

SUFFICIENCY AND DUALITY IN MINMAX FRACTIONAL SUBSET PROGRAMMING INVOLVING GENERALIZED TYPE-I FUNCTIONS

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ABSTRACT. A discrete minmax fractional subset programming problem is considered. Various parametric and parameter-free global sufficient optimality conditions and duality results are discussed under generalized $(\mathcal{F}, \alpha, \rho, \theta)$ -V-type-I n -set functions.

AMS Mathematics Subject Classification : 90C30, 90C32, 90C46, 90C47
Key words and phrases : Minmax fractional subset programming; Generalized $(\mathcal{F}, \alpha, \rho, \theta)$ -V-type-I n -set functions; Sufficient optimality conditions; Duality

1. Introduction

The notion of duality for generalized linear fractional programming problem with point-functions was initiated by Von Neumann [10] in the context of an economic equilibrium problem. Recently, various optimality conditions, duality results, and computational algorithms for several classes of generalized fractional programs have been appeared in the related literature. A fairly extensive list of references pertaining to different aspects of generalized fractional programming problems is given in [14, 15].

In this paper, we consider the following discrete minmax subset programming problem:

$$(\mathbf{P}) \text{ Minimize } \max_{1 \leq i \leq k} \frac{F_i(T)}{G_i(T)}$$

subject to $H_j(T) \leq 0$, $j \in M$, $T \in \mathcal{A}^n$,

where \mathcal{A}^n is the n -fold product of σ -algebra \mathcal{A} of subsets of a given set X , $F_i, G_i, i \in K = \{1, 2, \dots, k\}$ and $H_j, j \in M = \{1, 2, \dots, m\}$ are real-valued

Received July 20, 2007. Revised January 9, 2008. Accepted January 13, 2008.

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functions defined on \mathcal{A}^n , and for each $i \in K$, $F_i(T) \geq 0$ and $G_i(T) > 0$ for all $T \in \mathcal{A}^n$ such that $H_j(T) \leq 0$, $j \in M$.

Zalmai [11] established necessary and sufficient optimality conditions and various duality results for (P). A Lagrangian-type dual problem was constructed for (P) in [12] via a Gordan-type transposition theorem and appropriate duality results were proved. Preda [9] extended the concept of ρ -convexity [11] to (F, ρ) -convexity and obtained duality theorems. Bhatia and Kumar [3] derived sufficient optimality conditions and duality results for different combinations of the problem function by using ρ -convexity. In [2], Bector and Singh discussed optimality and duality theorems involving generalized b -vexity assumptions, however, the Lagrangian-type dual problem was discussed in [5]. Lai and Liu [6] presented parameter-free necessary and sufficient optimality conditions for (P). They constructed also two parameter-free dual models and discussed duality results. Zalmai [14] obtained nonparametric sufficient optimality conditions and duality results for minmax programming problems under generalized $(\mathcal{F}, \rho, \theta)$ -convexity assumptions. A number of parametric and parameter-free sufficient optimality conditions and duality results were discussed in [15] under generalized $(\mathcal{F}, \alpha, \rho, \theta)$ -V-convexity [13]. Recently, Ahmad and Sharma [1] derived sufficient optimality conditions for a multiobjective subset programming problem under $(\mathcal{F}, \alpha, \rho, d)$ -type I functions.

In this paper, motivated by Zalmai [14, 15] and Mishra [7], we present parametric and parameter-free sufficient optimality conditions for (P) under generalized $(\mathcal{F}, \alpha, \rho, \theta)$ -V-type-I n -set functions. Moreover, appropriate duality theorems are proved for parametric and parameter-free dual models of (P).

2. Notations and preliminaries

Let (X, \mathcal{A}, μ) be a finite atomless measure space with $L_1(X, \mathcal{A}, \mu)$ separable, and let d be the pseudometric on \mathcal{A}^n defined by

$$d(T, Y) = \left[\sum_{p \in N} \mu^2(T_p \Delta Y_p) \right]^{\frac{1}{2}}, \quad T_p, Y_p \in \mathcal{A}, \quad p \in N = \{1, 2, \dots, n\},$$

where Δ denotes the symmetric difference; thus, (\mathcal{A}^n, d) is a pseudometric space. For $h \in L_1(X, \mathcal{A}, \mu)$ and $S \in \mathcal{A}$ with characteristic function $\chi_S \in L_\infty(X, \mathcal{A}, \mu)$, the integral $\int_S h \, d\mu$ will be denoted by $\langle h, \chi_S \rangle$.

Corley [4] introduced the notion of differentiability for n -set functions as:

A function $F : \mathcal{A} \rightarrow R$ is said to be differentiable at T^* , if there exists $DF(T^*) \in L_1(X, \mathcal{A}, \mu)$, the derivative of F at T^* , such that for each $T \in \mathcal{A}$,

$$F(T) = F(T^*) + \langle DF(T^*), \chi_T - \chi_{T^*} \rangle + V_F(T, T^*),$$

where $V_F(T, T^*)$ is $o(d(T, T^*))$, that is, $\lim_{d(T, T^*) \rightarrow 0} \frac{V_F(T, T^*)}{d(T, T^*)} = 0$.

A function $G : \mathcal{A}^n \rightarrow R$ is said to have a partial derivative at $T^* = (T_1^*, T_2^*, \dots, T_n^*) \in \mathcal{A}^n$ with respect to its p th argument, if the function

$$F(T_p) = G(T_1^*, \dots, T_{p-1}^*, T_p, T_{p+1}^*, \dots, T_n^*)$$

has derivative $DF(T_p^*)$, $p \in N$; in that case, the p th partial derivative of G at T^* is defined to be $D_p G(T^*) = DF(T_p^*)$, $p \in N$.

A function $G : \mathcal{A}^n \rightarrow R$ is said to be differentiable at T^* , if all the partial derivatives $DG_p(T^*)$, $p \in N$, exist and

$$G(T) = G(T^*) + \sum_{p \in N} \langle DG_p(T^*), \chi_{T_p} - \chi_{T_p^*} \rangle + W_G(T, T^*),$$

where $W_G(T, T^*)$ is $o(d(T, T^*))$ for all $T \in \mathcal{A}^n$.

It was shown in [8] that for any triplet $(T, Y, \lambda) \in \mathcal{A} \times \mathcal{A} \times [0, 1]$, there exist sequences $\{T_k\}$ and $\{Y_k\}$ in \mathcal{A} such that

$$\chi_{T_k} \xrightarrow{w^*} \lambda \chi_{T \setminus Y} \text{ and } \chi_{Y_k} \xrightarrow{w^*} (1 - \lambda) \chi_{Y \setminus T} \tag{1}$$

imply

$$\chi_{T_k \cup Y_k \cup (T \cap Y)} \xrightarrow{w^*} \lambda \chi_T + (1 - \lambda) \chi_Y, \tag{2}$$

where $\xrightarrow{w^*}$ denotes weak* convergence of elements in $L_\infty(X, \mathcal{A}, \mu)$, and $T \setminus Y$ is the complement of T relative to Y . The sequence $\{V_k(\lambda)\} = \{T_k \cup Y_k \cup (T \cap Y)\}$ satisfying (1) and (2) is called the *Morris sequence* associated with (T, Y, λ) .

Definition 1. A function $F : \mathcal{A}^n \rightarrow R$ is said to be (strictly) convex if for every $(T, Y, \lambda) \in \mathcal{A}^n \times \mathcal{A}^n \times [0, 1]$, there exists a Morris sequence $\{V_k(\lambda)\}$ in \mathcal{A}^n such that

$$\limsup_{k \rightarrow \infty} F(V_k(\lambda)) (<) \leq \lambda F(T) + (1 - \lambda) F(Y).$$

It was shown in [4, 8] that if a differentiable function $F : \mathcal{A}^n \rightarrow R$ is (strictly) convex, then

$$F(T) (>) \geq F(Y) + \sum_{p \in N} \langle D_p F(Y), \chi_{T_p} - \chi_{Y_p} \rangle \forall T, Y \in \mathcal{A}^n.$$

Definition 2. A function $\mathcal{F}(T, T^*; \cdot) : L_1^n(X, \mathcal{A}, \mu) \rightarrow R$ is said to be sublinear with respect to its third argument, if for fixed $T, T^* \in \mathcal{A}^n$, and for every $f, g \in L_1^n(X, \mathcal{A}, \mu)$ and $c \in R_+ \equiv [0, \infty)$,

$$\mathcal{F}(T, T^*; f + g) \leq \mathcal{F}(T, T^*; f) + \mathcal{F}(T, T^*; g)$$

and

$$\mathcal{F}(T, T^*; cf) = c \mathcal{F}(T, T^*; f).$$

Consider the following generalized subset programming problem:

(P1) Minimize $\max_{1 \leq i \leq k} F_i(T)$

subject to $T \in X_\circ$,

where $X_o = \{T \in \mathcal{A}^n \mid H_j(T) \leq 0, j \in M\}$ denote the set of all feasible solutions of (P).

The following definitions [7] are needed in the sequel:

Let $\mathcal{F}(T, T^*; \cdot) : L_1^n(X, \mathcal{A}, \mu) \rightarrow R$ be a sublinear functional, $\theta : \mathcal{A}^n \times \mathcal{A}^n \rightarrow \mathcal{A}^n \times \mathcal{A}^n$ be a function such that $T \neq T^* \Rightarrow \theta(T, T^*) \neq (0, 0)$ and let the functions $F : \mathcal{A}^n \rightarrow R^k$ and $H : \mathcal{A}^n \rightarrow R^m$ with components $F_i, i \in K$, and $H_j, j \in M$, respectively, be differentiable at $T^* \in \mathcal{A}^n$.

Definition 3. (F, H) is said to be $(\mathcal{F}, \alpha, \rho, \theta)$ -V-type-I at $T^* \in \mathcal{A}^n$, if there exist vectors $\alpha = (\alpha_1^1, \alpha_2^1, \dots, \alpha_k^1, \alpha_1^2, \alpha_2^2, \dots, \alpha_m^2)$ and $\rho = (\rho_1^1, \rho_2^1, \dots, \rho_k^1, \rho_1^2, \rho_2^2, \dots, \rho_m^2) \in R^{k+m}$, where $\alpha_i^1, \alpha_j^2 : \mathcal{A}^n \times \mathcal{A}^n \rightarrow R_+ \setminus \{0\}$, and $\rho_i^1, \rho_j^2 \in R$ for $i \in K, j \in M$, such that for each $T \in X_o$, and for all $i \in K, j \in M$

$$\begin{aligned} F_i(T) - F_i(T^*) &\geq \mathcal{F}(T, T^*; \alpha_i^1(T, T^*)DF_i(T^*)) + \rho_i^1 d^2(\theta(T, T^*)), \\ -H_j(T^*) &\geq \mathcal{F}(T, T^*; \alpha_j^2(T, T^*)DH_j(T^*)) + \rho_j^2 d^2(\theta(T, T^*)). \end{aligned}$$

Definition 4. (F, H) is said to be $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ -V-pseudoquasi type-I at $T^* \in \mathcal{A}^n$, if there exist vectors $\bar{\alpha} = (\bar{\alpha}_1^1, \bar{\alpha}_2^1, \dots, \bar{\alpha}_k^1, \bar{\alpha}_1^2, \bar{\alpha}_2^2, \dots, \bar{\alpha}_m^2)$ and $\bar{\rho} = (\bar{\rho}^1, \bar{\rho}^2) \in R^2$, where $\bar{\alpha}_i^1, \bar{\alpha}_j^2 : \mathcal{A}^n \times \mathcal{A}^n \rightarrow R_+ \setminus \{0\}$ for $i \in K, j \in M$, such that for each $T \in X_o$,

$$\begin{aligned} \mathcal{F}\left(T, T^*; \sum_{i \in K} DF_i(T^*)\right) &\geq -\bar{\rho}^1 d^2(\theta(T, T^*)) \Rightarrow \sum_{i \in K} \bar{\alpha}_i^1 F_i(T) \geq \sum_{i \in K} \bar{\alpha}_i^1 F_i(T^*), \\ -\sum_{j \in M} \bar{\alpha}_j^2 H_j(T^*) \leq 0 &\Rightarrow \mathcal{F}\left(T, T^*; \sum_{j \in M} DH_j(T^*)\right) \leq -\bar{\rho}^2 d^2(\theta(T, T^*)). \end{aligned}$$

If in the above definition, the first inequality is satisfied as

$$\mathcal{F}\left(T, T^*; \sum_{i \in K} DF_i(T^*)\right) \geq -\bar{\rho}^1 d^2(\theta(T, T^*)) \Rightarrow \sum_{i \in K} \bar{\alpha}_i^1 F_i(T) > \sum_{i \in K} \bar{\alpha}_i^1 F_i(T^*),$$

then we say that (F, H) is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ -V-strictly pseudoquasi type-I at T^* .

Definition 5. (F, H) is said to be $(\mathcal{F}, \tilde{\alpha}, \tilde{\rho}, \theta)$ -V-quasi strictly pseudo type-I at $T^* \in \mathcal{A}^n$, if there exist vectors $\tilde{\alpha} = (\tilde{\alpha}_1^1, \tilde{\alpha}_2^1, \dots, \tilde{\alpha}_k^1, \tilde{\alpha}_1^2, \tilde{\alpha}_2^2, \dots, \tilde{\alpha}_m^2)$, and $\tilde{\rho} = (\tilde{\rho}^1, \tilde{\rho}^2) \in R^2$, where $\tilde{\alpha}_i^1, \tilde{\alpha}_j^2 : \mathcal{A}^n \times \mathcal{A}^n \rightarrow R_+ \setminus \{0\}$, for $i \in K, j \in M$, such that for each $T \in X_o$,

$$\begin{aligned} \sum_{i \in K} \tilde{\alpha}_i^1 F_i(T) \leq \sum_{i \in K} \tilde{\alpha}_i^1 F_i(T^*) &\Rightarrow \mathcal{F}\left(T, T^*; \sum_{i \in K} DF_i(T^*)\right) \leq -\tilde{\rho}^1 d^2(\theta(T, T^*)), \\ \mathcal{F}\left(T, T^*; \sum_{j \in M} DH_j(T^*)\right) &\geq -\tilde{\rho}^2 d^2(\theta(T, T^*)) \Rightarrow -\sum_{j \in M} \tilde{\alpha}_j^2 H_j(T^*) > 0. \end{aligned}$$

If in the above definition, the first inequality is satisfied as

$$\sum_{i \in K} \tilde{\alpha}_i^1 F_i(T) < \sum_{i \in K} \tilde{\alpha}_i^1 F_i(T^*) \Rightarrow \mathcal{F}\left(T, T^*; \sum_{i \in K} DF_i(T^*)\right) \leq -\tilde{\rho}^1 d^2(\theta(T, T^*)),$$

then we say that (F, H) is $(\mathcal{F}, \tilde{\alpha}, \tilde{\rho}, \theta)$ -V-prestrictquasi strictlypseudo type-I at T^* .

We next recall a set of necessary optimality conditions and other related results which form the basis for our discussion of sufficiency criteria for (P).

Theorem 2.1 [11]. *Assume that $F_i, G_i, i \in K$, and $H_j, j \in M$, are differentiable at $T^* \in \mathcal{A}^n$, and that for each $i \in K$, there exists $\hat{T}^i \in \mathcal{A}^n$ such that*

$$H_j(T^*) + \sum_{p \in N} \langle D_p H_j(T^*), \chi_{\hat{T}^i_p} - \chi_{T^*_p} \rangle < 0, \quad j \in M. \tag{3}$$

If T^* is an optimal solution of (P), then there exist $\lambda^* \in R$, $\mu^* \in U = \left\{ \mu \in R^k_+ : \sum_{i \in K} \mu_i = 1 \right\}$, and $\nu^* \in R^m_+$ such that

$$\sum_{p \in N} \left\langle \sum_{i \in K} \mu_i^* [D_p F_i(T^*) - \lambda^* D_p G_i(T^*)] + \sum_{j \in M} \nu_j^* D_p H_j(T^*), \chi_{T^*_p} - \chi_{\hat{T}^i_p} \right\rangle \geq 0, \\ \forall T \in \mathcal{A}^n, \\ \mu_i^* [F_i(T^*) - \lambda^* G_i(T^*)] = 0, \quad i \in K, \quad \nu_j^* H_j(T^*) = 0, \quad j \in M.$$

For brevity, we shall henceforth refer to an $T^* \in X_0$ satisfying (3) as a regular feasible solution of (P).

It is easily seen that one obtains the following parameter-free version of Theorem 2.1 by eliminating the parameter λ^* and redefining the multipliers associated with the inequality constraints.

Theorem 2.2. *Assume that $F_i, G_i, i \in K$, and $H_j, j \in M$, are differentiable at $T^* \in \mathcal{A}^n$. If T^* is a regular optimal solution of (P), then there exist $\mu^* \in U$ and $\nu^* \in R^m_+$ such that*

$$\sum_{p \in N} \left\langle \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*) D_p F_i(T^*) - \Theta(T^*, \mu^*) D_p G_i(T^*)] + \sum_{j \in M} \nu_j^* D_p H_j(T^*), \chi_{T^*_p} - \chi_{\hat{T}^i_p} \right\rangle \geq 0, \quad \forall T \in \mathcal{A}^n, \\ \mu_i^* [\Gamma(T^*, \mu^*) F_i(T^*) - \Theta(T^*, \mu^*) G_i(T^*)] = 0, \quad i \in K, \\ \psi(T^*) \equiv \max_{1 \leq i \leq k} \frac{F_i(T^*)}{G_i(T^*)} = \frac{\Theta(T^*, \mu^*)}{\Gamma(T^*, \mu^*)}, \quad \nu_j^* H_j(T^*) = 0, \quad j \in M,$$

where $\Theta(T^*, \mu^*) = \sum_{i \in K} \mu_i^* F_i(T^*)$ and $\Gamma(T^*, \mu^*) = \sum_{i \in K} \mu_i^* G_i(T^*)$.

Finally, we state following lemma that provides an alternative expression for the objective function of (P).

Lemma 2.1 [11]. *For each $T \in \mathcal{A}^n$, one has*

$$\max_{1 \leq i \leq k} \frac{F_i(T)}{G_i(T)} = \max_{\mu \in U} \frac{\sum_{i \in K} \mu_i F_i(T)}{\sum_{i \in K} \mu_i G_i(T)}.$$

In overall treatment of sufficiency and duality theorems, it is assumed that the functions $F_i, G_i, i \in K$, and $H_j, j \in M$ are differentiable on \mathcal{A}^n .

3. Parametric sufficient optimality conditions

In this section, we establish the parametric sufficient optimality conditions for (P). For stating optimality Theorems 3.2 and 3.3, we use the real-valued functions $\mathcal{B}_i(\cdot, \lambda^*, \mu^*)$ and $\mathcal{C}_j(\cdot, \nu^*)$ defined, for fixed λ^*, μ^* , and ν^* on \mathcal{A}^n , and for all $i \in K, j \in M$ by

$$\mathcal{B}_i(T, \lambda^*, \mu^*) = \mu_i^*[F_i(T) - \lambda^*G_i(T)], \text{ and } \mathcal{C}_j(T, \nu^*) = \nu_j^*H_j(T).$$

Theorem 3.1. *Let $T^* \in X_0$ and let there exist $\mu^* \in U, \nu^* \in R_+^m$ and $\lambda^* \in R_+$ such that*

$$\mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^*[DF_i(T^*) - \lambda^*DG_i(T^*)] + \sum_{j \in M} \nu_j^*DH_j(T^*) \right) \geq 0, \forall T \in \mathcal{A}^n, \tag{1}$$

$$\mu_i^*[F_i(T^*) - \lambda^*G_i(T^*)] = 0, \ i \in K, \tag{2}$$

$$\nu_j^*H_j(T^*) = 0, \ j \in M. \tag{3}$$

If

- (i) $[(F_1(\cdot) - \lambda^*G_1(\cdot), \dots, F_k(\cdot) - \lambda^*G_k(\cdot)), (H_1(\cdot), \dots, H_m(\cdot))]$ is $(\mathcal{F}, \alpha, \rho, \theta)$ - V -type-I at T^* ,
- (ii) $\alpha_1^1 = \alpha_2^1 = \dots = \alpha_k^1 = \alpha_1^2 = \alpha_2^2 = \dots = \alpha_m^2 = \delta$, and
- (iii) $\sum_{i \in K} \mu_i^* \rho_i^1 + \sum_{j \in M} \nu_j^* \rho_j^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. The inequality (1) along with the sublinearity of \mathcal{F} implies

$$\mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^*[DF_i(T^*) - \lambda^*DG_i(T^*)] \right) + \mathcal{F} \left(T, T^*; \sum_{j \in M} \nu_j^*DH_j(T^*) \right) \geq 0. \tag{4}$$

By hypothesis (i), we get

$$\begin{aligned} & (F_i(T) - \lambda^*G_i(T)) - (F_i(T^*) - \lambda^*G_i(T^*)) \\ & \geq \mathcal{F}(T, T^*; \alpha_i^1(T, T^*)[DF_i(T^*) - \lambda^*DG_i(T^*)]) + \rho_i^1 d^2(\theta(T, T^*)), \ i \in K, \\ & -H_j(T^*) \geq \mathcal{F}(T, T^*; \alpha_j^2(T, T^*)DH_j(T^*)) + \rho_j^2 d^2(\theta(T, T^*)), \ j \in M. \end{aligned}$$

On multiplying the first inequality by $\mu_i^* \geq 0$, second by $\nu_j^* \geq 0$, and using (2), (3) and (ii), we obtain

$$\begin{aligned} \mu_i^*(F_i(T) - \lambda^*G_i(T)) & \geq \mathcal{F}(T, T^*; \delta(T, T^*)\mu_i^*[DF_i(T^*) - \lambda^*DG_i(T^*)]) \\ & \quad + \mu_i^*\rho_i^1 d^2(\theta(T, T^*)), \ i \in K, \\ 0 & \geq \mathcal{F}(T, T^*; \delta(T, T^*)\nu_j^*DH_j(T^*)) + \nu_j^*\rho_j^2 d^2(\theta(T, T^*)), \ j \in M, \end{aligned}$$

which on being summarized yield

$$\sum_{i \in K} \mu_i^* (F_i(T) - \lambda^* G_i(T)) \geq \mathcal{F} \left(T, T^*; \delta(T, T^*) \sum_{i \in K} \mu_i^* [DF_i(T^*) - \lambda^* DG_i(T^*)] \right) + \mathcal{F} \left(T, T^*; \delta(T, T^*) \sum_{j \in M} \nu_j^* DH_j(T^*) \right) + \left(\sum_{i \in K} \mu_i^* \rho_i^1 + \sum_{j \in M} \nu_j^* \rho_j^2 \right) d^2(\theta(T, T^*)).$$

This inequality in view of (iii), (4), $\delta(T, T^*) > 0$, and the sublinearity of \mathcal{F} gives

$$\sum_{i \in K} \frac{\mu_i^*}{\delta(T, T^*)} (F_i(T) - \lambda^* G_i(T)) \geq 0.$$

As $\delta(T, T^*) > 0$, the above inequality reduces to

$$\sum_{i \in K} \mu_i^* (F_i(T) - \lambda^* G_i(T)) \geq 0. \tag{5}$$

Now from Lemma 2.1, we have

$$\psi(T) \equiv \max_{1 \leq i \leq k} \frac{F_i(T)}{G_i(T)} = \max_{\mu \in U} \frac{\sum_{i \in K} \mu_i F_i(T)}{\sum_{i \in K} \mu_i G_i(T)} \geq \frac{\sum_{i \in K} \mu_i^* F_i(T)}{\sum_{i \in K} \mu_i^* G_i(T)} \geq \lambda^*, \text{ (by (5)).} \tag{6}$$

Hence, in view of (2) and (6), we conclude that T^* is an optimal solution of (P). \square

Theorem 3.2. *Let $T^* \in X_0$ and let there exist $\mu^* \in U, \nu^* \in R_+^m$ and $\lambda^* \in R_+$ satisfying (1) to (3). If*

- (i) $[(\mathcal{B}_1(\cdot, \lambda^*, \mu^*), \dots, \mathcal{B}_k(\cdot, \lambda^*, \mu^*)), (\mathcal{C}_1(\cdot, \nu^*), \dots, \mathcal{C}_m(\cdot, \nu^*))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V -pseudoquasi type-I at T^* , and
- (ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. The inequality (3) and $\bar{\alpha}_j^2(T, T^*) > 0, j \in M$ imply

$$\sum_{j \in M} \bar{\alpha}_j^2(T, T^*) \nu_j^* H_j(T^*) = 0. \tag{7}$$

From (7) and hypothesis (i), we obtain

$$\mathcal{F} \left(T, T^*; \sum_{j \in M} \nu_j^* DH_j(T^*) \right) + \bar{\rho}^2 d^2(\theta(T, T^*)) \leq 0. \tag{8}$$

The inequality (8) along with (4) and hypothesis (ii) yields

$$\mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^* [DF_i(T^*) - \lambda^* DG_i(T^*)] \right) + \bar{\rho}^1 d^2(\theta(T, T^*)) \geq 0,$$

which because of hypothesis (i) gives

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^*(F_i(T) - \lambda^* G_i(T)) \geq \sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^*(F_i(T^*) - \lambda^* G_i(T^*)).$$

From the above inequality and (2), we have

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^*(F_i(T) - \lambda^* G_i(T)) \geq 0. \tag{9}$$

By virtue of Lemma 2.1, we have

$$\begin{aligned} \psi(T) &\equiv \max_{1 \leq i \leq k} \frac{F_i(T)}{G_i(T)} = \max_{1 \leq i \leq k} \frac{\bar{\alpha}_i^1(T, T^*) F_i(T)}{\bar{\alpha}_i^1(T, T^*) G_i(T)} \quad (\text{as } \bar{\alpha}_i^1(T, T^*) > 0, i \in K) \\ &= \max_{\mu \in U} \frac{\sum_{i \in K} \mu_i \bar{\alpha}_i^1(T, T^*) F_i(T)}{\sum_{i \in K} \mu_i \bar{\alpha}_i^1(T, T^*) G_i(T)} \\ &\geq \frac{\sum_{i \in K} \mu_i^* \bar{\alpha}_i^1(T, T^*) F_i(T)}{\sum_{i \in K} \mu_i^* \bar{\alpha}_i^1(T, T^*) G_i(T)} \geq \lambda^*, \quad (\text{by (9)}). \end{aligned} \tag{10}$$

Hence, in view of (2) and (10), we conclude that T^* is an optimal solution of (P). \square

The proof of the next theorem is analogues to Theorem 3.2, and hence being omitted.

Theorem 3.3. *Let $T^* \in X_o$ and let there exist $\mu^* \in U, \nu^* \in R_+^m$ and $\lambda^* \in R_+$ satisfying (1) to (3). If*

- (i) $[(\mathcal{B}_1(\cdot, \lambda^*, \mu^*), \dots, \mathcal{B}_k(\cdot, \lambda^*, \mu^*)), (\mathcal{C}_1(\cdot, \nu^*), \dots, \mathcal{C}_m(\cdot, \nu^*))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V -prestrict quasi strictly pseudo type-I at T^* , and
- (ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$,

then T^* is an optimal solution of (P).

In order to prove next sufficient optimality theorems, we introduce some additional notations. Let $\{I_o, I_1, \dots, I_r\}$ be a partition of the index set M , thus $I_\beta \subseteq M$ for each $\beta \in \{0, 1, \dots, r\}$, $I_\beta \cap I_\gamma = \emptyset$, if $\beta \neq \gamma$ and $\cup_{\beta=0}^r I_\beta = M$. For fixed λ^*, μ^* and ν^* , we define the real-valued functions on \mathcal{A}^n as:

$$\mathcal{E}_i(T, \lambda^*, \mu^*, \nu^*) = \mu_i^* \left[F_i(T) - \lambda^* G_i(T) + \sum_{j \in I_o} \nu_j^* H_j(T) \right], \quad i \in K,$$

and

$$\mathcal{L}_\beta(T, \nu^*) = \sum_{j \in I_\beta} \nu_j^* H_j(T), \quad \beta = 1, 2, \dots, r.$$

Theorem 3.4. *Let $T^* \in X_o$ and let there exist $\mu^* \in U, \nu^* \in R_+^m$ and $\lambda^* \in R_+$ satisfying (1) to (3). If*

- (i) $[(\mathcal{E}_1(\cdot, \lambda^*, \mu^*, \nu^*), \dots, \mathcal{E}_k(\cdot, \lambda^*, \mu^*, \nu^*)), (\mathcal{L}_1(\cdot, \nu^*), \dots, \mathcal{L}_r(\cdot, \nu^*))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V -pseudoquasi type-I at T^* , and

$$(ii) \bar{\rho}^1 + \sum_{\beta=1}^r \bar{\rho}_\beta^2 \geq 0,$$

then T^* is an optimal solution of (P).

Proof. From (3) and $\bar{\alpha}_j^2(T, T^*) > 0, j \in M$, we have $\sum_{j \in I_\beta} \bar{\alpha}_j^2(T, T^*) \nu_j^* H_j(T^*) = 0, \beta = 1, 2, \dots, r$, which along with hypothesis (i) gives

$$\mathcal{F} \left(T, T^*; \sum_{j \in I_\beta} \nu_j^* DH_j(T^*) \right) \leq -\bar{\rho}_\beta^2 d^2(\theta(T, T^*)), \beta = 1, 2, \dots, r. \tag{11}$$

By (1) and the sublinearity of \mathcal{F} , we obtain

$$\begin{aligned} & \mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^* [DF_i(T^*) - \lambda^* DG_i(T^*)] + \sum_{j \in I_o} \nu_j^* DH_j(T^*) \right) \\ & \quad + \sum_{\beta=1}^r \mathcal{F} \left(T, T^*; \sum_{j \in I_\beta} \nu_j^* DH_j(T^*) \right) \geq 0, \\ & \mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^* [DF_i(T^*) - \lambda^* DG_i(T^*)] + \sum_{j \in I_o} \nu_j^* DH_j(T^*) \right) \\ & \quad - \sum_{\beta=1}^r \bar{\rho}_\beta^2 d^2(\theta(T, T^*)) \geq 0, \text{ (by (11)).} \end{aligned}$$

As hypothesis (ii) holds, and $\sum_{i \in K} \mu_i^* = 1$, the above inequality becomes

$$\begin{aligned} & \mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^* [DF_i(T^*) - \lambda^* DG_i(T^*)] + \sum_{j \in I_o} \nu_j^* DH_j(T^*) \right) \\ & \quad + \bar{\rho}^1 d^2(\theta(T, T^*)) \geq 0. \end{aligned} \tag{12}$$

Inequality (12) together with hypothesis (i) implies

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mathcal{E}_i(T, \lambda^*, \mu^*, \nu^*) \geq \sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mathcal{E}_i(T^*, \lambda^*, \mu^*, \nu^*),$$

which in view of (2) and (3) yields

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^* \left[F_i(T) - \lambda^* G_i(T) + \sum_{j \in I_o} \nu_j^* H_j(T) \right] \geq 0.$$

Since $T \in X_o$ and $\nu^* \in R_+^m$, we get

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^* (F_i(T) - \lambda^* G_i(T)) \geq 0,$$

which is identical to (9). Therefore, following the proof of Theorem 3.2, we conclude that T^* is an optimal solution of (P). \square

Theorem 3.5. Let $T^* \in X_0$ and let there exist $\mu^* \in U$, $\nu^* \in R_+^m$ and $\lambda^* \in R_+$ satisfying (1) to (3). If

(i) $[(\mathcal{E}_1(\cdot, \lambda^*, \mu^*, \nu^*), \dots, \mathcal{E}_k(\cdot, \lambda^*, \mu^*, \nu^*)), (\mathcal{L}_1(\cdot, \nu^*), \dots, \mathcal{L}_r(\cdot, \nu^*))]$ is $(\mathcal{F}, \tilde{\alpha}, \tilde{\rho}, \theta)$ -V-prestrictquasi strictlypseudo type-I at T^* , and

$$(ii) \tilde{\rho}^1 + \sum_{\beta=1}^r \tilde{\rho}_\beta^2 \geq 0,$$

then T^* is an optimal solution of (P).

Proof. The proof follows on the similar lines of Theorem 3.4. □

4. Parameter-free sufficient optimality conditions

In this section, we discuss parameter-free versions of the parametric sufficient optimality conditions for (P) obtained in Section 3. For stating optimality Theorems 4.2 - 4.5, we use the functions $\mathcal{C}_j(\cdot, \nu^*)$ and $\mathcal{L}_\beta(\cdot, \nu^*)$ defined in Section 3, and the real-valued functions $\Lambda_i(\cdot, T^*, \mu^*)$ and $\Pi_i(\cdot, T^*, \mu^*, \nu^*)$ defined, for fixed T^* , μ^* , and ν^* on \mathcal{A}^n , and for all $i \in K$, by

$$\Lambda_i(T, T^*, \mu^*) = \mu_i^* [\Gamma(T^*, \mu^*)F_i(T) - \Theta(T^*, \mu^*)G_i(T)],$$

and

$$\Pi_i(T, T^*, \mu^*, \nu^*) = \mu_i^* [\Gamma(T^*, \mu^*)F_i(T) - \Theta(T^*, \mu^*)G_i(T) + \sum_{j \in I_0} \nu_j^* H_j(T)].$$

Theorem 4.1. Let $T^* \in X_0$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ such that

$$\mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*)DF_i(T^*) - \Theta(T^*, \mu^*)DG_i(T^*)] + \sum_{j \in M} \nu_j^* DH_j(T^*) \right) \geq 0, \forall T \in \mathcal{A}^n, \tag{1}$$

$$\mu_i^* [\Gamma(T^*, \mu^*)F_i(T^*) - \Theta(T^*, \mu^*)G_i(T^*)] = 0, i \in K, \tag{2}$$

$$\psi(T^*) \equiv \max_{1 \leq i \leq k} \frac{F_i(T^*)}{G_i(T^*)} = \frac{\Theta(T^*, \mu^*)}{\Gamma(T^*, \mu^*)}, \tag{3}$$

$$\nu_j^* H_j(T^*) = 0, j \in M. \tag{4}$$

If

(i) $[(\Gamma(T^*, \mu^*)F_1(\cdot) - \Theta(T^*, \mu^*)G_1(\cdot), \dots, \Gamma(T^*, \mu^*)F_k(\cdot) - \Theta(T^*, \mu^*)G_k(\cdot)), (H_1(\cdot), \dots, H_m(\cdot))]$ is $(\mathcal{F}, \alpha, \rho, \theta)$ -V-type-I at T^* ,

(ii) $\alpha_1^1 = \alpha_2^1 = \dots = \alpha_k^1 = \alpha_1^2 = \alpha_2^2 = \dots = \alpha_m^2 = \delta$, and

(iii) $\sum_{i \in K} \mu_i^* \rho_i^1 + \sum_{j \in M} \nu_j^* \rho_j^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. The inequality (1) along with the sublinearity of \mathcal{F} implies

$$\mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*)DF_i(T^*) - \Theta(T^*, \mu^*)DG_i(T^*)] \right)$$

$$+\mathcal{F}\left(T, T^*; \sum_{j \in M} \nu_j^* DH_j(T^*)\right) \geq 0. \tag{5}$$

By hypothesis (i), we get

$$\begin{aligned} &(\Gamma(T^*, \mu^*)F_i(T) - \Theta(T^*, \mu^*)G_i(T)) - (\Gamma(T^*, \mu^*)F_i(T^*) - \Theta(T^*, \mu^*)G_i(T^*)) \\ &\geq \mathcal{F}(T, T^*; \alpha_i^1(T, T^*)[\Gamma(T^*, \mu^*)DF_i(T^*) - \Theta(T^*, \mu^*)DG_i(T^*)]) + \rho_i^1 d^2(\theta(T, T^*)), \\ &\quad -H_j(T^*) \geq \mathcal{F}(T, T^*; \alpha_j^2(T, T^*)DH_j(T^*)) + \rho_j^2 d^2(\theta(T, T^*)), i \in K, j \in M. \end{aligned}$$

On multiplying the first inequality by $\mu^* \geq 0$, second by $\nu^* \geq 0$, and using (2), (4) and (ii), we obtain

$$\begin{aligned} \mu_i^*(\Gamma(T^*, \mu^*)F_i(T) - \Theta(T^*, \mu^*)G_i(T)) &\geq \mathcal{F}(T, T^*; \delta(T, T^*)\mu_i^*[\Gamma(T^*, \mu^*)DF_i(T^*) \\ &\quad - \Theta(T^*, \mu^*)DG_i(T^*)]) + \mu_i^*\rho_i^1 d^2(\theta(T, T^*)), i \in K, \\ 0 &\geq \mathcal{F}(T, T^*; \delta(T, T^*)\nu_j^*DH_j(T^*)) + \nu_j^*\rho_j^2 d^2(\theta(T, T^*)), j \in M. \end{aligned}$$

Taking summation over $i \in K$ and $j \in M$, respectively and then adding to obtain

$$\begin{aligned} &\sum_{i \in K} \mu_i^*(\Gamma(T^*, \mu^*)F_i(T) - \Theta(T^*, \mu^*)G_i(T)) \\ &\geq \mathcal{F}(T, T^*; \delta(T, T^*) \sum_{i \in K} \mu_i^*[\Gamma(T^*, \mu^*)DF_i(T^*) - \Theta(T^*, \mu^*)DG_i(T^*)]) \\ &+\mathcal{F}(T, T^*; \delta(T, T^*) \sum_{j \in M} \nu_j^*DH_j(T^*)) + (\sum_{i \in K} \mu_i^*\rho_i^1 + \sum_{j \in M} \nu_j^*\rho_j^2)d^2(\theta(T, T^*)), \end{aligned}$$

which in view of (5), (iii), $\delta(T, T^*) > 0$ and the sublinearity of \mathcal{F} gives

$$\sum_{i \in K} \frac{\mu_i^*}{\delta(T, T^*)}(\Gamma(T^*, \mu^*)F_i(T) - \Theta(T^*, \mu^*)G_i(T)) \geq 0.$$

Since $\delta(T, T^*) > 0$, the above inequality becomes

$$\sum_{i \in K} \mu_i^*(\Gamma(T^*, \mu^*)F_i(T) - \Theta(T^*, \mu^*)G_i(T)) \geq 0. \tag{6}$$

From Lemma 2.1, we get

$$\begin{aligned} \psi(T) &\equiv \max_{1 \leq i \leq k} \frac{F_i(T)}{G_i(T)} = \max_{\mu \in U} \frac{\sum_{i \in K} \mu_i F_i(T)}{\sum_{i \in K} \mu_i G_i(T)} \\ &\geq \frac{\sum_{i \in K} \mu_i^* F_i(T)}{\sum_{i \in K} \mu_i^* G_i(T)} \geq \frac{\Theta(T^*, \mu^*)}{\Gamma(T^*, \mu^*)} = \psi(T^*), \text{ (by (6) and (3)).} \end{aligned} \tag{7}$$

Hence T^* is an optimal solution of (P). □

Theorem 4.2. Let $T^* \in X_0$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ satisfying (1) to (4). If

- (i) $[(\Lambda_1(\cdot, T^*, \mu^*), \dots, \Lambda_k(\cdot, T^*, \mu^*)), (\mathcal{C}_1(\cdot, \nu^*), \dots, \mathcal{C}_m(\cdot, \nu^*))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V -pseudoquasi type-I at T^* , and
- (ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. Following the proof of Theorem 3.2, we get

$$\mathcal{F} \left(T, T^*; \sum_{j \in M} \nu_j^* DH_j(T^*) \right) + \bar{\rho}^2 d^2(\theta(T, T^*)) \leq 0,$$

which by the virtue of (5) and hypothesis (ii) implies

$$\mathcal{F}(T, T^*; \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*) DF_i(T^*) - \Theta(T^*, \mu^*) DG_i(T^*)]) + \bar{\rho}^1 d^2(\theta(T, T^*)) \geq 0.$$

This inequality together with hypothesis (i) gives

$$\begin{aligned} & \sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^* (\Gamma(T^*, \mu^*) F_i(T) - \Theta(T^*, \mu^*) G_i(T)) \\ & \geq \sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^* (\Gamma(T^*, \mu^*) F_i(T^*) - \Theta(T^*, \mu^*) G_i(T^*)), \end{aligned}$$

or

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^* (\Gamma(T^*, \mu^*) F_i(T) - \Theta(T^*, \mu^*) G_i(T)) \geq 0, \quad (\text{by (2)}). \quad (8)$$

By Lemma 2.1, (3) and (8), it follows that

$$\begin{aligned} \psi(T) & \equiv \max_{1 \leq i \leq k} \frac{F_i(T)}{G_i(T)} = \max_{1 \leq i \leq k} \frac{\bar{\alpha}_i^1(T, T^*) F_i(T)}{\bar{\alpha}_i^1(T, T^*) G_i(T)} \quad (\text{as } \bar{\alpha}_i^1(T, T^*) > 0, i \in K) \\ & = \max_{\mu \in U} \frac{\sum_{i \in K} \mu_i \bar{\alpha}_i^1(T, T^*) F_i(T)}{\sum_{i \in K} \mu_i \bar{\alpha}_i^1(T, T^*) G_i(T)} \\ & \geq \frac{\sum_{i \in K} \mu_i^* \bar{\alpha}_i^1(T, T^*) F_i(T)}{\sum_{i \in K} \mu_i^* \bar{\alpha}_i^1(T, T^*) G_i(T)} \geq \frac{\Theta(T^*, \mu^*)}{\Gamma(T^*, \mu^*)} = \psi(T^*). \end{aligned}$$

Hence $\psi(T) \geq \psi(T^*)$, which shows that T^* is an optimal solution of (P). \square

Theorem 4.3. Let $T^* \in X_o$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ satisfying (1) to (4). If

- (i) $[(\Lambda_1(\cdot, T^*, \mu^*), \dots, \Lambda_k(\cdot, T^*, \mu^*)), (\mathcal{C}_1(\cdot, \nu^*), \dots, \mathcal{C}_m(\cdot, \nu^*))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V -prestrictquasi strictlypseudo type-I at T^* , and
- (ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. The proof follows on the similar lines of Theorem 4.2. \square

Theorem 4.4. Let $T^* \in X_o$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ satisfying (1) to (4). If

- (i) $[(\Pi_1(\cdot, T^*, \mu^*, \nu^*), \dots, \Pi_k(\cdot, T^*, \mu^*, \nu^*)), (\mathcal{L}_1(\cdot, \nu^*), \dots, \mathcal{L}_r(\cdot, \nu^*))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V -pseudoquasi type-I at T^* , and
- (ii) $\bar{\rho}^1 + \sum_{\beta=1}^r \bar{\rho}_\beta^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. Following Theorem 3.4, one can get the inequality

$$\mathcal{F} \left(T, T^*; \sum_{j \in I_\beta} \nu_j^* DH_j(T^*) \right) \leq -\bar{\rho}_\beta^2 d^2(\theta(T, T^*)), \beta = 1, 2, \dots, r. \tag{9}$$

The inequality (1) and the sublinearity of \mathcal{F} give

$$\begin{aligned} \mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*) DF_i(T^*) - \Theta(T^*, \mu^*) DG_i(T^*)] + \sum_{j \in I_0} \nu_j^* DH_j(T^*) \right) \\ + \sum_{\beta=1}^r \mathcal{F} \left(T, T^*; \sum_{j \in I_\beta} \nu_j^* DH_j(T^*) \right) \geq 0, \end{aligned}$$

or

$$\begin{aligned} \mathcal{F} \left(T, T^*; \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*) DF_i(T^*) - \Theta(T^*, \mu^*) DG_i(T^*)] + \sum_{j \in I_0} \nu_j^* DH_j(T^*) \right) \\ - \sum_{\beta=1}^r \bar{\rho}_\beta^2 d^2(\theta(T, T^*)) \geq 0, \text{ (by (9))}, \end{aligned}$$

which in view of hypothesis (ii), and $\sum_{i \in K} \mu_i^* = 1$ yields that

$$\begin{aligned} \mathcal{F}(T, T^*; \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*) DF_i(T^*) - \Theta(T^*, \mu^*) DG_i(T^*)] + \sum_{j \in I_0} \nu_j^* DH_j(T^*)) \\ + \bar{\rho}^1 d^2(\theta(T, T^*)) \geq 0. \tag{10} \end{aligned}$$

The inequality (10) along with hypothesis (i) implies

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \Pi_i(T, T^*, \mu^*, \nu^*) \geq \sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \Pi_i(T^*, T^*, \mu^*, \nu^*),$$

which by (2) and (4) gives

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^* [\Gamma(T^*, \mu^*) F_i(T) - \Theta(T^*, \mu^*) G_i(T) + \sum_{j \in I_0} \nu_j^* H_j(T)] \geq 0.$$

Since $T \in X_0$ and $\nu^* \in R_+^m$, we get

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^* (\Gamma(T^*, \mu^*) F_i(T) - \Theta(T^*, \mu^*) G_i(T)) \geq 0.$$

Now, following the proof of Theorem 4.2, we obtain the required condition that T^* is an optimal solution of (P). □

Theorem 4.5. Let $T^* \in X_0$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ satisfying (1) to (4). If

(i) $[(\Pi_1(\cdot, T^*, \mu^*, \nu^*), \dots, \Pi_k(\cdot, T^*, \mu^*, \nu^*)), (\mathcal{L}_1(\cdot, \nu^*)), \dots, \mathcal{L}_r(\cdot, \nu^*)]$ is $(\mathcal{F}, \tilde{\alpha}, \bar{\rho}, \theta)$ -V-prestrictquasi strictlypseudo type-I at T^* , and

$$(ii) \tilde{\rho}^1 + \sum_{\beta=1}^r \tilde{\rho}_\beta^2 \geq 0,$$

then T^* is an optimal solution of (P).

Proof. The proof is analogues to that of Theorem 4.4. \square

5. Duality model I

In this section, duality theorems are proved for the following parametric dual problem:

(DI) Maximize λ
subject to

$$\mathcal{F} \left(T, Y; \sum_{i \in K} \mu_i [DF_i(Y) - \lambda DG_i(Y)] + \sum_{j \in M} \nu_j DH_j(Y) \right) \geq 0, \forall T \in \mathcal{A}^n, \quad (1)$$

$$\mu_i (F_i(Y) - \lambda G_i(Y)) \geq 0, \quad i \in K, \quad (2)$$

$$\nu_j H_j(Y) \geq 0, \quad j \in M, \quad (3)$$

$$Y \in \mathcal{A}^n, \lambda \in R_+, \mu \in U, \nu \in R_+^m,$$

In order to prove duality theorems, we use the functions $\mathcal{B}_i(\cdot, \lambda, \mu)$ and $\mathcal{C}_j(\cdot, \nu)$ introduced in Section 3.

Theorem 5.1 (Weak Duality). *Let T and (Y, λ, μ, ν) be the feasible solutions of (P) and (DI), respectively. If*

(i) $[(\mathcal{B}_1(\cdot, \lambda, \mu), \dots, \mathcal{B}_k(\cdot, \lambda, \mu)), (\mathcal{C}_1(\cdot, \nu), \dots, \mathcal{C}_m(\cdot, \nu))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ -V-pseudo quasi type-I at Y , and

$$(ii) \bar{\rho}^1 + \bar{\rho}^2 \geq 0,$$

then $\psi(T) \geq \lambda$.

Proof. From (3) and $\bar{\alpha}_j^2(T, Y) > 0, j \in M$, we have

$$- \sum_{j \in M} \bar{\alpha}_j^2(T, Y) \nu_j H_j(Y) \leq 0. \quad (4)$$

The inequality (4) and hypothesis (i) give

$$\mathcal{F}(T, Y; \sum_{j \in M} \nu_j DH_j(Y)) + \bar{\rho}^2 d^2(\theta(T, Y)) \leq 0. \quad (5)$$

The inequality (5) along with (1), the sublinearity of \mathcal{F} and hypothesis (ii) implies

$$\mathcal{F}(T, Y; \sum_{i \in K} \mu_i [DF_i(Y) - \lambda DG_i(Y)]) + \bar{\rho}^1 d^2(\theta(T, Y)) \geq 0,$$

which in view of hypothesis (i) gives

$$\sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mu_i (F_i(T) - \lambda G_i(T)) \geq \sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mu_i (F_i(Y) - \lambda G_i(Y)).$$

From the above inequality and (2), we obtain

$$\sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mu_i (F_i(T) - \lambda G_i(T)) \geq 0. \tag{6}$$

By virtue of Lemma 2.1, we have

$$\begin{aligned} \psi(T) &\equiv \max_{1 \leq i \leq k} \frac{F_i(T)}{G_i(T)} = \max_{1 \leq i \leq k} \frac{\bar{\alpha}_i^1(T, Y) F_i(T)}{\bar{\alpha}_i^1(T, Y) G_i(T)} \quad (\text{as } \bar{\alpha}_i^1(T, Y) > 0, i \in K) \\ &= \max_{a \in U} \frac{\sum_{i \in K} a_i \bar{\alpha}_i^1(T, Y) F_i(T)}{\sum_{i \in K} a_i \bar{\alpha}_i^1(T, Y) G_i(T)} \\ &\geq \frac{\sum_{i \in K} \mu_i \bar{\alpha}_i^1(T, Y) F_i(T)}{\sum_{i \in K} \mu_i \bar{\alpha}_i^1(T, Y) G_i(T)} \geq \lambda \quad (\text{by (6)}). \end{aligned}$$

Hence $\psi(T) \geq \lambda$. □

Theorem 5.2 (Weak Duality). *Let T and (Y, λ, μ, ν) be the feasible solutions of (P) and (DI), respectively. If*

(i) $[(\mathcal{B}_1(\cdot, \lambda, \mu), \dots, \mathcal{B}_k(\cdot, \lambda, \mu)), (C_1(\cdot, \nu), \dots, C_m(\cdot, \nu))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V - pre-strict quasistrictlypseudo type-I at Y , and

(ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$,

then $\psi(T) \geq \lambda$.

Proof. The proof follows on the similar lines of Theorem 5.1. □

Theorem 5.3 (Strong Duality). *Let T^* be a regular optimal solution of (P), let $\mathcal{F}(T, T^*; DF(T^*)) = \sum_{p \in N} \langle D_p F(T^*), \chi_{T_p} - \chi_{T_p^*} \rangle$ for any differentiable function $F : \mathcal{A}^n \rightarrow R$ and $T \in \mathcal{A}^n$, and assume that any of the weak duality theorems (Theorem 5.1 or 5.2) holds for all feasible solutions of (DI). Then there exist $\lambda^* \in R_+$, $\mu^* \in U$, and $\nu^* \in R_+^m$ such that $(T^*, \lambda^*, \mu^*, \nu^*)$ is an optimal solution of (DI) and the objective values of (P) and (DI) are equal.*

Proof. By Theorem 2.1, there exist $\lambda^* \in R_+$, $\mu^* \in U$, and $\nu^* \in R_+^m$ such that $(T^*, \lambda^*, \mu^*, \nu^*)$ is a feasible solution of (DI). Since $\psi(T^*) = \lambda^*$, it follows from the weak duality theorem (Theorem 5.1 or 5.2) that $(T^*, \lambda^*, \mu^*, \nu^*)$ is an optimal solution of (DI). □

Theorem 5.4 (Strict Converse Duality). *Let T^* be a regular optimal solution of (P) and let $(Y^*, \lambda^*, \mu^*, \nu^*)$ be an optimal solution of (DI) such that*

(i) $[(\mathcal{B}_1(\cdot, \lambda^*, \mu^*), \dots, \mathcal{B}_k(\cdot, \lambda^*, \mu^*)), (C_1(\cdot, \nu^*), \dots, C_m(\cdot, \nu^*))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V - strictlypseudoquasi type-I at Y^* , and

(ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$.

Also, suppose that, for any differentiable function $F : \mathcal{A}^n \rightarrow R$, $\mathcal{F}(T, T^; DF(T^*)) = \sum_{p \in N} \langle D_p F(T^*), \chi_{T_p} - \chi_{T_p^*} \rangle$, $T \in \mathcal{A}^n$. Then $Y^* = T^*$, that is, Y^* is an optimal solution of (P), and $\psi(T^*) = \lambda^*$.*

Proof. We assume $Y^* \neq T^*$ and exhibit a contradiction. Now, following the proof of weak duality (Theorem 5.1), we get inequality (6) as strict inequality so

that we obtain $\psi(T^*) > \lambda^*$, which contradicts the fact that $\psi(T^*) = \lambda^*$. Hence $Y^* = T^*$. \square

6. Duality model II

In this section, we present more general parametric dual model by making use of partitioning scheme introduced in Section 3.

(DII) Maximize λ
subject to

$$\mathcal{F} \left(T, Y; \sum_{i \in K} \mu_i [DF_i(Y) - \lambda DG_i(Y)] + \sum_{j \in M} \nu_j DH_j(Y) \right) \geq 0, \forall T \in \mathcal{A}^n, \quad (1)$$

$$\mu_i (F_i(Y) - \lambda G_i(Y) + \sum_{j \in I_o} \nu_j H_j(Y)) \geq 0, \quad i \in K, \quad (2)$$

$$\sum_{j \in I_\beta} \nu_j H_j(Y) \geq 0, \quad \beta = 1, 2, \dots, r, \quad (3)$$

$$Y \in \mathcal{A}^n, \lambda \in R_+, \mu \in U, \nu \in R_+^m.$$

We use the functions $\mathcal{E}_i(\cdot, \lambda, \mu, \nu)$ and $\mathcal{C}_j(\cdot, \nu)$ introduced in Section 3.

Theorem 6.1 (Weak Duality). *Let T and (Y, λ, μ, ν) be the feasible solutions of (P) and (DII), respectively. If*

(i) $[(\mathcal{E}_1(\cdot, \lambda, \mu, \nu), \dots, \mathcal{E}_k(\cdot, \lambda, \mu, \nu)), (\mathcal{L}_1(\cdot, \nu), \dots, \mathcal{L}_r(\cdot, \nu))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ -V-pseudo quasi type-I at Y , and

(ii) $\bar{\rho}^1 + \sum_{\beta=1}^r \bar{\rho}_\beta^2 \geq 0$,

then $\psi(T) \geq \lambda$.

Proof. From (3) and $\bar{\alpha}_j^2(T, Y) > 0, j \in M$, we have $-\sum_{j \in I_\beta} \bar{\alpha}_j^2(T, Y) \nu_j H_j(Y) \leq 0, \beta = 1, 2, \dots, r$, which along with hypothesis (i) gives

$$\mathcal{F}(T, Y; \sum_{j \in I_\beta} \nu_j DH_j(Y)) \leq -\bar{\rho}_\beta^2 d^2(\theta(T, Y)), \beta = 1, 2, \dots, r. \quad (4)$$

By (1) and the sublinearity of \mathcal{F} , we obtain

$$\begin{aligned} &\mathcal{F}(T, Y; \sum_{i \in K} \mu_i [DF_i(Y) - \lambda DG_i(Y)] + \sum_{j \in I_o} \nu_j DH_j(Y)) \\ &\quad + \sum_{\beta=1}^r \mathcal{F}(T, Y; \sum_{j \in I_\beta} \nu_j DH_j(Y)) \geq 0, \end{aligned}$$

or

$$\begin{aligned} &\mathcal{F}(T, Y; \sum_{i \in K} \mu_i [DF_i(Y) - \lambda DG_i(Y)] + \sum_{j \in I_o} \nu_j DH_j(Y)) \\ &\quad - \sum_{\beta=1}^r \bar{\rho}_\beta^2 d^2(\theta(T, Y)) \geq 0, \text{ (by (4)).} \end{aligned}$$

As hypothesis (ii) holds, and $\sum_{i \in K} \mu_i = 1$, the above inequality becomes

$$\mathcal{F}(T, Y; \sum_{i \in K} \mu_i [DF_i(Y) - \lambda DG_i(Y) + \sum_{j \in I_0} \nu_j DH_j(Y)]) + \bar{\rho}^1 d^2(\theta(T, Y)) \geq 0. \quad (5)$$

The inequality (5) together with hypothesis (i) implies

$$\sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mathcal{E}_i(T, \lambda, \mu, \nu) \geq \sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mathcal{E}_i(Y, \lambda, \mu, \nu),$$

which in view of (2) and (3) yields

$$\sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mu_i (F_i(T) - \lambda G_i(T) + \sum_{j \in I_0} \nu_j H_j(T)) \geq 0.$$

Since $T \in X_0$ and $\nu \geq 0$, we get

$$\sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mu_i (F_i(T) - \lambda G_i(T)) \geq 0,$$

Now by Theorem 5.1, we obtain $\psi(T) \geq \lambda$. □

Theorem 6.2 (Strong Duality). *Let T^* be a regular optimal solution of (P), let $\mathcal{F}(T, T^*; DF(T^*)) = \sum_{p \in N} \langle D_p F(T^*), \chi_{T_p} - \chi_{T_p^*} \rangle$ for any differentiable function $F : \mathcal{A}^n \rightarrow R$ and $T \in \mathcal{A}^n$, and assume that the assumptions of weak duality (Theorem 6.1) hold for all feasible solutions of (DII). Then there exist $\lambda^* \in R_+$, $\mu^* \in U$, and $\nu^* \in R_+^m$ such that $(T^*, \lambda^*, \mu^*, \nu^*)$ is an optimal solution of (DII) and the objective values of (P) and (DII) are equal.*

Proof. By Theorem 2.1, there exist $\lambda^* \in R_+$, $\mu^* \in U$, and $\nu^* \in R_+^m$ such that $(T^*, \lambda^*, \mu^*, \nu^*)$ is a feasible solution of (DII). Since $\psi(T^*) = \lambda^*$, it follows from the weak duality (Theorem 6.1) that $(T^*, \lambda^*, \mu^*, \nu^*)$ is an optimal solution of (DII). □

Theorem 6.3 (Strict Converse Duality). *Let T^* be a regular optimal solution of (P) and let $(Y^*, \lambda^*, \mu^*, \nu^*)$ be an optimal solution of (DII) such that*

(i) $[(\mathcal{E}_1(\cdot, \lambda^*, \mu^*, \nu^*), \dots, \mathcal{E}_k(\cdot, \lambda^*, \mu^*, \nu^*)), (\mathcal{L}_1(\cdot, \nu^*), \dots, \mathcal{L}_r(\cdot, \nu^*))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ -V-strictly pseudoquasi type-I at Y^* , and

(ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$.

Also, suppose that, for any differentiable function $F : \mathcal{A}^n \rightarrow R$, $\mathcal{F}(T, T^; DF(T^*)) = \sum_{p \in N} \langle D_p F(T^*), \chi_{T_p} - \chi_{T_p^*} \rangle$, $T \in \mathcal{A}^n$. Then $Y^* = T^*$, that is, Y^* is an optimal solution of (P), and $\psi(T^*) = \lambda^*$.*

Proof. The proof follows on the similar lines of Theorem 5.4. □

7. Duality model III

This section deals with the following parameter-free dual model for (P) and corresponding weak, strong and strict converse duality theorems.

(DIII) Maximize $\phi(Y, \mu, \nu) = \frac{\sum_{i \in K} \mu_i F_i(Y)}{\sum_{i \in K} \mu_i G_i(Y)}$

subject to

$$\mathcal{F} \left(T, Y; \sum_{i \in K} \mu_i [\Gamma(Y, \mu) DF_i(Y) - \Theta(Y, \mu) DG_i(Y)] + \sum_{j \in M} \nu_j DH_j(Y) \right) \geq 0, \forall T \in \mathcal{A}^n, \tag{1}$$

$$\mu_i [\Gamma(Y, \mu) F_i(Y) - \Theta(Y, \mu) G_i(Y)] \geq 0, i \in K, \tag{2}$$

$$\nu_j H_j(Y) \geq 0, j \in M. \tag{3}$$

$$Y \in \mathcal{A}^n, \mu \in U, \nu \in R_+^m.$$

We use the functions $\Lambda_i(\cdot, T, \mu)$ and $\mathcal{C}_j(\cdot, \nu)$ introduced in Section 4. Throughout this section, we assume that $\Theta(Y, \mu) \geq 0$ and $\Gamma(Y, \mu) > 0$ for all Y and μ such that (Y, μ, ν) is a feasible solution of the considered dual problem.

Theorem 7.1 (Weak Duality). *Let T and (Y, μ, ν) be the feasible solutions of (P) and (DIII), respectively. If*

(i) $[(\Lambda_1(\cdot, T, \mu), \dots, \Lambda_k(\cdot, T, \mu)), (\mathcal{C}_1(\cdot, \nu), \dots, \mathcal{C}_m(\cdot, \nu))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V - pseudo quasi type-I at Y , and (ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$, then $\psi(T) \geq \phi(Y, \mu, \nu)$.

Proof. By (3) and $\bar{\alpha}_j^2(T, Y) > 0, j \in M$, we get $-\sum_{j \in M} \bar{\alpha}_j^2(T, Y) \nu_j H_j(Y) \leq 0$,

which along with hypothesis (i) yields

$$\mathcal{F}(T, Y; \sum_{j \in M} \nu_j DH_j(Y)) + \bar{\rho}^2 d^2(\theta(T, Y)) \leq 0. \tag{4}$$

The inequality (4) together with (1), sublinearity of \mathcal{F} , and hypothesis (ii) implies

$$\mathcal{F}(T, Y; \sum_{i \in K} \mu_i [\Gamma(Y, \mu) DF_i(Y) - \Theta(Y, \mu) DG_i(Y)]) + \bar{\rho}^1 d^2(\theta(T, Y)) \geq 0,$$

which by virtue of hypothesis (i) gives

$$\begin{aligned} & \sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mu_i (\Gamma(Y, \mu) F_i(T) - \Theta(Y, \mu) G_i(T)) \\ & \geq \sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mu_i (\Gamma(Y, \mu) F_i(Y) - \Theta(Y, \mu) G_i(Y)), \end{aligned}$$

or

$$\sum_{i \in K} \bar{\alpha}_i^1(T, Y) \mu_i (\Gamma(Y, \mu) F_i(T) - \Theta(Y, \mu) G_i(T)) \geq 0, \text{ (by (2)).} \tag{5}$$

From Lemma 2.1, we have

$$\begin{aligned} \psi(T) & \equiv \max_{1 \leq i \leq k} \frac{F_i(T)}{G_i(T)} = \max_{1 \leq i \leq k} \frac{\bar{\alpha}_i^1(T, Y) F_i(T)}{\bar{\alpha}_i^1(T, Y) G_i(T)} \text{ (as } \bar{\alpha}_i^1(T, Y) > 0, i \in K) \\ & = \max_{a \in U} \frac{\sum_{i \in K} a_i \bar{\alpha}_i^1(T, Y) F_i(T)}{\sum_{i \in K} a_i \bar{\alpha}_i^1(T, Y) G_i(T)} \end{aligned}$$

$$\geq \frac{\sum_{i \in K} \mu_i \bar{\alpha}_i^1(T, Y) F_i(T)}{\sum_{i \in K} \mu_i \bar{\alpha}_i^1(T, Y) G_i(T)} \geq \frac{\Theta(Y, \mu)}{\Gamma(Y, \mu)}, \text{ (by (5)).}$$

Hence $\psi(T) \geq \frac{\Theta(Y, \mu)}{\Gamma(Y, \mu)} = \phi(Y, \mu, \nu)$. \square

Theorem 7.2 (Weak Duality). *Let T and (Y, μ, ν) be the feasible solutions of (P) and $(DIII)$, respectively. If*

(i) $[(\Lambda_1(\cdot, T, \mu), \dots, \Lambda_k(\cdot, T, \mu)), (C_1(\cdot, \nu), \dots, C_m(\cdot, \nu))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V - prestrict quasistrictlypseudo type-I at Y , and

(ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$,

then $\psi(T) \geq \phi(Y, \mu, \nu)$.

Proof. The proof follows on the similar lines of Theorem 7.1. \square

Theorem 7.3 (Strong Duality). *Let T^* be a regular optimal solution of (P) , let $\mathcal{F}(T, T^*; DF(T^*)) = \sum_{p \in N} \langle D_p F(T^*), \chi_{T_p} - \chi_{T_p^*} \rangle$ for any differentiable function $F: \mathcal{A}^n \rightarrow R$ and $T \in \mathcal{A}^n$, and assume that the assumptions of any of the weak duality theorems (Theorem 7.1 or 7.2) hold for all feasible solutions of $(DIII)$. Then there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ such that (T^*, μ^*, ν^*) is an optimal solution of $(DIII)$ and the objective values of (P) and $(DIII)$ are equal.*

Proof. By Theorem 2.2, there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ such that (T^*, μ^*, ν^*) is a feasible solution of $(DIII)$. Since $\psi(T^*) = \phi(Y^*, \mu^*, \nu^*)$, it follows from the weak duality theorem (Theorem 7.1 or 7.2) that (T^*, μ^*, ν^*) is an optimal solution of $(DIII)$. \square

Theorem 7.4 (Strict Converse Duality). *Let T^* be a regular optimal solution of (P) and (Y^*, μ^*, ν^*) be an optimal solution of $(DIII)$ such that*

(i) $[(\Lambda_1(\cdot, T^*, \mu^*), \dots, \Lambda_k(\cdot, T^*, \mu^*)), (C_1(\cdot, \nu^*), \dots, C_m(\cdot, \nu^*))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$ - V - strictlypseudo quasi type-I at Y^* , and

(ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0$.

Also, suppose that, for any differentiable function $F: \mathcal{A}^n \rightarrow R$, $\mathcal{F}(T, T^; DF(T^*)) = \sum_{p \in N} \langle D_p F(T^*), \chi_{T_p} - \chi_{T_p^*} \rangle$, $T \in \mathcal{A}^n$. Then $Y^* = T^*$, that is, Y^* is an optimal solution of (P) , and $\psi(T^*) = \phi(Y^*, \mu^*, \nu^*)$.*

Proof. The proof is similar to that of Theorem 5.4. \square

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