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CHARACTERIZATION OF MONOIDS BY REGULAR RIGHT ACTS

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ABSTRACT. The purpose of this paper is to continue the investigation of monoids over which various properties of acts happen to coincide. Special concern is to characterize monoids by regular right acts. In particular there are given some characterizations of monoids over which all right acts that satisfy condition (E)(or condition (P)) are regular.

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1. Introduction

For many years, a fruitful area of research in semigroup theory has been the investigation of properties connected with projectivity, flatness of acts over monoids. Many authors investigated the conditions on a monoid which are necessary and sufficient to make these properties of right acts coincide. But the problems of characterization of monoids by regular acts were quite a few considered.

Liu([7]) characterized monoids over which all flat acts are regular. In [12], Tran characterized monoids by regular acts and the involving results are as follow:

(i) every S-act is regular if and only if $S = \{1\}$ or $S = \{1, 0\}$;

(ii) every nontrivial cyclic S-act is regular if and only if $S = \{1, 0\}$ or S is a group of prime order;

(iii) regularity coincides with projectivity if and only if S is a group;

(iv) regularity coincides with strongly faithfulness if and only if S is left cancellative.

There are still however a number of open problems for characterization of monoids by regular right acts. Thus we continue to study of monoids over which

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certain distinct pairs of right acts in fact coincide, being chiefly interested in regular right acts. The main purpose of this article is dealing with the relationship between the regularity and condition (P)(or condition (E)) of right acts.

Throughout this paper S will denote a monoid. We refer the reader to [3] for basic definitions and terminologies relating to semigroup and acts over monoids.

A *right S-act A* over a monoid S is a set on which S acts unitarily from the right in the usual way, that is, $(as)t = a(st)$ and $a1 = a$ where $a \in A, s, t \in S$ and 1 is the identity of S. According to Knauer([4]), a right S-act A is *projective* if and only if A is the direct sum of cyclic subacts which each direct summand is isomorphic to principal right ideal of S generated by an idempotent. If for every $a \in S$, there is an element x in S such that $axa = a$, then the semigroup S is said to be *von Neumann regular.* Extending the notion of this von Neumann regularity of semigroup, we define regularity of acts over a monoid S. An right S-act A is called *regular* if for any element $a \in S$, there is an S-homomorphism $f : aS \to S$ such that $af(a) = a$. Obviously this notion is an extension of von Neumann regularity, but a monoid S which is regular as an right S-act need not be von Neumann regular. A right S-act A is said to *satisfy condition (P)* if whenever $au = a'u'$ with $a, a' \in A,\ u,\ u' \in S$, there exist $a'' \in A,\ s, s' \in S$ with $a = a''s, a' = a''s'$ and $su = s'u'$. A right S-act A is said to *satisfy condition(E)* if whenever $au = au'$ with $a \in A,\ u, u' \in S$, there exist $a'' \in A,\ s \in S$ with $a = a''s$ and $su = su'$.

2. Monoids over which all right acts that satisfy condition (E) are regular

Moon([8]) showed that all right S-acts satisfy condition (E) if and only if $S = \{1\}$ or $\{1, 0\}$. Tran([12]) also showed that $S = \{1\}$ or $\{1, 0\}$ if and only if all right S-acts are regular. We now characterize monoids over which the class of regular right acts coincides with the class of right acts satisfying condition (E).

Lemma 1. *Every regular right act over a monoid S satisfies condition (E).*

Proof. Let A be a regular right S-act and let $a \in A, s, t \in S$ such that $as = at$. If A is regular then there is an idempotent element e of S such that $ae = a$ and $es = et$. Hence we take $a' = a$ in A and $u = e$ in S. Then it is clear that A satisfies condition (E) since $a = ae = a'u$ and $us = es = et = ut$. $\square$

Note. There is an S-act satisfying condition (E) which is not regular. For example, Z_4 satisfies condition (E) but it is not regular as an Z_4-act. Hence it is natural to investigate monoids over which all S-acts that satisfy condition (E) are regular.

Theorem 1. *Let S be a monoid. Then the following statements are equivalent;*
(*i*) *S is either a group or a band.*
(*ii*) *All right S-acts that satisfy condition (E) are regular.*

Proof. (*i*) $\Rightarrow$ (*ii*) Let A be an S-act that satisfies condition (E) and let $a \in A$. If $as = at$ for $s, t \in S$, then there is an element $a' \in A$ and $u \in S$ such that $a = a'u$ and $us = ut$.

If S is a group then $us = ut$ implies $s = t$, hence we can define an S-isomorphism $f : aS \to S$ by $f(as) = s$ for all s in S. Thus A is regular since $af(a) = a$. we now assume that S is a band and define a S-homomorphism $f : aS \to S$ by $f(as) = us$ where $a = a'u$. Since $u^2 = u$ and $af(a) = au = (a'u)u = a'u = a$, A is regular.

(*ii*) $\Rightarrow$ (*i*) Let I be the set of non-right invertible elements of S. If I is empty then all elements of S are right invertible so that S is a group. If I is not empty then I is a proper right ideal of S since $1 \notin I$.

Let x, y, z be symbols not representing elements of S and define $A(I) = ((S \setminus I) \times \{x, y\}) \cup (I \times \{z\})$ with a right S-action on $A(I)$ by

$$(s, u)t = \begin{cases} (\text{st, u}) & \text{if } st \notin I, \\ (\text{st, z}) & \text{if } st \in I \end{cases} \tag{1}$$

and

$$(s, z)t = (st, z)$$

where $s, t \in S$ and $u \in \{x, y\}$. Then $A(I)$ is a right S-act that satisfies condition (E) and for any $s \in S$, (s, z) is in $A(I)$.

If all right acts that satisfy condition (E), then $A(I)$ is regular and then there exists an S-homomorphism $f : (s, z)S \to S$ such that

$(s, z)f(s, z) = (s, z)$ and hence $sf(s, z) = s$. Since $s = 1s = f(1, z)s = f(s, z)$, it implies that $s = sf(s, z) = ss = s^2$. Thus S is a band. □

3. Monoids over which all right acts that satisfy condition (E) are projective

A right act is said to be *strongly flat* if it satisfies both condition (P) and condition (E). It is well known that any projective act is strongly flat so that it satisfies condition (E).

In [12], Tran showed that S is a group if and only if the class of regular acts coincides with the class of projective acts. Thus we investigate the monoid over which the class of acts satisfying condition (E) coincides with the class of projective acts.

Lemma 2. ([1]) *All right S-acts satisfying condition (E) are strongly flat if and only if S is a group.*

Theorem 2. *Let S be a monoid. Then the following statements are equivalent;*
(i) S is a group.
(ii) A right S-act A is projective if and only if A satisfies condition (E).

Proof. $(i) \Rightarrow (ii)$. Assume that S is a group and let A be a right S-act. If A is projective then it is strongly flat. Hence it satisfies both condition (P) and condition (E). If A satisfies condition (E) where S is a group then it is regular so that it is projective by theorem 1 and [12].

$(ii) \Rightarrow (i)$. Assume that the class of projective right S-acts coincides with the class of right S-acts that satisfy condition (E), and let I be the set of non-right invertible elements of S, If I is empty then all elements of S are right invertible so that S is a group.

If I is not empty then I is a proper right ideal of S and then the set $A(I)$ is an right S-act that satisfies condition (E) and fails to satisfy condition (P). But if $A(I)$ is projective by assumption then it should satisfy condition (P) so that it is strongly flat. Hence S is a group by lemma 2. □

Lemma 3. ([8]) *Let S be a monoid and $x \in S$. Then the cyclic right S-act of the form $S/\rho(x, x^2)$ satisfies condition (E) if and only if $x = x^2$ or x is right invertible.*

Theorem 3. *Let S be a monoid. Then all right acts that satisfy condition (E) are projective if and only if $S = \{1\}$.*

Proof. If $S = \{1\}$ then it is clear that all S-acts that satisfy condition (E) are projective , hence we only prove that the converse holds. Assume that all S-acts satisfying condition (E) are projective and let x be any element of S. If x is right invertible then the cyclic right S-act

$$S/\rho(x, x^2) = S/\rho(x, 1)$$

satisfies condition (E) and hence it is projective by assumption. Since every projective right S-act is strongly flat, $x^{n+1} = x^n$ for some $n > 0$ and then $x = 1$. Let I be the set of all non right invertible elements of S. Then $I = S \setminus \{1\}$.

If I is nonempty then the right S-act $A(I)$ is projective by assumption so that it is the direct sum of cyclic subacts. Hence the direct summand

$$((S \setminus I) \times \{x, y\}) = \{(1, x), (1, y)\}$$

of $A(I)$ should be the cyclic subact or the direct sum of cyclic subacts. Hence if it is cyclic generated by $(1, x)$ then $(1, y) = (1, x)t$ for some $t \in S$.

If $t = 1$ then $(1, y) = (1, x)$. If $t \neq 1$ then $(1, y) = (t, z)$. Thus it is not generated by $(1, x)$. Similarly it is not generated by $(1, y)$. Moreover it can not be written as the direct sum of cyclic subacts by a similar argument. Thus $I = S \setminus \{1\}$ must be empty and then $S = \{1\}$. □

4. Monoids over which all right acts satisfy condition (P) are regular

In this section we are dealing with relationship between regularity and condition (P) for right acts.

Theorem 4. *Every cyclic regular right act over a monoid S satisfies condition (P).*

Proof. Let A be a cyclic right act over a monoid S, *that is*, $A = xS$ for some $x \in A$, and assume that $as = a't$ for $a, a' \in A$ and $s, t \in S$. If $a, a' \in A$ then there are p and q in S such that

$$a = xp, a' = xq.$$

If A is regular, then there is an S-homomorphism $f : xS \to S$ such that $xf(x) = x$. Take $a'' = x \in A$ and $u = f(a), v = f(a')$ in S. Then

$$a''u = xf(a) = xf(x)p = xp = a, a''v = xf(a') = xf(x)q = xq = a'$$

and

$$us = f(a)s = f(as) = f(a't) = f(a')t = vt.$$

Thus A satisfies condition (P). □

Lemma 4. ([10]) *All right S-acts satisfy condition (P) if and only if S is a group.*

A monoid S is called *right PP* if every principal right ideal of S is projective. It can be shown that S is right PP if and only if for every $x \in S$, there exists $e^2 = e \in S$ such that $xe = x$ and $xu = xv$ implies $eu = ev$. Every regular and every left cancellative monoid is right PP.

Lemma 5. ([6]) *S is a (von Neumann) regular semigroup if and only if S is right PP and all regular acts are flat.*

Theorem 5. *If S is right PP and all regular S-acts satisfy condition (P) then S is von Neumann regular. In particular, the converse also holds if $|E(S)| = 1$ where $E(S)$ is the set of idempotent elements of S.*

Proof. Assume that S is right PP and all regular right S-acts satisfy condition (P). Then S is von Neumann regular by theorem 5 since all right S-acts that satisfy condition (P) are flat.

For the converse we assume that S is von Neumann regular with $|E(S)| = 1$. If S is von Neumann regular then S is clearly right PP and for every $a \in S$, there is some $a' \in S$ such that $a = aa'a$. Since both aa' and $a'a$ belong to $E(S)$, $aa' = 1 = a'a$ so that S is a group.

Thus it is clear that all regular right S-acts satisfy condition (P) by lemma 4. □

Note. It is said to *satisfy condition* (A) for right acts if all right S-acts satisfy *ACC* (*ascending chain condition*) for cyclic subacts.

Theorem 6. *For a monoid S, the followings are equivalent;*

(i) all elements of $S \setminus \{1\}$ are right zeroes.

(ii) S satisfies condition (A) and all right S-acts that satisfy condition (P) are regular.

Proof. Assume that every element of $S \setminus \{1\}$ is a right zero. Then for every $x, y \in S \setminus \{1\}$, $xy = y$ and $yx = x$, hence there are only two cyclic subacts xS and S for all $x \in S$. Thus S satisfies condition (A).

Let A be an S-act that satisfies condition (P) and for any $a \in A$, let I be the set $\{s \in S \setminus \{1\} | as = a\}$. We first assume that I is empty. Then $at \neq a$ for all $t \in S \setminus \{1\}$, so we can define a map

$$f : aS \to S \text{ by } f(at) = t.$$

If $as = at$ for any s, t in S then there are a'' in A and u, v in S such that

$$a = a''u = a''v \text{ and } us = vt.$$

If $s = 1$ and $t \neq 1$ then $a = as = at$, so t is in I. Similarly if $s \neq 1$ and $t = 1$ then $as = at = a$, s is in I. Since these contradict to our assumption for I, both s and t should be not equal to 1. But if they are right zeroes then $us = vt$ implies $s = t$.

Moreover since $af(a) = a$, it is easily seen that A is regular. Assume that I is nonempty and let a be in $S \setminus \{1\}$. Since $as = a$, we define a map

$$f : aS \to sS \text{ by } f(a) = s \text{ and } f(at) = t$$

for all $t \in S \setminus \{1\}$. If $at = at'$ for $t, t' \in S \setminus \{1\}$ then it is easily seen that $t = t'$ by the same arguments before. Thus f is a well defined S-homomorphism with $af(a) = as = a$ and then A is regular.

Assume that S satisfies condition (A) and all right S-acts that satisfy condition (P) are regular, and let $x \in S \setminus \{1\}$. Since a monoid S satisfies condition (P) as an S-act, it is a regular right S-act by assumption and then there exists an S-homomorphism $f : xS \to S$ such that $xf(x) = x$. Thus if $u = f(x)$ then

$$u^2 = u \text{ and } xu = xf(x) = x$$

so that xS is isomorphic to uS. But if S satisfies condition (A) for cyclic subacts then xS is actually equal to uS and hence we have that $xu = x = ux$. Let n be the least positive integer such that x^n is a right zero and assume that $n > 2$. If x^n is a right zero then

$$xx^n = x^n = xx^{n-1} \Rightarrow ux^n = ux^{n-1} \Rightarrow x^n = x^{n-1}$$

since $ux = x$. If $n = 2$ then x^2 is a right zero, hence

$$x^3 = x^2 \Rightarrow ux^2 = ux \Rightarrow x^2 = x.$$

Since both cases contradict to the choice of n, n should be equal to 1, *i.e.*, x is a right zero. □

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