 by setting $G = A^{k_1}$ and $\tilde{G} = B^{k_2}$.

Lemma 2.3. *Let $A \in \mathbb{C}^{n\times n}$, $Ind(A) = k_1$, $B \in \mathbb{C}^{m\times m}$, and $Ind(B) = k_2$. And let $D \in \mathbb{C}^{n\times m}$ satisfy*

$$R(D) \subset R(A^{k_1+1}, B^{k_2+1}).$$

Then the restricted matrix equation

$$AXB = D, \quad R(X) \subset R(A^{k_1}),\ N(X) \supset N(B^{k_2})$$

has a unique solution $X = A_d D B_d$.

We introduce first two formulas of regular inverse which will be used later.

Theorem 2.1. *Let A, T, S and G be as in Lemma 2.1 and*

$$AT \oplus S = \mathbb{C}^m.$$

Let U_1, V_1 be matrices of full column rank whose columns form bases for $N(GA)$ and $N((GA)^*)$, respectively. We define

$$E = U_1 V_1^*.$$

Then

$$(GA + E)^{-1} = (GA)_g + E_g. \tag{2.1}$$

Proof. By hypothesis, we deduce that

$$R(E) = R(U_1 V_1^*) = R(U_1) = N(GA), \tag{2.2}$$

$$N(E) = N(U_1V_1^*) = N(V_1^*) = [R(V_1)]^{\perp} = [N(GA)^*]^{\perp} = R(GA). \quad (2.3)$$

Thus

$$(GA)E = 0, \ E(GA) = 0.$$

Since $\text{rank}(E^2) = \text{rank}(E)$, we know that $\text{Ind}(E) = 1$. From Lemma 1.2 (d), (2.2) and (2.3), we have

$$(GA)E_g = 0, \ E(GA)_g = 0. \quad (2.4)$$

Using (2.2)-(2.4), Lemma 1.2 (e) and (f), we obtain

$$\begin{aligned} & (GA+E)((GA)_g + E_g) \\ = \ & (GA)(GA)_g + (GA)E_g + E(GA)_g + EE_g \\ = \ & P_{R(GA),N(GA)} + P_{N(GA),R(GA)} \\ = \ & I. \end{aligned}$$

Therefore $(GA+E)^{-1} = (GA)_g + E_g$. □

Theorem 2.2. *Let B, $\tilde{T}$, $\tilde{S}$ and $\tilde{G}$ be as in Lemma 2.1 and*

$$B\tilde{T} \oplus \tilde{S} = \mathbb{C}^p.$$

Suppose that U_2, V_2 be matrices of full column rank whose columns form bases for $N(B\tilde{G})$ and $N((B\tilde{G})^*)$, respectively. We define

$$F = U_2V_2^*.$$

Then $(B\tilde{G}+F)^{-1} = (B\tilde{G})_g + F_g$.

Proof. The proof is similar to Theorem 2.1. □

Now, the equivalent matrix equation of (1.2) is presented as follows.

Theorem 2.3 *Given the matrices A, B, D, E, F and the subspaces T, S, $\tilde{T}$, $\tilde{S}$ as in Theorem 2.1 and Theorem 2.2. Suppose that matrices $G \in \mathbb{C}^{n\times m}$ and $\tilde{G} \in \mathbb{C}^{q\times p}$ satisfy $R(G) = T$, $N(G) = S$, $R(\tilde{G}) = \tilde{T}$, $N(\tilde{G}) = \tilde{S}$. If $D \in R(AG, \tilde{G}B)$, then the restricted matrix equation (1.2) is equivalent to the nonsingular matrix equation*

$$(GA+E)X(B\tilde{G}+F) = GD\tilde{G}. \quad (2.5)$$

Proof. It has been proved in [10] that (1.2) has a unique solution $A^{(2)}_{T,S}DB^{(2)}_{\tilde{T},\tilde{S}}$. From Theorem 2.1 and Theorem 2.2, we have that $(GA+E)$ and $(B\tilde{G}+F)$ are nonsingular. So (2.5) has a unique solution. From assumptions, we have

$$D = AGY\tilde{G}B, \ \text{for some } Y \in \mathbb{C}^{m\times q}$$

and

$$N(E) = R(GA), \ R(F) = N(B\tilde{G}).$$

Thus

$$EGA = 0, \ B\tilde{G}F = 0,$$
$$E_gGD = E_gGAGY\tilde{G}B = 0,$$

and $D\tilde{G}F_g = AGY\tilde{G}B\tilde{G}F_g = 0$. Hence

$$\begin{aligned} X &= (GA+E)^{-1}GD\tilde{G}(B\tilde{G}+F)^{-1} \\ &= (GA)_gGD\tilde{G}(B\tilde{G})_g + (GA)_gGD\tilde{G}F_g + E_gGD\tilde{G}(B\tilde{G})_g + E_gGD\tilde{G}F_g \\ &= A^{(2)}_{T,S}DB^{(2)}_{\tilde{T},\tilde{S}}. \end{aligned}$$

□

Corollary 2.1. *Let $A \in \mathbb{C}^{m\times n}_r$, $B \in \mathbb{C}^{p\times q}_s$, $D \in \mathbb{C}^{m\times q}$. Let $U \in \mathbb{C}^{n\times(n-r)}_{n-r}$ and $V \in \mathbb{C}^{p\times(p-s)}_{p-s}$ be matrices whose columns form bases for $N(A)$ and $N(B)$, respectively. Define $E = UU^*$ and $F = VV^*$. If $D \in R(AA^*, B^*B)$, then the restricted matrix equation*

$$AXB = D, \quad R(X) \subset R(A^*),\ N(X) \supset N(B^*) \tag{2.6}$$

is equivalent to the nonsingular matrix equation

$$(A^*A + E)X(BB^* + F) = A^*DB^*. \tag{2.7}$$

Corollary 2.2. *Let $A \in \mathbb{C}^{n\times n}$, $Ind(A) = k_1$, $B \in \mathbb{C}^{m\times m}$, $Ind(B) = k_2$, $rank(A^{k_1}) = r$, $rank(B^{k_2}) = s$, and $D \in \mathbb{C}^{n\times m}$. Let $U_1, V_1 \in \mathbb{C}^{n\times(n-r)}_{n-r}$ be matrices whose columns form bases for $N(A^{k_1})$ and $N(A^{k_1^*})$, respectively. Let $U_2, V_2 \in \mathbb{C}^{m\times(m-s)}_{m-s}$ be matrices whose columns form bases for $N(B^{k_2})$ and $N(B^{k_2^*})$, respectively. We define $E = U_1V_1^*$ and $F = U_2V_2^*$. If*

$$D \in R(A^{k_1+1}, B^{k_2+1}),$$

then the restricted matrix equation

$$AXB = D, \quad R(X) \subset R(A^{k_1}), N(X) \supset N(B^{k_2}), \tag{2.8}$$

is equivalent to the nonsingular matrix equation

$$(A^{k_1+1} + E)X(B^{k_2+1} + F) = A^{k_1}DB^{k_2}. \tag{2.9}$$

3. Condensed Cramer rule

In this section, we present a condensed Cramer rule for the solution of (1.2) and condensed determinantal expressions for $A^{(2)}_{T,S}A$, $AA^{(2)}_{T,S}$.

Lemma 3.1[10]. *Let $A \in \mathbb{C}^{n\times n}_n$, $B \in \mathbb{C}^{m\times m}_m$ and $D \in \mathbb{C}^{n\times m}$ be given. Then the unique solution $X = (x_{ij}) \in \mathbb{C}^{n\times m}$ of the matrix equation*

$$AXB = D$$

is given by

$$x_{ij} = \frac{\sum_{l=1}^{m} det(A(i \to d_l))det(B(e_l^T \leftarrow j))}{det(A)det(B)}, \ (i = 1, \cdots, n; j = 1 \cdots, m), \tag{3.1}$$

where d_l is the lth column of D and e_l is the lth column of the $m \times m$ identity matrix.

Theorem 3.1. *Let A, B, D, T, S, $\tilde{T}$, $\tilde{S}$, U_1, V_1, U_2, V_2, E, F be as in Theorem 2.3. Then the unique solution $X = (x_{ij}) \in \mathbb{C}^{n\times p}$ of (1.2) is given by*

$$x_{ij} = \frac{\sum_{l=1}^{p} det((GA+E)(i \to d_l))det((B\tilde{G}+F)(e_l^T \leftarrow j))}{det(GA+E)det(B\tilde{G}+F)}, \quad (i = 1,\cdots,n; j = 1,\cdots,p),$$

where d_l is the lth column of $GD\tilde{G}$ and e_l is the lth column of the $p \times p$ identity matrix.

Proof. Since (1.2) is equivalent to the nonsingular matrix equation (2.5), we can derive a condensed Cramer rule for (1.2) by using Lemma 3.1 to (2.5). □

Remark. The result is better than Wang et al. in [2]. We note that $(UV)^{-1}$ is employed in their condensed Cramer rule.

Corollary 3.1. *Let A, B, D, U, V, E, F be as in Corollary 2.1. Then the unique solution $X = (x_{ij}) \in \mathbb{C}^{n\times p}$ of*

$$AXB = D, \ R(X) \subset R(A^*), \ N(X) \supset N(B^*),$$

is given by

$$x_{ij} = \frac{\sum_{l=1}^{p} det((A^*A+E)(i \to d_l))det((BB^*+F)(e_l^T \leftarrow j))}{det(A^*A+E)det(BB^*+F)}, \quad (i = 1,\cdots,n; j = 1,\cdots,p),$$

where d_l is the lth column of A^*DB^* and e_l is the lth column of the $p\times p$ identity matrix.

Corollary 3.2. *Let A, k_1, B, k_2, D, U_1, V_1, U_2, V_2, E, F be as in Corollary 2.2. Then the unique solution $X = (x_{ij}) \in \mathbb{C}^{n\times m}$ of*

$$AXB = D, \ R(X) \subset R(A^{k_1}), \ N(X) \supset N(B^{k_2}),$$

is given by

$$x_{ij} = \frac{\sum_{l=1}^{m} det((A^{k_1+1}+E)(i \to d_l))det((B^{k_2+1}+F)(e_l^T \leftarrow j))}{det(A^{k_1+1}+E)det(B^{k_2+1}+F)}, \quad (i = 1,\cdots,n; j = 1,\cdots,m),$$

where d_l is the lth column of $A^{k_1}DB^{k_2}$ and e_l is the lth column of the $m \times m$ identity matrix.

It is well known that if A is nonsingular, then the inverse of A is given by

$$A_{i,j}^{-1} = \frac{\det(A(i \to e_j))}{\det(A)}, \ i,j = 1,2,\cdots,n, \tag{3.2}$$

where e_j is the jth column of identity matrix. We can obtain (3.2) immediately by following lemma with $D = I$.

Lemma 3.2[2]. *Let $A \in \mathbb{C}_n^{n\times n}$ and $D \in \mathbb{C}^{n\times m}$. Then the unique solution $Y = (y_{i,j}) \in \mathbb{C}^{n\times m}$ of the matrix equation $AY = D$ is given by*

$$y_{i,j} = \frac{det(A(i \to d_j))}{det(A)}, \ (i = 1, 2, \cdots, n; j = 1, 2, \cdots, m),$$

where d_j is the jth column of D.

The determinantal expression of the regular inverse can be extended to the generalized inverses $A^\dagger$, A_d and A_g, but $A^{(2)}_{T,S}$. The reason can be found in [12]. These results offer a useful tool for the theory and computation of generalized inverse. By using Theorem 2.1, Theorem 2.2 and Lemma 3.2, the condensed determinantal expressions of projectors $AA^{(2)}_{T,S}$ and $A^{(2)}_{T,S}A$ are given as follows.

Theorem 3.2. *Let A, G, E be as in Theorem 3.1. Then*

(a) The matrix equation $(GA + E)X = GA$ has the unique solution $X = A^{(2)}_{T,S}A$ which is given by

$$(A^{(2)}_{T,S}A)_{i,j} = \frac{det((GA + E)(i \to d_j))}{det(A)}, \ (i = 1, 2, \cdots, n; j = 1, 2, \cdots, n), \quad (3.3)$$

where d_j is the jth column of GA.

(b) The matrix equation $X(AG+E) = AG$ has the unique solution $X = AA^{(2)}_{T,S}$ which is given by

$$(AA^{(2)}_{T,S})_{i,j} = \frac{det((AG + E)(d_i^T \leftarrow j))}{det(A)}, \ (i = 1, 2, \cdots, m; j = 1, 2, \cdots, m), \quad (3.4)$$

where d_i^T is the ith row of AG.

Proof. (a) From Theorem 2.1, we know that

$$X = (GA + E)^{-1}GA = (GA)_g GA + E_g GA = A^{(2)}_{T,S}A.$$

Employing Lemma 3.2, we get (3.3). In a similar manner, we can establish (3.4) by using of Theorem 2.2. □

As the corollary of Theorem 3.2, we can get condensed determinantal expressions of the projectors $AA^\dagger$, $A^\dagger A$, AA_d, A_dA.

4. Algorithm

In this section, we present a method for computing the solution of matrix equation (1.2) based on Theorem 2.3. This gives the uniform method for computing the special cases (2.6) and (2.8).

First of all, we need an algorithm for computing the basis of $N(A)$ [2].

A matrix $H \in \mathbb{C}^{m\times m}$ is said to be in Hermite echelon form if its elements h_{ij} satisfy the following conditions:

1. $h_{ij} = 0$, $i > j$,
2. h_{ii} is either 0 or 1,
3. if $h_{ii} = 0$ then $h_{ik} = 0$ for every k, $1 \le k \le m$,

4. if $h_{ii} = 1$ then $h_{ki} = 0$ for every $k \neq i$.

For a given matrix $A \in \mathbb{C}^{m\times m}$, the Hermite echelon form H_A obtained by row reducing A is unique; $N(A) = N(H_A) = R(I - H_A)$ and a basis for $N(A)$ is the set of nonzero columns of $I - H_A$.

Algorithm 1. *Let $GA \in \mathbb{C}_r^{m\times m}$. This algorithm is designed for computing $U \in \mathbb{C}_{m-r}^{m\times(m-r)}$ whose columns form a basis for $N(GA)$.*

1. Row reduce GA to its Hermite echelon form H_{GA}.

2. Form $I - H_{GA}$, and select the nonzero columns $u_1, u_2, \cdots, u_{m-r}$ from this matrix, $U = (u_1, u_2, \cdots, u_{m-r})$. In the same manner, we can obtain the basis matrices of $N((GA)^*)$, $N(B\tilde{G})$ and $N((B\tilde{G})^*)$.

Algorithm 2.

1. *Formulate $GA + E$, $B\tilde{G} + F$ and $GD\tilde{G}$.*
2. *Execute elementary row operations on the pair*

$$\begin{pmatrix} GA + E, & GD\tilde{G} \end{pmatrix}$$

and transform it into

$$\begin{pmatrix} I, & A_{T,S}^{(2)} D\tilde{G} \end{pmatrix}.$$

3. *Execute elementary row operations on the pair*

$$\begin{pmatrix} (B\tilde{G} + F)^T, & (A_{T,S}^{(2)} D\tilde{G})^T \end{pmatrix}$$

and transform it into

$$\begin{pmatrix} I, & (A_{T,S}^{(2)} D B_{\tilde{T},\tilde{S}}^{(2)})^T \end{pmatrix}.$$

5. Conclusion Remark

In this paper, we present condensed Cramer rule for the solution of restricted matrix equation. The condensed Cramer rules of this paper are different from those in [2] because the $(UV)^{-1}$ is not employed in our results which is important in numerical computation.

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