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UNIQUENESS THEOREMS FOR ENTIRE FUNCTIONS WHICH SHARING VALUES WITH THEIR FIRST DERIVATIVES

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Abstract. In this paper, we get another proof of L. A. Rubel and C. C. Yang's theorem. It is more vivid technique. By using the relation between normal families and shared values we prove the theorem.

1. Introduction

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane. We say that $f(z)$ and $g(z)$ share the finite values a CM, IM and DM provided that $f(z)=a$ and $g(z)=a$ have the same zeros counting multiplicities, ignoring multiplicities and one point or more has different multiplicities, respectively. The uniqueness problem was first solved by R. Nevanlinna. In 1926, R. Nevanlinna proved if two nonconstant meromorphic functions f and g share five values IM, then $f \equiv g$. The resent studies have been much more specialized. In 1976, L. A. Rubel and C. C. Yang [10] proved if an entire function f shares two finite values CM with its derivatives, then $f \equiv f'$. L. A. Rubel and C. C. Yang state that they do not know what can be said if, in the hypothesis of theorem, the word entire is replaced by the word meromorphic, or if CM is replaced by IM. If both are replaced simultaneously, then $f \equiv f'$ is not always true. The result has been generalized to sharing values IM by G. G. Gundersen [4] and by E. Mues - N. Steinmets [8] independently. The condition "f and g share four values CM" has been weakened to "f and g share two values CM and two values IM" by G. G. Gundersen [2, 3] as well as by E. Mues and S. Wang [7, 12]. But whether the condition can be weakened to "f and g share three values IM and another value CM" or not, is still an open question.

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2. Notations on auxiliary results

The basic results and notations of R. Nevanlinna's value distribution theory have become an indispensable tool in the study of uniqueness problems. We discuss the fundamental properties of these notations. Define $n(t,0)$ and $n(t,\infty)$ as the number of zeros and poles in $|z| \leq t$, respectively, counted accordingly to multiplicity. Let $f(z)$ be meromorphic $|z| \leq r < \infty$ and have zeros $a_1, a_2, \cdots, a_N$ and poles $b_1, b_2, \cdots, b_M$ in $|z| < r$, repeated according to multiplicity. If $f(z) \neq 0, \infty$ and $z = \rho e^{i\theta}$, $0 \leq \rho < r$, then

$$\begin{aligned}\log|f(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\phi})| \frac{r^2-\rho^2}{r^2 - 2r\rho\cos(\theta-\phi)+\rho^2} d\phi \\ &\quad + \sum_{i=1}^{N} \log\left|\frac{r(z-a_i)}{r^2-\bar{a}_i z}\right| - \sum_{j=1}^{M} \log\left|\frac{r(z-b_j)}{r^2-\bar{b}_j z}\right|.\end{aligned}$$

The case when $z = 0$ is called Jensen's formula.

Definition 2.1 (Jensen's formula)**.**

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\phi})| d\phi - \sum_{i=1}^{N} \log\frac{r}{|a_i|} + \sum_{j=1}^{M} \log\frac{r}{|b_j|}.$$

Definition 2.2. (1) (Counting function). For a meromorphic function f, we define

$$N(r,f) = N(r,\infty) = \int_0^r \frac{n(t,\infty)}{t} dt = \sum_{j=1}^{M} \log\frac{r}{|b_j|},$$

$$N(r,\frac{1}{f}) = \int_0^r \frac{n(t,0)}{t} dt = \sum_{i=1}^{N} \log\frac{r}{|a_i|}.$$

(2) (Proximity function). For a meromorphic function f, we define

$$m(r,f) = m(r,\infty) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\phi})| d\phi$$

where $\log^+ x = max(0, \log x)$, $(\log^+ 0 = 0)$.

Definition 2.3. (1) (Characteristic function). The *R. Nevanlinna characteristic function* of a meromorphic function f be defined as

$$T(r,f) = m(r,f) + N(r,f). \tag{1}$$

(2) We shall call an *error term* and denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o(1)T(r,f)$$

as $r \longrightarrow \infty$, possibly outside of a set of finite linear measure.

Then Jensen's fomula becomes simply

$$T(r,f) = T(r,\frac{1}{f}) + \log|f(0)|. \tag{2}$$

Example. Let $f(z) = e^z = e^{r\cos\theta + i\sin\theta}$, for $z = re^{i\theta}$. Then

$$m(r,f) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\theta d\theta = \frac{r}{\pi}, \quad N(r,f) = 0,$$

so that

$$T(r,f) = m(r,f) + N(r,f) = \frac{r}{\pi}.$$

Introducing the notation

$$m(r,a) = m(r,\frac{1}{f-a}), \quad N(r,a) = N(r,\frac{1}{f-a}).$$

Theorem 2.4 ([5], First fundamental theorem)**.** *If $f(z)$ is meromorphic in $|z| < R \leq \infty$, then for any $a \in \mathbf{C}$ and $0 < r < R$,*

$$m(r,a) + N(r,a) = T(r,\frac{1}{f-a}) = T(r,f) - \log|f(0) - a| + \epsilon(r,a)$$

where $\epsilon(r,a) \leq \log^+|a| + \log 2$.

The theorem states that the sum of these two quantities remains the same up to a bounded term, specially

$$m(r,a) + N(r,a) = T(r,f) + O(1)$$

as $r \longrightarrow R$, where a can be finite or infinite.

Theorem 2.5 ([5], Fundamental inequality)**.** *Suppose that $f(z)$ is a nonconstant meromorphic function in $|z| \leq r$. Let $a_1, a_2, \cdots, a_q$, where $q \geq 2$ distinct finite complex numbers, $\delta > 0$, and suppose that $|a_\mu - a_\nu| \geq \delta$ for $1 \leq \mu < \nu \leq q$.*
Then

$$m(r,\infty) + \sum_{\nu=1}^{q} m(r,a_\nu) \leq 2T(r,f) - N_1(r) + S(r),$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N(r,\frac{1}{f'}) + 2N(r,f) - N(r,f')$$

and

$$S(r) = m(r,\frac{f'}{f}) + m(r,\sum_{\nu=1}^{q}\frac{f'}{f-a_\nu}) - q\log^+\frac{3q}{\delta} + \log 2 + \log\frac{1}{|f'(0)|},$$

with modification if $f(0) = 0$ or ∞, or $f'(0) = 0$.

The term N_1 measures the number of multiple points of $f(z)$, whereas $S(r)$ plays the role of a negligible error term.

Lemma 2.6 ([5]). *Let $z_1, z_2, \cdots, z_n$ be $n \geq 1$ points in the plane and let $\delta(z)$ be the least of the distances $|z - z_\nu|, \nu = 1$ to n. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{r}{\delta(re^{i\theta})} d\theta \leq 2\log n + \frac{1}{2}.$$

The following lemma is based on [5], and it plays a basic role to prove our result.

Lemma 2.7 ([5]). *Suppose that $f(z)$ is meromorphic in $|z| \leq R$, that $0 < r < R$, and that $f(0) \neq 0, \infty$. Then*

$$\begin{aligned} m(r, \frac{f'}{f}) \quad &< \quad 4\log^+ T(r,f) + 4\log^+\log^+ \frac{1}{|f(0)|} + 5\log^+ R \\ &\quad +6\log^+ \frac{1}{R-r} + \log^+ \frac{1}{r} + 14. \end{aligned}$$

3. Some lemmas

Let $\mathbf{D}$ be a domain in $\mathbf{C}$ and $\mathcal{F}$ be a family of holomorphic functions in $\mathbf{D}$. $\mathcal{F}$ is said to be normal in $\mathbf{D}$, in the sense of Montel [11].

Definition 3.1. A family $\mathcal{F}$ of holomorphic functions on a domain $\mathbf{D} \subset \mathbf{C}$ is *normal* in $\mathbf{D}$ if every sequence of functions $f_n \subset \mathcal{F}$ contains either a subsequence which converges to a limit function $f \not\equiv \infty$ uniformly on each compact subset of $\mathbf{D}$, or a subsequence which converges uniformly to ∞ on each compact subset.

For the proof of our results, we need the following lemmas.

Lemma 3.2 ([9, 11]). *Let $\mathcal{F}$ be a family of holomorphic functions in a domain $\mathbf{D}$, and let a, b two distinct finite complex numbers. If, for any $f \in \mathcal{F}$, f and f' share a, b IM, then $\mathcal{F}$ is normal in $\mathbf{D}$.*

Lemma 3.3 ([1]). *Let f be an entire function, M a positive number. If $f^{\#}(z) \leq M$ for any $z \in \mathbf{C}$, then f is of exponential type. Here, as usual,*

$$f^{\#}(z) = \frac{|f'(z)|}{(1 + |f(z)|^2)}$$

is the spherical derivative.

Lemma 3.4 ([6], Marty's Theorem). *A family $\mathcal{F}$ of meromorphic functions on a domain $\mathbf{D}$ is normal if and only if for each compact subset $\mathbf{K} \subset \mathbf{D}$, there exists a constant $M = M(\mathbf{K})$ such that the spherical derivative*

$$f^{\#}(z) = \frac{|f'(z)|}{(1 + |f(z)|^2)} \leq M,$$

$z \in \mathbf{K}, f \in \mathcal{F}$, that is, $f^{\#}$ is locally bounded.

4. Uniqueness theorem

Let f be a nonconstant meromorphic function in the complex plane and let $\mathbf{S}$ be a set of complex numbers. Put

$$E(\mathbf{S}, f) = \bigcup_{a\in S} \{z : f(z) - a = 0\}$$

where a zero of multiplicity m is counted m times in the set.

Remark. Let

$$f(z) = \frac{2Ae^{2z}}{e^{2z} - B}, \quad A \neq 0, \quad B \neq 0.$$

Then we have

$$f'(z) = \frac{-4ABe^{2z}}{(e^{2z} - B)^2}.$$

We know that f share 0 and A(by DM) with f'.

By using the new method and notation, we prove the L. A. Rubel and C. C. Yang's Theorem.

Theorem 4.1. *Let a and b be distinct nonzero complex numbers, and let f be a nonconstant entire function. If $E(a, f) = E(a, f')$ and $E(b, f) = E(b, f')$, then $f \equiv f'$.*

Proof. Suppose that $E(a, f) = E(a, f')$ and $E(b, f) = E(b, f')$ where a, b are two nonzero distinct finite complex numbers. Set

$$\phi(z) = \frac{[f'(z) - a][f'(z) - b]}{[f(z) - a][f(z) - b]}.$$

Then by $E(a, f) = E(a, f')$ and $E(b, f) = E(b, f')$, there exists an entire function h satisfying

$$\phi(z) = \frac{[f'(z) - a][f'(z) - b]}{[f(z) - a][f(z) - b]} = e^{h(z)}.$$

By using the result of Lemma 2.6, we get

$$\begin{aligned} m(r, \phi) &\leq m(r, \frac{f'}{f - a}) + m(r, \frac{-a}{f - a}) + m(r, \frac{f'}{f - b}) + m(r, \frac{-b}{f - b}) \\ &= S(r, f), \end{aligned} \tag{3}$$

$$N(r, \phi) = 0$$

and hence

$$T(r, \phi) = m(r, \phi) + N(r, \phi) = S(r, f). \tag{4}$$

Let us now show that f is of exponential type. Set

$$\mathcal{F} = \{f(z + w) = f_w(z); w \in \mathbf{C}\}.$$

Then $\mathcal{F}$ is a family of holomorphic functions on the unit disc $\triangle$. By the assumption, for any function $g(z) = f(z+w)$ we have g and g' share a, b IM. Hence by Lemma 3.2, $\mathcal{F}$ is normal in $\triangle$. Thus by Lemma 3.4, there exists $M > 0$ satisfying $f^{\#}(z) \leq M$ for all $z \in \mathbf{C}$. By Lemma 3.3, f is of exponential type. Therefore, $T(r, f) = O(r)$, whence $S(r, f) = O(\log r)$. It then follows from (4) that ϕ is a polynomial, so by (3) ϕ must be a nonzero constant A. Hence

$$\frac{[f'(z) - a][f'(z) - b]}{[f(z) - a][f(z) - b]} = A,$$

that is,

$$[f'(z) - a][f'(z) - b] = A[f(z) - a][f(z) - b]. \tag{5}$$

Differentiating the two sides of (5), we obtain

$$f''(2f' - a - b) = Af'(2f - a - b). \tag{6}$$

We claim that $f' \neq 0$. Indeed, suppose that $f'(z_0) = 0$ and $f(z) = f(z_0) + A_n(z - z_0)^n + \cdots$, where $A_n \neq 0, n \geq 2$. Then the left-hand side (6) vanishes at z_0 to order $n - 2$, while the right-hand side vanishes to the order at least $n - 1$. This is a contradiction. Hence $f'(z) = BCe^{cz}$ and $f(z) = D + Be^{cz}$, where $B \neq 0, C \neq 0$ and D are constants. Since $E(a, f) = E(a, f')$ and $E(b.f) = E(b, f')$, we have $C = 1$ and $D = 0$.

In the above discussion we have shown that $f \equiv f'$. This completes the proof of the theorem. □

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