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SOME BOUNDED OPERATORS IN SPACES OF TYPE W^{Φ}

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ABSTRACT. For some generalized N -function Φ , some Hölder type inequalities and bounded operators on spaces of type $W_M^{\Omega,\Phi}$ generalizing the W^p-spaces due to Pathak and Upadhyay are obtained.

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For a nondecreasing right-continuous function a with $a(0) = 0$, $a(t) > 0$ if $t > 0$ and $a(\infty) = \infty$, define $M(x) = \int_0^{|x|} a(t)dt$, which is called an N-function. We know that M is continuous, convex and $\lim_{|x|\to 0} M(x)/x = 0$. We define $v(s) = \sup_{a(t)\le s} t$, $s \ge 0$ and $\Omega(y) = \int_0^{|y|} v(s)ds$. Then Ω is an N- function and $\Omega(y) = \sup_x \left\{x|y| - M(x)\right\}$. We call (M, Ω) is a complementary pair of N-functions. In the sequel, let M, Ω and Φ be N- functions.

Now the class K_M^{Φ} is defined as the set of all differentiable functions $\varphi(x)$ satisfying

$$\|\varphi\|_{M,q}^{\Phi} = inf\left\{\lambda \ge 0 \;\left|\; \int_{-\infty}^{\infty} \Phi\left(\frac{1}{\lambda}\left|e^{M(ax)}\varphi^{(q)}(x)\right|\right) dx \le 1\right.\right\} < \infty$$

for each nonnegative integer q where the positive constant a depends upon the function φ. In general K_M^{Φ} is not a vector space.

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The spaces W_M^{Φ} is defined to be the linear convex hull of the class K_M^{Φ}. W_M^{Φ} is a Banach space under the norm $\|\cdot\|_{M,q}^{\Phi}$ and can be regarded as the union of countably normed spaces $W_{M,a}^{\Phi}$ of all infinitly differentiable functions φ, which for any $\delta > 0$ satisfy

$$\|\varphi\|_{M,q,a}^{\Phi} = inf\left\{\lambda \mid \int_{-\infty}^{\infty} \Phi\left(\frac{1}{\lambda}\left|e^{M[(a-\delta)x]}\varphi^{(q)}(x)\right|\right) dx \le 1\right\}.$$

for $q = 0, 1, 2, ...$

The class $K^{\Omega,\Phi}$ is defined to be the set of all entire functions $\varphi(z)$, $z = x + iy$ satisfying, for $k = 0, 1, 2, \dots$,

$$\|\varphi\|^{\Omega,k,\Phi} = \sup_y inf\left\{\lambda \middle| \int_{-\infty}^{\infty} \Phi\left(\frac{1}{\lambda}\left|e^{-\Omega(by)}z^k\varphi(z)\right|\right) dx \le 1\right\} < \infty.$$

The spaces $W^{\Omega,\Phi}$ is defined to be the linear convex hull of the class $K^{\Omega,\Phi}$ with the norm $\|\cdot\|^{\Omega,k,\Phi}$. The space $W^{\Omega,b,\Phi}$ is the set of all functions φ in $W^{\Omega,\Phi}$ with the norm $(k = 0, 1, 2, ...)$

$$\|\varphi\|^{\Omega,k,b,\Phi} = \sup_y inf\left\{\lambda \middle| \int_{-\infty}^{\infty} \Phi\left(\frac{1}{\lambda}\left|e^{-\Omega[(b+\rho)y]}z^k\varphi(z)\right|\right) dx \le 1\right\}.$$

We denote by $K_M^{\Omega,\Phi}$ the set of all entire analytic functions $\varphi(z), z = x + iy$ with the norm

$$\|\varphi\|_M^{\Omega,\Phi} = \sup_y inf\left\{\lambda \mid \int_{-\infty}^{\infty} \Phi\left(\frac{1}{\lambda}\left|e^{[M(ax)-\Omega(by)]}\varphi(z)\right|\right) dx \le 1\right\} < \infty.$$

The space $W_M^{\Omega,\Phi}$ is the convex hull of the class $K_M^{\Omega,\Phi}$ with the norm $\|\cdot\|_M^{\Omega,\Phi}$ and can also be represented as a union of countably normed linear spaces. We denote by $W_{M,a}^{\Omega,b,\Phi}$ the set of all functions belonging to the spaces $W_M^{\Omega,\Phi}$ with the norm

$$\|\varphi\|_{M,a}^{\Omega,b,\Phi} = \sup_y inf\left\{\lambda \mid \int_{-\infty}^{\infty} \Phi\left(\frac{1}{\lambda}\left|e^{M[(a-\delta)x]-\Omega[(b+\rho)y]}\varphi(z)\right|\right) dx \le 1\right\}.$$

In the sequel we denote by $A \cdot B$ the collection of all products $f_1 \cdot f_2$ for any functions $f_1 \in A$ and $f_2 \in B$, and for the simplicity of notation, let $[b, k, f] =| e^{-\Omega[(b+\frac{\rho}{2})y]}z^k f(z) \mid$.

If $\Phi(x) = x^p, 1 \le p < \infty$, we have

$$W_M^{\Phi} = W_M^p, W_{M,a}^{\Phi} = W_{M,a}^p, W^{\Omega,\Phi} = W^{\Omega,p} \text{ and } W^{\Omega,b,\Phi} = W^{\Omega,b,p}[3].$$

Theorem 1. *(a) For any $\varphi(z) \in W_M^{\Phi}$, the differentiation $\acute{\varphi}(x)$ and the multiplication $x\varphi(x)$ by x of $\varphi(x)$ belongs to the space W_M^{Φ}.*

(b) For any $\varphi(z) \in W^{\Omega,\Phi}$, the differentiation $\acute{\varphi}(z)$ and the multiplication $z\varphi(z)$ by z of $\varphi(z)$ belongs to the space $W^{\Omega,\Phi}$.

(c) For any $\varphi(z) \in W_M^{\Omega,\Phi}$, the differentiation $\acute{\varphi}(z)$ and the multiplication $z\varphi(z)$ by z of $\varphi(z)$ belongs to the space $W_M^{\Omega,\Phi}$.

Proof. (a)For any $\varphi(x) \in W_M^{\Phi}$, $\mid \varphi^{(q)}(x) \mid e^{M(ax)} \leq C_q$ implies that $\mid \varphi^{(q+1)}(x) \mid e^{M(ax)} \leq C_{q+1}$ and

$$\begin{aligned} \mid [x\varphi(x)]^{(q)} \mid &\leq \Big(\mid x \mid C_q + qC_{q-1} \Big) e^{-M(ax)} \\ &\leq C_q C_\delta e^{-M[(a-\delta)x]} + qC_{q-1} e^{-M(ax)} \leq C_q^{'} e^{-M[(a-\delta)x]}, \end{aligned}$$

where $C_q^{'} = C_q C_\delta + qC_{q-1}$. Hence $\acute{\varphi}(x) \in W_M^{\Phi}$ and $x\varphi(x) \in W_M^{\Phi}$.

(b) For any $\varphi(z) \in W^{\Omega,\Phi}$, $\mid z^k \varphi(z) \mid e^{-\Omega(by)} \leq C_k$. Since $\mid z^{k-1}\varphi(z) \mid e^{-\Omega[(b+r)y]} \leq C_{k-1}$ and

$$\mid [z^k \varphi(z)]\acute{} \mid \leq \frac{1}{r} C_k e^{-\Omega(b(y+r))} \leq \frac{1}{r} C_k e^{\Omega[(b+r)y]+C_r} \leq C_{kr} e^{\Omega[(b+r)y]},$$

we have

$$\begin{aligned} \mid z^k \acute{\varphi}(z) \mid &\leq \mid [z^k \varphi(z)] \mid + k \mid z^{k-1}\varphi(z) \mid \\ &\leq C_{kr} e^{\Omega[(b+r)y]} + kC_{k-1} e^{\Omega[(b+r)y]} \leq C_{kr}^{'} e^{\Omega[(b+r)y]}, \end{aligned}$$

which means that $\acute{\varphi}(z) \in W^{\Omega,\Phi}$. Also $\mid z^{k+1}\varphi(z) \mid e^{-\Omega(by)} \leq C_{k+1}$, which implies $z\varphi(z) \in W^{\Omega,\Phi}$.

(c) For any $\varphi(z) \in W_M^{\Omega,\Phi}$, since $\mid \acute{\varphi}(z) \mid e^{-M[(x-r)a]-\Omega[b(y+r)]} \leq \frac{C}{r}$ and $\mid z\varphi(z) \mid e^{M[(a-r)x]-\Omega[(b+r)y]} \leq C_r$, we have

$$\mid \acute{\varphi}(z) \mid e^{M[(a-r)x]-\Omega[(b+r)y]} \leq C_r \quad \text{and} \quad \mid z\varphi(z) \mid e^{\Omega[(a-\delta)x]-\Omega[(b+\rho)y]} \leq C_{\delta\rho},$$

which implies $\acute{\varphi}(z) \in W_{M,a}^{\Omega,b,\Phi}$ and $z\varphi(z) \in W_{M,a}^{\Omega,b,\Phi}$. □

By the convexity of $\Phi_i (i = 1, 2, 3)$, we have the following lemma;

Lemma 2. [2] *If N-functions* $\Phi_i (i = 1, 2, 3)$ *satisfy the inequality*

$$\limsup_{x\to\infty} \Phi_1^{-1}(x)\Phi_2^{-1}(x)/\Phi_3^{-1}(x) < \infty$$

for any $x \geq 0$, *then for* $f_1 \in W^{\Omega,b_0,\Phi_1}$ *and* $f_2 \in W^{\Omega,b,\Phi_2}$, *we have* $f_1 f_2 \in W^{\Omega,b_0+b,\Phi_3}$, *that is,*

$$W^{\Omega,b_0,\Phi_1} \cdot W^{\Omega,b,\Phi_2} \subset W^{\Omega,b_0+b,\Phi_3}$$

and

$$\|f_1 f_2\|^{\Omega,k_0+k,b_0+b,\Phi_3} \leq 2\|f_1\|^{\Omega,k_0,b_0,\Phi_1}\|f_2\|^{\Omega,k,b,\Phi_2}$$

Lemma 3. *If N-functions* $\Phi_i (i = 1, 2, 3)$ *satisfy the inequality, for any* $x \geq 0$ *and some positive constant* α, $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \alpha\Phi_3^{-1}(x)$, *then for nonnegative* x *and* y, *we have* $\Phi_3\left(\frac{xy}{\alpha}\right) \leq \Phi_1(x) + \Phi_2(y)$, *where* $\Phi_i^{-1}(x) = inf\{\Phi_i(t) > x\}$.

Proof. By the definition of the inverse, we have $\Phi_i(\Phi_i^{-1}(x)) \leq x \leq \Phi_i^{-1}(\Phi_i(x))$. Let $x, y \in R^+$ be arbitrarily fixed. Then $\Phi_1(x) \leq \Phi_2(y)$ or its order would be reversed. In the first case, we have

$$xy \leq \Phi_1^{-1}(\Phi_1(x))\Phi_2^{-1}(\Phi_2(y)) \leq \Phi_1^{-1}(\Phi_2(y))\Phi_2^{-1}(\Phi_2(y)) \leq \alpha\Phi_3^{-1}(\Phi_2(y))$$

Hence $\Phi_3\left(\frac{xy}{\alpha}\right) \leq \Phi_2(y)$. If the second case is true, we get $\Phi_3\left(\frac{xy}{\alpha}\right) \leq \Phi_1(x)$, so

$$\Phi_3\left(\frac{xy}{\alpha}\right) \leq max\{\Phi_1(x), \Phi_2(y)\} \leq \Phi_1(x) + \Phi_2(y),$$

which completes the proof. □

Theorem 4. *If N-functions* $\Phi_i (i = 1, 2, 3)$ *satisfy the inequality, for any* $x \geq 0$ *and some positive constant* α, $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \alpha\Phi_3^{-1}(x)$, *then for* $f_1 \in W^{\Omega,b_0,\Phi_1}$ *and* $f_2 \in W^{\Omega,b,\Phi_2}$, *we have* $f_1 f_2 \in W^{\Omega,b_0+b,\Phi_3}$, *that is,*

$$W^{\Omega,b_0,\Phi_1} \cdot W^{\Omega,b,\Phi_2} \subset W^{\Omega,b_0+b,\Phi_3}$$

and

$$\|f_1 f_2\|^{\Omega,k_0+k,b_0+b,\Phi_3} \leq 2\alpha\|f_1\|^{\Omega,k_0,b_0,\Phi_1}\|f_2\|^{\Omega,k,b,\Phi_2}$$

Proof. Without loss of generality, we may assume that

$$\| f_1 \|^{\Omega,0,b_0,\Phi_1} = \| f_2 \|^{\Omega,k,b,\Phi_2} = 1.$$

By the Lemma 3, we have the following inequalities; for any ε,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Phi_3\left(\frac{1}{2\alpha(1+\varepsilon)^2}[b_0+b, k_0+k, f_1f_2]\right)dx \\
&\leq \int_{-\infty}^{\infty} \frac{1}{2}\Phi_3\left(\frac{1}{\alpha(1+\varepsilon)^2}[b_0,k_0,f_1]\cdot[b,k,f_2]\right)dx \\
&\leq \frac{1}{2}\int_{-\infty}^{\infty}\Phi_1\left(\frac{1}{1+\varepsilon}[b_0,k_0,f_1]\right)dx + \frac{1}{2}\int_{-\infty}^{\infty}\Phi_2\left(\frac{1}{1+\varepsilon}[b,k,f_2]\right)dx \\
&\leq \frac{1}{2}\left(\|f_1\|^{\Omega,k_0,b_0,\Phi_1} + \|f_2\|^{\Omega,k,b,\Phi_2}\right) \leq \frac{1}{2}+\frac{1}{2} = 1,
\end{aligned}$$

where $(f_1f_2)(z) = f_1(z)f_2(z)$; which implies that

$$\|f_1f_2\|^{\Omega,k_0+k,b_0+b,\Phi_3} \leq 2\alpha(1+\varepsilon)^2\|f_1\|^{\Omega,k_0,b_0,\Phi_1}\|f_2\|^{\Omega,k,b,\Phi_2},$$

which completes the proof. □

Lemma 5. *For some constant α_i and the coresponding complementary pairs (M_i, Φ_i), the followings are equivalent;*

(*a*) $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \alpha\Phi_3^{-1}(x)$;
(*b*) $\Phi_3(\alpha_1 xy) \leq \Phi_1(x) + \Phi_2(y)$ *for some* $\alpha > 0$, *and* $x, y \geq x_0 \geq 0$;
(*c*) $M_1(\alpha_2 yz) \leq \Phi_2(y) + M_3(z)$, $y, z \geq x_2 \geq 0$;
(*d*) $M_2(\alpha_3 xz) \leq \Phi_1(x) + M_3(z)$, $x, z \geq x_3 \geq 0$.

Proof. By the Theorem 4 and Lemma 5, this is proved. □

By the Lemma 3 and properties of the coresponding complementary pairs of N-functions, we have the following corollary;

Corollary 6. *For some constant c_i and the coresponding complementary pairs $(M_i, \Phi_i)(i = 1, 2, 3)$ of N-functions in Lemma 5, if the inequality $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \alpha\Phi_3^{-1}(x)$ holds, then we have the followings;*

(*a*) *for* $f_1 \in W^{\Omega,b_0,\Phi_1}$ *and* $f_2 \in W^{\Omega,b,\Phi_2}$, $f_1f_2 \in W^{\Omega,b_0+b,\Phi_3}$, *that is,*

$$W^{\Omega,b_0,\Phi_1} \cdot W^{\Omega,b,\Phi_2} \subset W^{\Omega,b_0+b,\Phi_3}$$

and

$$\|f_1f_2\|^{\Omega,k_0+k,b_0+b,\Phi_3} \leq \frac{2}{c_1}\|f_1\|^{\Omega,k_0,b_0,\Phi_1}\|f_2\|^{\Omega,k,b,\Phi_2}.$$

(b) *for* $f_1 \in W^{\Omega,b_0,\Phi_2}$ *and* $f_2 \in W^{\Omega,b,M_3}$, $f_1 f_2 \in W^{\Omega,b_0+b,M_1}$, *that is,*

$$W^{\Omega,b_0,\Phi_2} \cdot W^{\Omega,b,M_3} \subset W^{\Omega,b_0+b,M_1}$$

and

$$\|f_1 f_2\|^{\Omega,k_0+k,b_0+b,M_1} \leq \frac{2}{c_2}\|f_1\|^{\Omega,k_0,b_0,\Phi_2}\|f_2\|^{\Omega,k,b,M_3}.$$

(c) *for* $f_1 \in W^{\Omega,b_0,\Phi_1}$ *and* $f_2 \in W^{\Omega,b,M_3}$, $f_1 f_2 \in W^{\Omega,b_0+b,M_1}$, *that is,*

$$W^{\Omega,b_0,\Phi_1} \cdot W^{\Omega,b,M_3} \subset W^{\Omega,b_0+b,M_2}$$

and

$$\|f_1 f_2\|^{\Omega,k_0+k,b_0+b,M_2} \leq \frac{2}{c_3}\|f_1\|^{\Omega,k_0,b_0,\Phi_1}\|f_2\|^{\Omega,k,b,M_3}.$$

Theorem 7. *Let* $\Phi_i, i = 1,2,3$ *be N-functions such that* $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \alpha\Phi_3^{-1}(x)$ *and* $f(z)$ *be an entire analytic function satisfying*

$$\| (1+ | x |^h)^{-1} f(z) \|^{\Omega,k_0,b_0,\Phi_1} = D_{\Phi_1} < \infty.$$

Then we have $\varphi f \in W^{\Omega,k,b_0+b,\Phi_3}$ *for all* $\varphi \in W^{\Omega,k,b,\Phi_2}$.

Proof. By Theorem 4, we have, for any ε,

$$\begin{aligned}
&\left\|\frac{\varphi f}{(1+\varepsilon)^2}\right\|^{\Omega,k_0+k,b_0+b,\Phi_3} \\
&\leq 2\alpha\left\|\frac{1}{1+\varepsilon}(1+|x|^h)^{-1} f(z)\right\|^{\Omega,k_0,b_0,\Phi_1}\left\|\frac{1}{1+\varepsilon}(1+|x|^h)\varphi(z)\right\|^{\Omega,k,b,\Phi_2} \\
&\leq 2\alpha\left\|\frac{1}{1+\varepsilon}(1+|x|^h)^{-1} f(z)\right\|^{\Omega,k_0,b_0,\Phi_1}\left\|\frac{1}{1+\varepsilon}(\varphi(z)+|x|^h\varphi(z))\right\|^{\Omega,k,b,\Phi_2} \\
&\leq 2\alpha D_{\Phi_1}(\| \varphi \|^{\Omega,k,b,\Phi_2} + \| \varphi \|^{\Omega,k+h,b,\Phi_2}) < \infty,
\end{aligned}$$

which impliies that $\varphi f \in W^{\Omega,k_0+k,b_0+b,\Phi_3}$ for all $\varphi \in W^{\Omega,k,b,\Phi_2}$. □

If $\Phi(x) = x^p, 1 \leq p < \infty$, we have $W_M^{\Omega,\Phi} = W_M^{\Omega,p}$, $W_{M,a}^{\Omega,b,\Phi} = W_{M,a}^{\Omega,b,p}$[2, 3].

Theorem 8. *If $\Phi_i(i=1,2,3)$ are N-functions such that the inequality $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \alpha\Phi_3^{-1}(x)$ for any $x \geq 0$ and $f(z)$ is an entire function satisfying*

$$\sup_y inf\left\{\lambda \Big| \int_{-\infty}^{\infty} \Phi_1\left(\frac{1}{\lambda} \mid e^{-[M(a_0x)+\Omega(b_0y)]}f(z)\Big|\right) dx \leq 1\right\} = D_{\Phi_1} < \infty.$$

Then $\varphi f \in W_{M,a-a_0}^{\Omega,b+b_0,\Phi_3}$ for all $\varphi \in W_{M,a}^{\Omega,b,\Phi_2}$.

Proof. By the simillar argument as the proof of Theorem 4, We have

$$\begin{aligned}
&\left\| \frac{1}{\alpha(1+\epsilon)^2}\varphi f \right\|_{M,a-a_0}^{\Omega,b+b_0,\Phi_3} \\
&= sup_y inf\Big\{\lambda \mid \int_{-\infty}^{\infty} \Phi_3(\frac{1}{\lambda(1+\epsilon)^2} \mid e^{[M((a-a_0-\delta)x)-\Omega((b+b_0+\rho)y)]} \\
&\quad \times f(z)\frac{1}{\alpha}\varphi(z) \mid)dx \leq 1\Big\} \\
&\leq \sup_y inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi_3(\frac{1}{\lambda(1+\varepsilon)^2} \mid e^{[M((a-\delta)x)-M(a_0x)-\Omega((b+\rho)y)-\Omega(b_0y)]} \\
&\quad \times f(z)\frac{1}{\alpha}\varphi(z) \mid)dx \leq 1\} \\
&\leq \sup_y inf\left\{\lambda \Big| \int_{-\infty}^{\infty} \Phi_1\left(\frac{1}{\lambda(1+\varepsilon)} \mid e^{-[M(a_0x)+\Omega(b_0y)]}f(z) \mid\right) dx \leq 1\right\} \\
&\quad + \sup_y inf\left\{\lambda \Big| \int_{-\infty}^{\infty} \Phi_2\left(\frac{1}{1+\varepsilon} \mid e^{[M((a-\delta)x)-\Omega((b+\rho)y)]}\varphi(z) \mid\right) dx \leq 1\right\} \\
&\leq \left(D_{\Phi_1} + \parallel \varphi \parallel_{M,a}^{\Omega,b,\Phi_2}\right) < \infty.
\end{aligned}$$

□

Lemma 9. *For some constant $\alpha > 0$, if N-functions $\Phi_i(i=1,2,3)$ satisfy the inequality*

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \alpha x\Phi_3^{-1}(x) \cdots\cdots (*)$$

for any $x \geq 0$, then for nonnegative x and y, we have

$$\frac{xy}{\alpha} \leq \Phi_1(x)\Phi_3^{-1}(\Phi_2(y)) + \Phi_2(y)\Phi_3^{-1}(\Phi_1(x)),$$

where $\Phi_i^{-1}(x) = \{t \mid \Phi_i(t) > x\}$ for all n.

Proof. By the similar way as in the proof of Lemma 3, this is proved; for given $x \geq 0$ and $y \geq 0$, either $\Phi_1 \leq \Phi_2$ or its reversed order hold. Suppose $\Phi_1 \leq \Phi_2$. Since $x/\Phi_i^{-1}(x)$ is increasing and $\Phi_i(\Phi_i^{-1}(x)) \leq x \leq \Phi_i^{-1}(\Phi_i(x))$ for $i = 1,2,3$, we have

$$\begin{aligned}\frac{xy}{\alpha} &\leq \frac{y\Phi_1^{-1}(\Phi_1(x))\Phi_2^{-1}(\Phi_1(x))}{\alpha\Phi_2^{-1}(\Phi_1(x))} \leq \frac{y\Phi_1(x)\Phi_3^{-1}(\Phi_1(x))}{\alpha\Phi_2^{-1}(\Phi_1(x))} \\ &\leq \frac{y\Phi_2(y)\Phi_3^{-1}(\Phi_1(x))}{\Phi_2^{-1}(\Phi_2(y))} \leq \Phi_2(y)\Phi_3^{-1}(\Phi_1(x))\end{aligned}$$

If $\Phi_1(x) > \Phi_2(y)$, then

$$\frac{xy}{\alpha} \leq \Phi_1(x)\Phi_3^{-1}(\Phi_2(y)).$$

In both cases, for nonnegative x and y,

$$\begin{aligned}\frac{xy}{\alpha} &\leq \max\left\{\Phi_2(y)\Phi_3^{-1}(\Phi_1(x)), \Phi_1(x)\Phi_3^{-1}(\Phi_2(y))\right\} \\ &\leq \Phi_1(x)\Phi_3^{-1}(\Phi_2(y)) + \Phi_2(y)\Phi_3^{-1}(\Phi_1(x))\end{aligned}$$

□

Theorem 10. *For some constant $\alpha > 0$, if N-functions $\Phi_i (i = 1,2,3)$ satisfy the inequality $(*)$ in Lemma 7, then for $f_1 \in W^{\Omega,b_0,\Phi_1}$ and $f_2 \in W^{\Omega,b,\Phi_2}$, we have $f_1 f_2 \in W^{\Omega,b_0+b,\Phi_3}$, that is,*

$$W^{\Omega,b_0,\Phi_1} \cdot W^{\Omega,b,\Phi_2} \subset W^{\Omega,b_0+b,\Phi_3}$$

and

$$\|f_1 f_2\|^{\Omega,k_0+k,b_0+b,\Phi_3} \leq 2\alpha\|f_1\|^{\Omega,k_0,b_0,\Phi_1}\|f_2\|^{\Omega,k,b,\Phi_2}$$

Proof. Without loss of generality, we may assume that

$$\| f_1 \|^{\Omega,k_0,b_0,\Phi_1} = \| f_2 \|^{\Omega,k,b,\Phi_2} = 1.$$

Then by the convexity of Φ_3 and the condition (1), we have the following inequalities;

$$\begin{aligned}&\int_{-\infty}^{\infty} \Phi_3\left(\frac{1}{2\alpha(1+\varepsilon)^2}[b_0+b, k_0+k, f_1 f_2]\right) dx \\ &\leq \frac{1}{2}\int_{-\infty}^{\infty} \Phi_3\left(\frac{1}{1+\epsilon}\Phi_1([b_0,k_0,f_1])\Phi_3^{-1}(\Phi_2(\frac{1}{1+\epsilon}[b,k,f_2]))\right) dx \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} \Phi_3\left(\Phi_2(\frac{1}{1+\epsilon}[b,k,f_2])\Phi_3^{-1}(\Phi_1(\frac{1}{1+\epsilon}[b_0,k_0,f_1]))\right) dx \\ &= \frac{1}{2}I_1 + \frac{1}{2}I_2 (say)\end{aligned}$$

By symmetry it suffices to consider one of them, say I_1.

$$\begin{aligned} I_1 &\leq \frac{\int_{-\infty}^{\infty} \Phi_3(\Phi_1(\frac{1}{1+\epsilon}[b_0, k_0, f_1])\Phi_3^{-1}(\Phi_2(\frac{1}{1+\epsilon}[b, k, f_2])))dx}{\int_{-\infty}^{\infty} \Phi_1(\frac{1}{1+\epsilon}[b_0, k_0, f_1])dx} \\ &\leq \frac{(\int_{-\infty}^{\infty} \Phi_1(\frac{1}{1+\epsilon}[b_0, k_0, f_1])dx)(\int_{-\infty}^{\infty} \Phi_2(\frac{1}{1+\epsilon}[b, k, f_2])dx)}{\int_{-\infty}^{\infty} \Phi_1(\frac{1}{1+\epsilon}[b_0, k_0, f_1])dx} \\ &\leq \int_{-\infty}^{\infty} \Phi_2(\frac{1}{1+\epsilon}[b, k, f_2])dx \\ &\leq \int_{-\infty}^{\infty} \Phi_2([b, k, f_2])dx \leq \|f_2\|^{\Omega,k,b,\Phi_2} \leq 1. \end{aligned}$$

Similarly $I_2 \leq 1$, so this implies, for any ε,

$$\int_{-\infty}^{\infty} \Phi_3\left(\frac{1}{2\alpha(1+\varepsilon)^2}[b_0+b, k_0+k, f_1 f_2]\right) dx \leq \|f_1\|^{\Omega,k_0,b,\Phi_2} \leq 1.$$

This shows that $f_1 f_2 \in W^{\Omega,b_0+b,\Phi_3}$, $W^{\Omega,b_0,\Phi_1} \cdot W^{\Omega,b,\Phi_2} \subset W^{\Omega,b_0+b,\Phi_3}$ and

$$\|f_1 f_2\|^{\Omega,k_0+k,b_0+b,\Phi_3} \leq 2\alpha(1+\varepsilon)^2 \|f_1\|^{\Omega,k_0,b_0,\Phi_1} \|f_2\|^{\Omega,k,b,\Phi_2},$$

which completes the proof. □

Theorem 11. *If $\Phi_i (i = 1, 2)$ are monotone nondecreasing N-functions such that*

$$\int_0^1 (\Phi_1^{-1}(t)\Phi_2^{-1}(t)/t^2)dt < \infty$$

and, for some constant α,

$$\Phi_3^{-1}(x) = \frac{1}{\alpha} \int_0^x (\Phi_1^{-1}(t)\Phi_2^{-1}(t)/t^2)dt,$$

then, for $f_1 \in W^{\Omega,b_0,\Phi_1}$ and $f_2 \in W^{\Omega,b,\Phi_2}$, we have $f_1 f_2 \in W^{\Omega,b_0+b,\Phi_3}$, that is,

$$W^{\Omega,b_0,\Phi_1} \cdot W^{\Omega,b,\Phi_2} \subset W^{\Omega,b_0+b,\Phi_3}$$

and

$$\|f_1 f_2\|^{\Omega,k_0+k,b_0+b,\Phi_3} \leq 2\alpha\|f_1\|^{\Omega,k_0,b_0,\Phi_1}\|f_2\|^{\Omega,k,b,\Phi_2}$$

Proof. Since $\Phi_1^{-1}(t)/t$ and $\Phi_2^{-1}(t)/t$ are nonincreasing it follows that $\Phi_3(t)$ is concave and $\Phi_3^{-1}(0) = 0$. Therefore Φ_3 is a N-function and

$$\Phi_1^{-1}(t) = \int_0^x \left(\Phi_1^{-1}(t)\Phi_2^{-1}(t)/t^2\right) dt \geq x(\Phi_1^{-1}(t)/t)(\Phi_2^{-1}(t)/t),$$

so that $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq x\Phi_3^{-1}(x)$. By the Theorem 8, this is proved. □

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