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SAOR METHOD FOR FUZZY LINEAR SYSTEM

SHU-XIN MIAO* AND BING ZHENG

ABSTRACT. In this paper, the symmetric accelerated overrelaxation (SAOR) method for solving $n \times n$ fuzzy linear system is discussed, then the convergence theorems in the special cases where matrix S in augmented system $SX = Y$ is H-matrices or consistently ordered matrices and or symmetric positive definite matrices are also given out. Numerical examples are presented to illustrate the theory.

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1. Introduction

Linear system have important applications to many branches of science and engineering. However, in many applications such as many real-world engineering problem, we usually can not be sure that the problems are perfect. Especially, if they are known through some measurements they necessarily contain unknown parameters. Therefore, these problems and operations-research algorithms are designed for fuzzy data other than crisp data. The fuzzy system, which can formulate uncertainty in actual environment, play an essential role in such cases [1-5].

The fuzzy data have been applied to various fields more and more, such as mathematics, physics, statistics, engineering and social science, and lots of modeling techniques control problems and operations-research algorithms have been designed for them since the concept of *fuzzy number* and arithmetic operations with these numbers are first introduced and then investigated by Zadeh [1-3], it is immensely important to establish mathematical models and numerical procedures for fuzzy linear systems and solve them.

We can refer to [6-9] for looking through these applications. One of the major applications using fuzzy numbers arithmetic is treating systems of simultaneous linear equations whose parameters are all or partially represented by fuzzy

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numbers. Friedman, Ma and Kandel [10] propose a general model for solving $n \times n$ fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector, and present conditions for the existence of a unique fuzzy solution to the system. In recent, the iteration methods, which are effective and then become more attractive than direct methods because of storage requirements and preservation of sparsity, for solving such fuzzy linear systems have been investigated in many papers [11,15-23].

In this paper, we consider the *symmetric accelerated overrelaxation* (SAOR) method for solving the FLS presented by Friedman et al. [10], and the convergence theorem in the special cases where matrix S in $SX = Y$ is H-matrices or consistently ordered matrices and or symmetric positive definite matrices are given out. The structure of this paper is organized as follows:

In Section 2 we recall the fuzzy linear systems and its solution. In Section 3 the SAOR method for FLS are proposed and give some convergence theorems in the special cases. The procedure is illustrated with numerical examples in Section 4 and conclusions are drawn in Section 5.

2. Fuzzy linear system and its solution

2.1. Fuzzy number and fuzzy linear system. Following [12] we represent a fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0, 1]$, and
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

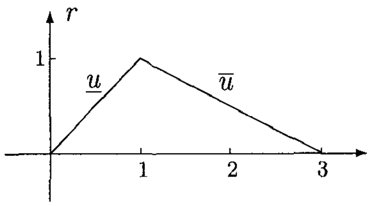

Figure 1. A fuzzy number

For example, the fuzzy number $(r, 3-2r)$ is shown in Figure 1. A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$.

By appropriate definitions the fuzzy number space $\{(\underline{u}(r), \overline{u}(r))\}$ becomes a convex cone E^1 which is then embedded isomorphically and isometrically into a Banach space.

Definition 2.1. The $n \times n$ linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2, \\ \quad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n, \end{cases} \tag{2.1}$$

where the coefficient matrix $A = (a_{ij})$, $1 \le i, j \le n$ is a crisp matrix and $y_i \in E^1$, $1 \le i \le n$, is called a *fuzzy linear system* (FLS).

2.2. The solution to FLS. To define a *solution* to the system (2.1) we should recall the arithmetic operations of arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r))$, $0 \le r \le 1$, and real number k,

(1) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

(2) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$, and

(3) $kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \ge 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$

Definition 2.2. A fuzzy number vector $X = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ given by

$$x_i = (\underline{x}_i(r), \overline{x}_i(r)), \quad 1 \le i \le n, 0 \le r \le 1,$$

is called a *solution* of the fuzzy linear system (2.1) if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{y}_i. \end{cases} \tag{2.2}$$

Using the embedding method given in [10], from (2.2), Friedman et al. extend FLS (2.1) to a $2n \times 2n$ crisp linear system

$$SX = Y \tag{2.3}$$

where $S = (s_{kl})$, s_{kl} are determined as follows

$$\begin{array}{llll} a_{ij} \ge 0 \Rightarrow & s_{ij} = a_{ij}, & s_{i+n,\, j+n} = a_{ij}, & \\ a_{ij} < 0 \Rightarrow & s_{i,\, j+n} = -a_{ij}, & s_{i+n,\, j} = -a_{ij}, & \end{array} \quad 1 \le i, j \le n,$$

and any s_{kl} which is not determined by the above items is zero, $1 \le k, l \le 2n$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \quad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}.$$

In terms of [10], we know that S has the following structure

$$\begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}$$

where S_1, $S_2 \ge 0$, $A = S_1 - S_2$, and (2.3) can be rewritten as follows

$$\begin{cases} S_1 \underline{X} - S_2 \overline{X} = \underline{Y}, \\ S_2 \underline{X} - S_1 \overline{X} = -\overline{Y}, \end{cases}$$

where

$$\underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix}, \overline{X} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix}, \underline{Y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_n \end{bmatrix}, \overline{Y} = \begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \\ \vdots \\ \overline{y}_n \end{bmatrix}.$$

The following theorem implies that when FLS (2.1) has a unique solution.

Theorem 2.1 ([10]). *The matrix S is nonsingular if and only if the matrices $A = S_1 - S_2$ and $S_1 + S_2$ are both nonsingular.*

Under the conditions of Theorem 2.1, the solution vector of (2.3)

$$X = S^{-1}Y \tag{2.4}$$

is thus unique but may still not be an *appropriate fuzzy vector*. By Theorem 2 of [10], we know that S^{-1} has the same structure as S, i.e.

$$S^{-1} = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}.$$

The following result provides a sufficient condition for the unique solution to be a *fuzzy vector*.

Theorem 2.2 ([11]). *The unique solution X of (2.4) is a fuzzy vector for arbitrary fuzzy vector Y, if S^{-1} is nonnegative.*

Restricting the discussion to triangular fuzzy numbers, i.e. $\underline{y}_i(r), \overline{y}_i(r)$ and consequently $\underline{x}_i(r), \overline{x}_i(r)$ are all linear functions of r, and having calculated X which solves (2.3), we can define the fuzzy solution to the original system given by (2.1) as follows.

Definition 2.3. Let $X = \{(\underline{x}_i(r), -\overline{x}_i(r)),\ 1 \leq i \leq n\}$ denote the unique solution of (2.3). The fuzzy number vector $U = \{(\underline{u}_i(r), \overline{u}_i(r)),\ 1 \leq i \leq n\}$ defined by

$$\begin{aligned} \underline{u}_i(r) &= \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\}, \\ \overline{u}_i(r) &= \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\} \end{aligned}$$

is called the *fuzzy solution* of $SX = Y$. If $(\underline{x}_i(r), \overline{x}_i(r))$, $1 \leq i \leq n$ are all fuzzy numbers then $\underline{u}_i(r) = \underline{x}_i(r)$, $\overline{u}_i(r) = \overline{x}_i(r)$, $1 \leq i \leq n$ and U is called a *strong fuzzy solution*, otherwise, U is called a *weak fuzzy solution.*

3. The SAOR iterative method for FLS

In this section we first present the point SAOR iterative method for FLS (2.3), then the convergence of SAOR iterative for FLS are discussed in some special cases.

3.1. Point splitting and the SAOR iterative scheme. For the case S is nonsingular, without loss of generality, assume that $s_{ii} > 0$, $i = 1, 2, \cdots 2n$, then we have the following point proper splitting of S: $S = D - L - U$, where

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}, L = \begin{bmatrix} L_1 & 0 \\ -S_2 & L_1 \end{bmatrix}, U = \begin{bmatrix} U_1 & -S_2 \\ 0 & U_1 \end{bmatrix}, \quad (3.1)$$

$D_1 = \mathrm{diag}(s_{ii})$, $i = 1, 2, \cdots n$, $D_1 - L_1 - U_1 = S_1$, and L_1 and U_1 are strictly lower and upper triangular matrices.

For $SX = Y$, the SAOR iterative scheme defined as :

$$\begin{aligned} (D - rL)X_{k+\frac{1}{2}} &= [(1-\omega)D + (\omega - r)L + \omega U]X_k + \omega Y, \\ (D - rU)X_{k+1} &= [(1-\omega)D + (\omega - r)U + \omega L]X_{k+\frac{1}{2}} + \omega Y, \end{aligned}$$

that is

$$X_{k+1} = H_{r,\omega}X_k + B, \quad (3.2)$$

where $X_k = \begin{bmatrix} \underline{X}_k \\ -\overline{X}_k \end{bmatrix}$, $k = 0, 1, \cdots$. r, ω are the real relaxation parameter,

$$\begin{aligned} H_{r,\omega} &= (D - rU)^{-1}[(1-\omega)D + (\omega - r)U + \omega L] \\ &\quad \times (D - rL)^{-1}[(1-\omega)D + (\omega - r)L + \omega U] \end{aligned}$$

is the iterative matrix and

$$\begin{aligned} B &= \omega\{(D - rU)^{-1}[(1-\omega)D + (\omega - r)U + \omega L](D - rL)^{-1} \\ &\quad + (D - rU)^{-1}\}Y. \end{aligned}$$

3.2. Convergence analysis of SAOR for FLS. In what follows we shall try to find, when matrix S in (2.3) be H-matrices, consistently ordered matrices and symmetric positive definite matrices, the restriction imposed on the parameters r and ω such that the SAOR method converges. In terms of the classical convergence theorems about SAOR concerned in [13,14,24], we can easily obtain the following convergence.

3.2.1. ***S*** **be *H*-matrices. Definition 3.1.** A matrix A is said to be H-matrices if there exist a nonnegative matrix P such that $m(A)$ is a nonsingular M-matrix and have the form: $m(A) = \alpha I - P$ with $\alpha > \rho(P)$, where $m(A)$ is the comparison matrix of A, $\rho(P)$ denotes the spectral radius of matrix P.

Theorem 3.1. *The matrix S in* (2.3) *is H-matrix if and only if the coefficient matrix A in* (2.1) *is H-matrix.*

For proof this theorem, we should first proof the following theorem.

Theorem 3.2. *The coefficient matrix A in* (2.1) *is H-matrix if and only if it is strictly diagonally dominant.*

Proof. Assume A is $H-matrix$, from the definition, then there exist a matrix $P \geq 0$ and $\alpha > \rho(P)$ such that $m(A) = \alpha I - P$ is a nonsingular $M-matrix$. Without loss of generality, we can let $\alpha > \min_{1\leq i\leq n} |a_{i,i}|$, then $P = \alpha I - m(A) \geq 0$. From the Gerschgorin theorem, we have:

$$\rho(P) \leq |\alpha - |a_{i,i}|| + \sum_{j=1,j\neq i}^{n} |a_{i,j}| = \alpha - (|a_{i,i}| - \sum_{j=1,j\neq i}^{n} |a_{i,j}|). \tag{3.3}$$

Since $\alpha > \rho(P)$, it follows from (3.3) that:

$$|a_{i,i}| - \sum_{j=1,j\neq i}^{n} |a_{i,j}| > 0,$$

this is imply A is strictly diagonally dominant.
Reversing the process, then we obtained the result and completed the proof. □

Allahviranloo provides a necessary and sufficient condition for S being a *strictly diagonally dominant matrix* in [11] as the following theorem.

Theorem 3.3. The matrix S *in* (2.3) *is strictly diagonally dominant if and only if the coefficient matrix* A *of equations* (2.1) *is strictly diagonally dominant.*

Proof Theorem 3.1. Follows by the results Allahviranloo provided and Theorem 3.2, we can easily obtain the result of this theorem. □

Now we give the convergence of the SAOR method in this case, it is the following theorem.

Theorem 3.4. *Assume that matrix* $S = (s_{i,j}) \in R^{n\times n}$ *be H-matrix and satisfy* $s_{i,i} \neq 0$, $i = 1, 2, \cdots, n$, $0 \leq r \leq \omega$, *then for*

$$0 < \omega < 2/[1 + \rho(|J|)],$$

where ω *is the relaxation parameter,* J *is Jacobi iteration matrix, and* $|J|$ *denotes the nonnegative matrix whose entries are the module of those of* J *and* $\rho(|J|)$ *is the spectral radius of* $|J|$, *the SAOR method* (3.2) *is convergent.*

By the results concerned in [13], the proof can be easily completed.

3.2.2. S be consistently ordered matrices. Definition 3.2. The matrix A is consistently one, if for $1, 2, ..., M$, there exist $W_1, W_2, ..., W_n$ with $\bigcup_{k=1}^{n} W_k = W$, $W_i \cap W_j = \Phi$ $(i \neq j)$ and for all non-diagonal elements of matrix $A_{M\times M}$ $a_{i,j} \neq 0 (i \neq j)$, (i, j) satisfy :
(i). If $i \in W_k$, then $j \in W_{k-1}$, when $j < i$;
(ii). If $i \in W_k$, then $j \in W_{k+1}$, when $j > i$.

Theorem 3.5. *If coefficient matrix* A *in* (2.1) *is a consistently ordered matrix with nonvanishing elements, then* S *in* (2.3) *is also a consistently ordered one.*

Proof. If A is a consistently ordered matrix, follows definition 3.2 for all $a_{i,j} \in A$ with $a_{i,j} \neq 0$ $(i \neq j)$, when $i \in W_k$, then:
(i). $j \in W_{k-1}$, when $j < i$;
(ii). $j \in W_{k+1}$, when $j > i$.
See that the entries relations between A and S is:

$$\begin{array}{llll} a_{ij} \geq 0 & \Rightarrow \quad s_{ij} = a_{ij}, & s_{i+n,\, j+n} = a_{ij}, & \\ a_{ij} < 0 & \Rightarrow \quad s_{i,\, j+n} = -a_{ij}, & s_{i+n,\, j} = -a_{ij}, & \end{array} \quad 1 \leq i, j \leq n.$$

So $s_{i,j}$ and $a_{i,j}$ have the same properties (where $s_{i,j}$ are the entries of matrix S), it is: $s_{i,j} \neq 0$ $(i \neq j)$, when $i \in W_k$, then:
(i). $j \in W_{k-1}$, when $j < i$;
(ii). $j \in W_{k+1}$, when $j > i$.
This implies that S is a consistent ordered matrix. □

Due to our above works, the following convergence theorem can obtain easily.

Theorem 3.6. *If S in (2.3) is a consistently ordered matrix with nonvanishing elements, all the eigenvalue of Jocobi iterative matrix $L_{0,1}$ are nonnegative real, and $\overline{\mu} = \rho(L_{0,1}) < 1$, then the eigenvalues of $H_{r,\omega}$ are nonnegative real number for all r, ω. If r, ω satisfy*

$$\begin{cases} 0 \leq r \leq 2, \\ 0 < \omega \leq 1 + \frac{r}{4-r} \end{cases}$$

and r, ω is not equal to 2 at the same time, then SAOR method is convergent.

The proof we references to [14].

3.2.3. S be symmetric positive definite matrix.

Furthermore when S is a symmetric positive definite matrix, then the convergence theorem of SAOR method concerned in [24] as below.

Theorem 3.7. *Suppose that S in (2.3) is a symmetric positive definite matrix, if ω and r satisfy:*

$$\begin{cases} 0 < \omega < 2, \\ \omega - \frac{2-\omega}{\overline{\mu}} < r < \omega + \frac{2-\omega}{\overline{\mu}} \end{cases}$$

then the SAOR method is convergent, where $\overline{\mu} = \rho(L_{0,1})$ is the spectral radius of the Jacobi iterative matrix.

K. Wang and B. Zheng provides a necessary and sufficient condition for S being a symmetric positive definite matric in [20] as the following theorem.

Theorem 3.8. *The matrix S in (2.3) is positive definite if and only if the coefficient matrix A of Eqs. (2.1) and the matrix $S_1 + S_2$ are positive definite.*

4. Examples

In this section we will give some numerical examples for illustrating the methods in this paper. All examples runs by MATLAB, and we present a stopping criterion with tolerance $\varepsilon > 0$ as follows:

$$\|X_{k+1} - X_k\| \le \varepsilon.$$

Since the fuzzy number we will use is trapezoidal fuzzy number, the norm of

$$X = \begin{bmatrix} \underline{X} \\ -\overline{X} \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix} = \begin{bmatrix} x_{1a} + x_{1b}r \\ x_{2a} + x_{2b}r \\ \vdots \\ x_{2na} + x_{2nb}r \end{bmatrix},$$

where x_{ia} and x_{ib} are crisp numbers, $i = 1, \cdots, 2n$, $0 \le r \le 1$, can be defined as

$$\|X\| = \max_i \{|x_{ia}|, |x_{ib}|\}. \qquad (*)$$

Example 4.1. Consider the 2×2 fuzzy linear system

$$\begin{cases} x_1 - x_2 = (r, 2 - r), \\ x_1 + 3x_2 = (4 + r, 7 - 2r). \end{cases}$$

The extended 4×4 matrix is

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

and the solution of the extended system is

$$\begin{aligned} X &= \begin{bmatrix} \underline{x}_1(r) \\ \underline{x}_2(r) \\ -\overline{x}_1(r) \\ -\overline{x}_2(r) \end{bmatrix} = S^{-1}Y \\ &= \begin{bmatrix} 1.1250 & -0.1250 & 0.3750 & -0.3750 \\ -0.3750 & 0.3750 & -0.1250 & 0.1250 \\ 0.3750 & -0.3750 & 1.1250 & -0.1250 \\ -0.1250 & 0.1250 & -0.3750 & 0.3750 \end{bmatrix} \begin{bmatrix} r \\ 4 + r \\ r - 2 \\ 2r - 7 \end{bmatrix} \\ &= \begin{bmatrix} 1.375 + 0.625r \\ 0.875 + 0.125r \\ -2.875 + 0.875r \\ -1.375 + 0.375r \end{bmatrix}. \end{aligned}$$

The exact solution is

$$\begin{cases} x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (1.375 + 0.625r,\ 2.875 - 0.875r), \\ x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (0.875 + 0.125r,\ 1.375 - 0.375r), \end{cases}$$

which is a strong fuzzy solution.

For the SAOR method, by (3.1), we have

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $L_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $U_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $S_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Applying on the iterative scheme (3.2), with taking $r = 1.2$, $\omega = 1.5$ and the initial value$X_0 = [0, 0, 0, 0]^T$ we obtain the approximate solution

$$X_{Approximate} = \begin{bmatrix} 0.6250r + 1.3750 \\ 0.1250r + 0.8750 \\ -0.8750r + 2.8750 \\ 1.3750 - 0.3750r \end{bmatrix},$$

i.e.

$$\begin{cases} x_1 = (1.3750 + 0.6250r,\ 2.8750 - 0.8750r), \\ x_2 = (0.8750 + 0.1250r,\ 1.3750 - 0.3750r). \end{cases}$$

The exact and approximate solutions are compared in Figure 2.

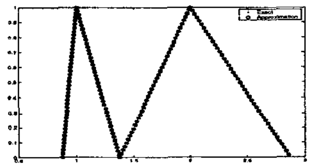

Figure 2. The exact and approximate solutions
The norm of vector X defined as $(*)$, $\varepsilon = 10^{-5}$

Example 4.2. Consider the 2×2 fuzzy linear system

$$\begin{cases} 3x_1 - 2x_2 = (1, 2 - r), \\ -2x_1 + 3x_2 = (2 + r, 5 - 2r). \end{cases}$$

The extended 4×4 matrix is

$$S = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 3 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 2 & 0 & 0 & 3 \end{bmatrix},$$

thus $S_1 + S_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. Both A and $S_1 + S_2$ are positive definite. By Theorem 3.8, S is positive definite, therefore, the SAOR method is convergent.

The exact and approximate solutions are

$$\begin{cases} x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (2.6 - 0.8r,\ 2.0 - 0.2r), \\ x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (2.0 + 0.2r,\ 3.4 - 1.2r), \end{cases}$$

and

$$\begin{cases} x_1 = (2.6000 - 0.8000r,\ 2.0000 - 0.2000r), \\ x_2 = (2.0000 + 0.2000r,\ 3.4000 - 1.2000r), \end{cases}$$

and x_1 is not a fuzzy number, so the fuzzy solution is a weak fuzzy solution given by

$$\begin{cases} u_1 = (\underline{u}_1(r), \overline{u}_1(r)) = (1.8,\ 2.6 - 0.8r), \\ u_2 = (\underline{u}_2(r), \overline{u}_2(r)) = (2.0 + 0.2r,\ 3.4 - 1.2r). \end{cases}$$

The exact and approximate solutions are compared in Figure 3.

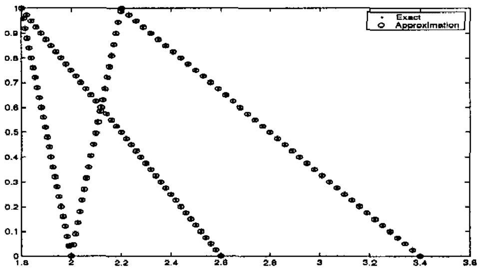

Figure 3. The exact and approximate solutions
The norm of vector X defined as $(*)$, $\varepsilon = 10^{-5}$

5. Conclusion

We present the SAOR methods for the $n \times n$ fuzzy linear system and obtain the convergence theorems of the iterative schemes in some special cases. If the proposed matrix S by Friedman et al. [10] is nonsingular, then for any initial vector X_0, the SAOR iteration will converge to $X = \begin{bmatrix} \underline{X} \\ -\overline{X} \end{bmatrix}$, the unique solution of $SX = Y$. The numerical examples show that the methods are effective and applicable for solving the fuzzy linear system.

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References

1. L.A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965) 338–353.
2. L.A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning*, Inform. Sci. **8** (1975) 199-249, 301–357, **9** (1975) 43-80.
3. L.A. Zadeh, *Fuzzy sets as a basis for a possibility theory*, Fuzzy Sets and Systems **1** (1978) 3–28.

4. J.U. Jeong, *Stability of a periodic solution for fuzzy differential equations*, J. Appl. Math. Comput. 13 (2003), 217–222.
5. H.M. Nehi, H.R. Maleki and M. Mashinchi, *A canonical representation for the solution of fuzzy linear system and fuzzy linear programming problem*, J. Appl. Math. Comput. 20 (2006), 345–354.
6. W. Pedrycz, *Relevancy of fuzzy models*, Inform. Sci. **52** (1990) 285–302.
7. W. Pedrycz, *Interfaces of fuzzy models: A study in fuzzy information processing*, Inform. Sci. **90** (1996) 231–280.
8. W. Pedrycz, A. Bargiela, *Fuzzy fractal dimensions and fuzzy model*, Inform. Sci. **153** (2003) 199–216.
9. A. Konar, U.K. Chakraborty, P.P. Wang, *Supervised learning on fuzzy Petri net*, Inform. Sci. **172** (2005) 397–416.
10. M. Friedman, M. Ma and A. Kandel, *Fuzzy linear systems*, Fuzzy Sets and Systems **96** (1998) 201–209.
11. T. Allahviranloo, *Numerical methods for fuzzy system of linear equations*, Appl. Math. Comput. **155** (2004) 493–502.
12. Cong-Xin Wu, Ming Ma, *Embedding problem of fuzzy number space: Part I.* Fuzzy Sets and Systems **44** (1991) 33–38.
13. Yong-Zhong Song, *On the convergence of the generalized AOR method*, Linear Algebra and Its Application **256** (1997) 199–218.
14. Zhang Yin, *The convergent of the SAOR method*, Mathematica Numerica Sinica **2** (1988) 201–204. (in Chinese)
15. T. Allahviranloo, *Successive over relaxation iterative method for fuzzy system of linear equations*, Appl. Math. Comput. **162** (2005) 189–196.
16. T. Allahviranloo, *The Adomian decomposition method for fuzzy system of linear equations*, Appl. Math. Comput. **163** (2005) 553–563.
17. S. Abbasbandy, A. Jafarian and R. Ezzati, *Conjugate gradient method for fuzzy symmetric positive definite system of linear equations*, Appl. Math. Comput. **171** (2005) 1184–1191.
18. S. Abbasbandy, R. Ezzati and A. Jafarian, *LU decomposition method for solving fuzzy system of linear equations*, Appl. Math. Comput. **172** (2006) 633–643.
19. S. Abbasbandy and A. Jafarian, *Steepest descent method for system of fuzzy linear equations*, Appl. Math. Comput. **175** (2006) 823–833.
20. Ke Wang, Bing Zheng, *Symmetric successive overrelaxation methods for fuzzy linear systems*, Appl. Math. Comput. **175** (2006) 891-901.
21. M. Dehghan, B. Hashemi, *Iterative solution of fuzzy linear systems*, Appl. Math. Comput. **175** (2006) 645–674.
22. T. Allahviranloo, E. Ahmady, N. Ahmady and Kh. Shams Alketaby, *Block Jacobi two-stage method with Gauss-Sidel inner iterations for fuzzy system of linear equations*, Appl. Math. Comput. **175** (2006) 1217–1228.
23. K. Wang and B. Zheng, *Block iterative methods for fuzzy linear systems*, J. Appl. Math. & Computing, **25** (2007) 119–136.
24. S.-X. Miao, B. Zheng and K. Wang, *Block SOR methods for fuzzy linear systems*, J. Appl. Math. & Computing, **26** (2008) 201–218.
25. Zhang Yin, *The FBAOR iterative method for linear systems*, Numerical Mathematics A Journal of Chinese Universities **4** (1987) 334–365. (in Chinese)

Shu-Xin Miao received his B.Sc from North West Normal University (China,2003) and and M.Sc from Lanzhou University (China, 2008). His main research area is Numerical Linear Algebra.

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, P. R. China
e-mail: miaoshx05@@lzu.cn

Bing Zheng received his B.Sc from North West Normal University (China,1983) and M.Sc from Anhui University (China, 1989). In 2003, he received his Ph.D from Shanghai University in China and now is working at School of Mathematics and Statistics of Lanzhou University in China as a associate professor. During 2001-2002, he is a visiting scholar at Indian statistical Institute-Delhi center. His main research area is Numerical Linear Algebra, including generalized inverses of matrix and operator in Banach or Hilbert spaces.

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, P. R. China
e-mail: bzheng@@lzu.edu.cn