

**NOTES ON IDEALS AND ORTHOGONAL GENERALIZED
(σ, τ)-DERIVATIONS**

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ABSTRACT. In this paper, some results of [6] concerning orthogonal (σ, τ) -derivations and generalized (σ, τ) -derivations are generalized for a nonzero ideal of a semiprime ring.

1. Introduction

Let R be a ring and σ, τ be two mappings from R into itself. A ring R is said to be a 2-torsion free if whenever $2x = 0$ with $x \in R$ implies that $x = 0$. A ring R is called a semiprime ring if for any $x \in R$, $xRx = (0)$ implies that $x = 0$. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$ and is called (σ, τ) -derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$.

During the last couple of decades, a lot of work has been done on prime or semiprime rings with derivation. In [3], Bresar defined the following notion. An additive mapping $D : R \rightarrow R$ is said to be a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that

$$D(xy) = D(x)y + xd(y), \text{ for all } x, y \in R.$$

In the view of above observation, the concept of generalized derivations includes the concept of derivations. Hence it should be interesting to extend some results concerning to these notions to generalized derivations. Inspired by the definition of (σ, τ) -derivation was extended as follows:

An additive mapping $D : R \rightarrow R$ is called a generalized (σ, τ) -derivation if there exists a (σ, τ) -derivation $d : R \rightarrow R$ such that $D(xy) = D(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. By the this notion, some results concerning generalized derivations are generalized for generalized (σ, τ) -derivations.

Two additive maps $d, g : R \rightarrow R$ are called orthogonal if

$$d(x)Rg(y) = 0 = g(y)Rd(x), \text{ for all } x, y \in R.$$

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The concept of orthogonal derivations was introduced by M. Brešar and Vukman in [2]. They presented several necessary and sufficient conditions for d and g to be orthogonal. Later the authors have investigated the properties orthogonal derivations in semiprime ring R or a nonzero ideal of R (see [2], [4]). The authors introduced orthogonal generalized derivations in [5] and orthogonal (σ, τ) -derivations in [6]. They also obtained some results concerning two orthogonal generalized derivations on a semiprime ring. In [1], E. Albaş has extended these results to orthogonal generalized derivations on a nonzero ideal I of R in [5]. In this paper, our aim is to show results in [6] to orthogonal generalized (σ, τ) -derivations on nonzero ideal of semiprime ring R .

Recall that if R is a semiprime ring, then $I \cap l(I) = 0$ or $I \cap r(I) = 0$, where $l(I)$ and $r(I)$ denote the left annihilator and the right annihilator of I , respectively.

Throughout this paper, R will be 2-torsion free semiprime ring, I a nonzero ideal of R , d and g are (σ, τ) -derivations of R such that $g\tau = \tau g$, $d\tau = \tau d$, $\sigma g = g\sigma$, $\sigma d = d\sigma$ while σ, τ automorphisms of R . We denote a generalized (σ, τ) -derivation $D : R \rightarrow R$ determined by a (σ, τ) -derivation d of R by (D, d) and let generalized (σ, τ) -derivations (D, d) and (G, g) such that $G\tau = \tau G$, $D\tau = \tau D$, $\sigma G = G\sigma$, $\sigma D = D\sigma$.

2. Preliminaries

Lemma 1 ([4, Lemma 1]). *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R and $a, b \in R$. Then the following conditions are equivalent.*

- (i) $axb = 0$ for all $x \in I$;
- (ii) $bxa = 0$ for all $x \in I$;
- (iii) $axb + bxa = 0$ for all $x \in I$.

Moreover, if one of these conditions is fulfilled and $l(I) = 0$, then $ab = ba = 0$ too.

Lemma 2 ([4, Lemma 3]). *Let R be a semiprime ring and I a nonzero ideal of R . Suppose that additive mappings f and h of R into itself satisfy $f(x)Ih(x) = 0$ for all $x \in I$. Then $f(x)Ih(y) = 0$ for all $x, y \in I$.*

Lemma 3 ([6, Theorem 1]). *Let R be a 2-torsion free semiprime ring. (σ, τ) -derivations d and g of R are orthogonal if and only if one of the following conditions holds:*

- (i) $dg = 0$;
- (ii) $gd = 0$;
- (iii) $dg + gd = 0$;
- (iv) $d(x)g(x) = 0$, for all $x \in R$;
- (v) dg is a (σ^2, τ^2) -derivation of R .

3. Results

Lemma 4. *Let $(D, d), (G, g)$ be generalized (σ, τ) -derivations of R and $l(I) = 0$. If $D(I)IG(I) = 0$, then $D(R)RG(R) = 0$.*

Proof. Suppose that $D(x)zG(y) = 0$ for all $x, y, z \in I$. By Lemma 1, we have

$$(1) \quad D(x)G(y) = G(y)D(x) = 0, \text{ for all } x, y \in I.$$

Replacing y by yr , $r \in R$ in (1) and using this equation, we obtain that

$$D(x)\tau(y)g(r) = 0, \text{ for all } x, y \in I, r \in R.$$

Since τ is an automorphism of R , we get

$$\tau^{-1}(D(x))y\tau^{-1}(g(r)) = 0, \text{ for all } x, y \in I, r \in R.$$

Applying Lemma 1 and using τ is an automorphism of R , we conclude that

$$g(r)D(x) = D(x)g(r) = 0, \text{ for all } x \in I, r \in R.$$

Letting x by xs , $s \in R$ in $g(r)D(x) = 0$ and using similarly techniques the above reduces to

$$(2) \quad g(r)d(s) = d(s)g(r) = 0, \text{ for all } r, s \in R.$$

Now, witting xr by x , $r \in R$ in (1), $G(y)D(x) = 0$ and using this equation, we find that

$$G(y)\tau(x)d(r) = 0, \text{ for all } x, y \in I, r \in R,$$

and so,

$$\tau^{-1}(G(y))x\tau^{-1}(d(r)) = 0, \text{ for all } x, y \in I, r \in R.$$

According to Lemma 1 and using τ is an automorphism of R , we get

$$(3) \quad G(y)d(r) = d(r)G(y) = 0, \text{ for all } y \in I, r \in R.$$

Replacing r by ry , $y \in I$ in (2), $d(s)g(r) = 0$, we obtain that

$$(4) \quad d(s)\tau(r)g(y) = 0, \text{ for all } y \in I, r, s \in R.$$

On the other hand, taking sy by y , $s \in R$ in (3), $d(r)G(y) = 0$ and using (4), we conclude that

$$d(r)G(s)\sigma(y) = 0, \text{ for all } y \in I, r, s \in R,$$

and so,

$$\sigma^{-1}(d(r))\sigma^{-1}(G(s))I = 0, \text{ for all } r, s \in R.$$

Since $l(I) = 0$ and σ is an automorphism of R , we see that

$$(5) \quad d(r)G(s) = 0, \text{ for all } r, s \in R.$$

Replacing x by rx and y by sy , $r, s \in R$ in (1), $D(x)G(y) = 0$, we have

$$\begin{aligned} 0 &= (D(r)\sigma(x) + \tau(r)d(x))(G(s)\sigma(y) + \tau(s)g(y)) \\ &= D(r)\sigma(x)G(s)\sigma(y) + \tau(r)d(x)G(s)\sigma(y) + D(r)\sigma(x)G(s)\sigma(y) \\ &\quad + \tau(r)d(x)\tau(s)g(y). \end{aligned}$$

Since σ, τ are automorphisms of 2-torsion free semiprime ring R and (5), (4) relations reduces to

$$D(r)\sigma(x)G(s)\sigma(y) = 0, \text{ for all } x, y \in I, r, s \in R,$$

and so,

$$\sigma^{-1}(D(r))x\sigma^{-1}(G(s))I = 0, \text{ for all } x \in I, r, s \in R.$$

By the hypothesis and σ is an automorphism of R , we arrive at

$$D(r)xG(s) = 0, \text{ for all } x \in I, r, s \in R.$$

Replacing x by $r'G(s)xD(r)r'$, $r' \in R$, we conclude that

$$D(r)r'G(s)ID(r)r'G(s)I = 0, \text{ for all } r, r', s \in R.$$

$D(r)r'G(s)I$ is a nilpotent right ideal of R . By the semiprimeness of R , we obtain that $D(r)r'G(s)I = 0$, $r, s, r' \in R$ and using $l(I) = 0$. Hence $D(r)r'G(s) = 0$, for all $r, s, r' \in R$. Thus, we have $D(R)RG(R) = 0$. \square

Corollary 1. *Let (D, d) , (G, g) be generalized (σ, τ) -derivations of R and $l(I) = 0$. If $D(I)\sigma(I)G(I) = 0$ or $D(I)\tau(I)G(I) = 0$, then $D(R)RG(R) = 0$.*

Lemma 5. *Let $l(I) = 0$. (σ, τ) -derivations d and g of R are orthogonal if and only if $d(x)g(y) + g(x)d(y) = 0$, for all $x, y \in I$.*

Proof. Suppose that $d(x)g(y) + g(x)d(y) = 0$ for all $x, y \in I$. Replacing y for yx in this equation, we have

$$\begin{aligned} 0 &= d(x)g(yx) + g(x)d(yx) \\ &= (d(x)g(y) + g(x)d(y))\sigma(x) + d(x)\tau(y)g(x) + g(x)\tau(y)d(x) \\ &= d(x)\tau(y)g(x) + g(x)\tau(y)d(x), \end{aligned}$$

and so,

$$\tau^{-1}(d(x))y\tau^{-1}(g(x)) + \tau^{-1}(g(x))y\tau^{-1}(d(x)) = 0, \text{ for all } x, y \in I.$$

By Lemma 1, we obtain that

$$\tau^{-1}(d(x))I\tau^{-1}(g(x)) = 0 = \tau^{-1}(g(x))I\tau^{-1}(d(x)), \text{ for all } x \in I.$$

According to Lemma 2, we get

$$\tau^{-1}(d(x))I\tau^{-1}(g(y)) = 0 = \tau^{-1}(g(y))I\tau^{-1}(d(x)), \text{ for all } x, y \in I.$$

We can write this,

$$d(x)\tau(I)g(y) = 0 = g(y)\tau(I)d(x), \text{ for all } x, y \in I.$$

We know that every d (σ, τ) -derivation is (d, d) generalized (σ, τ) -derivation. So, by Corollary 1, $d(R)RG(R) = 0 = g(R)Rd(R)$. Hence d and g are orthogonal.

Conversely, if d and g are orthogonal, then $d(r)Rg(s) = 0 = g(r)Rd(s)$ for all $r, s \in R$. In particular, we get $d(r)Ig(s) = 0 = g(r)Id(s)$ for all $r, s \in R$. By Lemma 1, $d(r)g(s) = g(r)d(s) = 0$ for all $r, s \in R$ and so, $d(x)g(y) =$

$g(x)d(y) = 0$ for all $x, y \in I$. Thus, we get $d(x)g(y) + g(x)d(y) = 0$ for all $x, y \in I$. \square

Theorem 1. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R such that $l(I) = 0$, d and g (σ, τ) -derivations of R . Then the following conditions are equivalent.*

- (i) d and g are orthogonal.
- (ii) $d(x)g(x) = 0$, for all $x \in I$.

Proof. (i) \Rightarrow (ii). Let d and g are orthogonal derivations. Then $d(r)g(r) = 0$, for all $r \in R$ by Lemma 3. In particular, $d(x)g(x) = 0$, for all $x \in I$.

(ii) \Rightarrow (i). A linearization of $d(x)g(x) = 0$, for all $x \in I$ gives

$$(6) \quad d(x)g(y) + d(y)g(x) = 0, \text{ for all } x, y \in I.$$

Replacing y by yz , $z \in I$ in (6), we see that

$$0 = d(x)g(y)\sigma(z) + d(x)\tau(y)g(z) + d(y)\sigma(z)g(x) + \tau(y)d(z)g(x).$$

Then by (6), $d(x)g(y) = -d(y)g(x)$ and $d(z)g(x) = -d(x)g(z)$ and so, the above relation reduces to

$$(7) \quad d(y)[\sigma(z), g(x)] = [\tau(y), d(x)]g(z), \text{ for all } x, y, z \in I.$$

Taking ry instead of y , $r \in R$ in (7) and using this equation, we obtain that

$$\begin{aligned} d(r)\sigma(y)[\sigma(z), g(x)] + \tau(r)d(y)[\sigma(z), g(x)] \\ = [\tau(r), d(x)]\tau(y)g(z) + \tau(r)[\tau(y), d(x)]g(z), \\ d(r)\sigma(y)[\sigma(z), g(x)] = [\tau(r), d(x)]\tau(y)g(z). \end{aligned}$$

Writing $\tau^{-1}(d(x))$ by r in the above relation and using $\tau d = d\tau$, we have

$$(8) \quad \tau^{-1}(d^2(x))\sigma(y)[\sigma(z), g(x)] = 0, \text{ for all } x, y, z \in I.$$

Taking rz by z , $r \in R$ in (8) and using (8), we conclude that

$$\tau^{-1}(d^2(x))\sigma(y)[\sigma(r), g(x)]\sigma(z) = 0, \text{ for all } x, y, z \in I, r \in R.$$

Since σ is an automorphism of R and $l(I) = 0$, we get

$$\tau^{-1}(d^2(x))\sigma(y)[r, g(x)] = 0, \text{ for all } x, y \in I, r \in R,$$

and so,

$$\sigma^{-1}(\tau^{-1}(d^2(x)))y\sigma^{-1}([r, g(x)]) = 0, \text{ for all } x, y \in I, r \in R.$$

Define a map $I_r : R \rightarrow R$ such that $I_r(x) = [r, x]$, for all $x \in R$. I_r is an additive map of R . We can write the above equation,

$$\sigma^{-1}\tau^{-1}d^2(x)y\sigma^{-1}I_r g(x) = 0, \text{ for all } x, y \in I, r \in R.$$

From Lemma 2 and Lemma 1, we obtain that

$$\sigma^{-1}\tau^{-1}d^2(x)\sigma^{-1}I_r g(z) = 0, \text{ for all } x, z \in I, r \in R.$$

That is,

$$0 = \tau^{-1}(d^2(x))[r, g(z)], \text{ for all } x, z \in I, r \in R.$$

Replacing r by sr , $s \in R$, we see that

$$(9) \quad 0 = \tau^{-1}(d^2(x))s[r, g(z)], \text{ for all } x, z \in I, r, s \in R.$$

Writing xv by $x, v \in I$ in (9) and using this relation, we conclude that

$$2d(x)\tau^{-1}\sigma(d(v))s[r, g(z)] = 0, \text{ for all } v, x, z \in I, r, s \in R.$$

Since R is 2-torsion free semiprime ring, we get

$$(10) \quad d(x)\tau^{-1}\sigma(d(v))s[r, g(z)] = 0, \text{ for all } v, x, z \in I, r, s \in R.$$

Taking vr' by $v, r' \in R$ in (10) and using this equation, we have

$$d(x)\sigma(v)\tau^{-1}\sigma(d(r'))s[r, g(z)] = 0, \text{ for all } v, x, z \in I, r, r', s \in R.$$

Since σ, τ are automorphisms of R such that $\sigma d = d\sigma$ and $\tau d = d\tau$, we obtain that

$$d(x)\tau^{-1}\sigma(\tau(v))d(r')s[r, g(z)] = 0, \text{ for all } x, z, v \in I, r, r', s \in R.$$

Replacing v for $v\sigma^{-1}(s')$, $s' \in R$ in this equation, we arrive at

$$(11) \quad d(x)\sigma(v)s'd(r')s[r, g(z)] = 0, \text{ for all } x, z, v \in I, r, r', s, s' \in R.$$

In particular, letting s' by $[r, g(z)]s'$, r' by x and s by $\sigma(v)$ in (11), we find that

$$d(x)\sigma(v)[r, g(z)]s'd(x)\sigma(v)[r, g(z)] = 0, \text{ for all } v, x, z \in I, r, r', s' \in R.$$

Using the semiprimeness of R , we have

$$d(x)\sigma(v)[r, g(z)] = 0, \text{ for all } v, x, z \in I, r \in R,$$

and so,

$$\sigma^{-1}(d(x))I\sigma^{-1}([r, g(z)]) = 0, \text{ for all } x, z \in I, r \in R.$$

By Lemma 1, we see that

$$(12) \quad d(x)[r, g(z)] = 0, \text{ for all } x, z \in I, r \in R.$$

Replacing r by rs , $s \in R$ in (12), we get

$$d(x)r[s, g(z)] = 0, \text{ for all } x, z \in I, r, s \in R.$$

Writing $d(x)$ by s in this equation yields $d(x)r[d(x), g(z)] = 0$, for all $x, z \in I, r \in R$. In particular,

$$d(x)g(z)r[d(x), g(z)] = 0 \text{ and also } g(z)d(x)r[d(x), g(z)] = 0,$$

for all $x, z \in I, r, s \in R$. Hence combining the last two relations, we arrive at

$$[d(x), g(z)]R[d(x), g(z)] = 0, \text{ for all } x, z \in I.$$

By the semiprimeness of R , we have $[d(x), g(z)] = 0$, for all $x, z \in I$. Thus $d(x)g(z) = g(z)d(x)$, for all $x, z \in I$. We know $d(x)g(z) + d(z)g(x) = 0$ from (6). So, we conclude that $d(x)g(z) + g(x)d(z) = 0$, for all $x, z \in I$. By Lemma 5, we get d and g are orthogonal. \square

We can give the following corollaries in view of Lemma 3.

Corollary 2. *Let d and g (σ, τ) -derivations of R and $l(I) = 0$. Then the following conditions are equivalent.*

- (i) d and g are orthogonal;
- (ii) $dg = 0$;
- (iii) $gd = 0$;
- (iv) $dg + gd = 0$;
- (v) $d(x)g(x) = 0$, for all $x \in I$;
- (vi) dg is a (σ^2, τ^2) -derivation of R .

Corollary 3. *Let $l(I) = 0$. If d is (σ, τ) -derivation of R such that $d(x)^2 = 0$ for all $x \in I$, then $d = 0$.*

Lemma 6. *Let (D, d) , (G, g) be generalized (σ, τ) -derivations of R and $l(I) = 0$. Then the following are equivalent.*

- (i) For any $x, y \in I$, the following relations hold:
 - (a) $D(x)G(y) + G(x)D(y) = 0$,
 - (b) $d(x)G(y) + g(x)D(y) = 0$;
- (ii) $D(x)G(y) = d(x)G(y) = 0$, for all $x, y \in I$;
- (iii) $D(x)G(y) = 0$, for all $x, y \in I$ and $dG = dg = 0$.

Proof. (i) \Rightarrow (ii). Replacing x by yx , $y \in I$ in (a) and using (b), we have

$$\begin{aligned} 0 &= D(yx)G(y) + G(yx)D(y) \\ &= D(y)\sigma(x)G(y) + G(y)\sigma(x)D(y) + \tau(y)\{d(x)G(y) + g(x)D(y)\} \\ &= D(y)\sigma(x)G(y) + G(y)\sigma(x)D(y). \end{aligned}$$

Using the same tricks in Lemma 5, we get

$$\sigma^{-1}(D(x))I\sigma^{-1}(G(y)) = 0 = \sigma^{-1}(G(x))I\sigma^{-1}(D(y)), \text{ for all } x, y \in I.$$

Applying Lemma 1 and using σ is an automorphism of R , we see that

$$(13) \quad D(x)G(y) = 0 = G(y)D(x), \text{ for all } x, y \in I.$$

Now, putting zx for x , $z \in I$ in (13), $G(y)D(x) = 0$ and using (13), we conclude that

$$\tau^{-1}(G(y))z\tau^{-1}(d(x)) = 0, \text{ for all } x, y, z \in I.$$

We have

$$G(y)d(x) = d(x)G(y) = 0, \text{ for all } x, y \in I.$$

by Lemma 1. The proof is completed.

(ii) \Rightarrow (i). Suppose that $D(x)G(y) = d(x)G(y) = 0$, for all $x, y \in I$. Replacing x by yx , $y \in I$ in $D(x)G(y) = 0$, reduces to

$$\sigma^{-1}(D(y))x\sigma^{-1}(G(y)) = 0, \text{ for all } x, y \in I,$$

and so,

$$\sigma^{-1}(D(y))x\sigma^{-1}(G(z)) = 0, \text{ for all } x, y, z \in I,$$

by Lemma 2. We obtain that $D(y)G(z) = G(z)D(y) = 0$, by Lemma 1, which shows (a).

Writting zy by y , $z \in I$ in $D(x)G(y) = 0$, we find that (b), similarly.

(ii) \Rightarrow (iii). Replacing y by yx in $d(x)G(y) = 0$, we have

$$\tau^{-1}(d(x))y\tau^{-1}(g(x)) = 0, \text{ for all } x \in I.$$

and so,

$$\tau^{-1}(d(x))y\tau^{-1}(g(z)) = 0, \text{ for all } x, y, z \in I.$$

by Lemma 2. That is

$$d(x)\tau(I)g(z) = 0, \text{ for all } x, y, z \in I.$$

Thus, we obtain that $d(r)Rg(s) = 0$, for all $r, s \in R$, by Corollary 1. Hence, d and g orthogonal by [2, Lemma 1] and so, $dg = 0$, by Lemma 3. Letting yx by x in $d(x)G(y) = 0$, we arrive at, $d(r)RG(s) = 0$, for all $r, s \in R$, similarly. Thus, $d(r)G(s) = G(s)d(r) = 0$ by [2, Lemma 1]. Using $d(r)RG(s) = 0$, $d(r)G(s) = 0$ and $\sigma d = d\sigma$, $\tau d = d\tau$, we see that

$$\begin{aligned} 0 &= d(d(r)r'G(s)) = d(d(r)r')\sigma(G(s)) + \tau(d(r)r')dG(s) \\ &= d(\tau(r))\tau(r')dG(s). \end{aligned}$$

We get

$$d(r)r'dG(s) = 0, \text{ for all } r, r', s \in R,$$

by τ is an automorphism of R . Replacing r by $G(s)$ in this equation, we have $dG(s)r'dG(s) = 0$, for all $r', s \in R$. Since R is semiprime, we conclude that, $dG = 0$.

(iii) \Rightarrow (ii). Since $dG = 0$, $dg = 0$ and $\sigma d = d\sigma$, $\tau d = d\tau$, for all $r, s \in R$, we have

$$\begin{aligned} 0 &= dG(rs) = d(G(r)\sigma(s) + \tau(r)g(s)) \\ &= dG(r)\sigma^2(s) + \tau(G(r))d(\sigma(s)) + d(\tau(r))\sigma(g(s)) + \tau^2(r)dg(s) \\ &= G(\tau(r))d(\sigma(s)) + d(\tau(r))g(\sigma(s)). \end{aligned}$$

Since σ, τ are automorphisms of R , we get

$$G(r)d(s) + d(r)g(s) = 0, \text{ for all } r, s \in R.$$

In particular,

$$(14) \quad G(x)d(y) + d(x)g(y) = 0, \text{ for all } x, y \in I.$$

Writting zy by y , $z \in I$ in $D(x)G(y) = 0$ and using the same techniques, we obtain that

$$g(y)D(x) = D(x)g(y) = 0, \text{ for all } x, y \in I.$$

Now, replacing x by xz , $z \in I$ in $g(y)D(x) = 0$ and using this equation, we find that

$$0 = g(y)D(xz) = g(y)D(x)\sigma(z) + g(y)\tau(x)d(z) = g(y)\tau(x)d(z),$$

and so,

$$\tau^{-1}(g(y))x\tau^{-1}(d(z)) = 0, \text{ for all } x, y, z \in I.$$

By Lemma 1 and τ is an automorphism of R , we arrive at

$$d(z)g(y) = g(y)d(z) = 0, \text{ for all } y, z \in I.$$

We see that $G(x)d(y) = 0$, for all $x, y \in I$, from (14). Writting zy by y in this relation and using this relation, we get

$$G(x)\tau(z)d(y) = 0, \text{ for all } x, y, z \in I.$$

By Corollary 1, we obtain that

$$G(r)r'd(s) = 0, \text{ for all } r, r', s \in R.$$

In particular, we get

$$G(y)zd(x) = 0, \text{ for all } x, y, z \in I.$$

Again applying Lemma 1, we conclude that $d(x)G(y) = 0$, for all $x, y \in I$. This complates the proof. \square

Theorem 2. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R such that $l(I) = 0$ and $(D, d), (G, g)$ be generalized (σ, τ) -derivations of R . Then the following are equivalent.*

- (i) (D, d) and (G, g) are orthogonal;
- (ii) For any $x, y \in I$, the following relations hold:
 - (a) $D(x)G(y) + G(x)D(y) = 0$,
 - (b) $d(x)G(y) + g(x)D(y) = 0$;
- (iii) $D(x)G(y) = d(x)G(y) = 0$, for all $x, y \in I$;
- (iv) $D(x)G(y) = 0$, for all $x, y \in I$ and $dG = dg = 0$.

Proof. (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) are clear by Lemma 6.

(iii) \Rightarrow (i). Writting xz by $x, z \in I$ in $D(x)G(y) = 0$ and using $d(x)G(y) = 0$, we have

$$D(x)\sigma(z)G(y) = 0, \text{ for all } x, y, z \in I.$$

By Corollary 1, we conclude that $D(R)RG(R) = 0$. Thus D and G are orthogonal.

(i) \Rightarrow (iii). Since (D, d) and (G, g) are orthogonal, we get

$$D(x)zG(y) = 0, \text{ for all } x, y, z \in I.$$

By Lemma 1, we have $D(x)G(y) = G(y)D(x) = 0$, for all $x, y \in I$. Replacing x by $xz, z \in I$ in $G(y)D(x) = 0$ and using this equation, we obtain that

$$G(y)\tau(x)d(z) = 0, \text{ for all } x, y, z \in I.$$

That is,

$$\tau^{-1}(G(y))x\tau^{-1}d(z) = 0, \text{ for all } x, y, z \in I.$$

Using Lemma 1 and τ is an automorphism, we conclude that, $d(z)G(y) = 0$, for all $y, z \in I$. \square

Corollary 4. *Let (D, d) be generalized (σ, τ) -derivation of R and $l(I) = 0$. If $D(x)D(y) = 0$, for all $x, y \in I$, then $D = d = 0$.*

Proof. Putting zy by y , $z \in I$ in the hypothesis, we see that

$$D(x)\tau(z)d(y) = 0, \text{ for all } x, y, z \in I,$$

and so,

$$\tau^{-1}(D(x))z\tau^{-1}(d(y)) = 0, \text{ for all } x, y, z \in I.$$

Using Lemma 1 and τ is an automorphism, we have $d(y)D(x) = 0$, for all $x, y \in I$. Replacing x by zx , we have $d(y)\tau(z)d(x) = 0$. We obtain that $d(R)Rd(R) = 0$, by Corollary 1. Since R is semiprime, $d = 0$. Writting xz by x in $D(x)D(y) = 0$ and using $d(y)D(x) = 0$, we see that $D(x)\sigma(z)D(y) = 0$, for all $x, y, z \in I$. Thus we get $D(R)RD(R) = 0$, from Corollary 1 and so, $D = 0$, by the semiprimeness of R . \square

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