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A COMMON FIXED POINT THEOREM FOR A SEQUENCE OF MAPS IN A GENERALIZED MENGER SPACE

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Abstract. The object of this paper is to establish a unique common fixed point theorem through weak compatibility for a sequence of self-maps satisfying a generalized contractive condition in a generalized Menger space. It improves and generalizes the result of Milovanovic-Arandelovic [2], Vasuki [10] and Sehgal and Bharucha-Reid [8]. All the results presented in this paper are new.

1. Introduction

Menger space is a generalization of metric spaces in which the distances between points are specified by probability distributions rather than numbers. The general notion was introduced by Menger [6] in 1942 and has been developed by a number of authors. Schweizer and Sklar [7], studied this concept and gave some fundamental results on this space. It has been observed by many authors that a contraction in metric space may be exactly translated into a probabilistic metric space endowed with the min. norm. In [8] Sehgal and Bharucha-Reid established Banach contraction principle in a complete Menger space, which is a milestone in developing fixed point theory in a Menger space.

Recently, Jungck and Rhoades [5] termed a pair of self-maps to be coincidentally commuting or equivalently weak compatible if they commute at their coincidence points. Precisely, commuting implies weak compatibility. But it is to be observed here (in Example 1) that a weak compatible pair needs not to be commuting in a Menger space.

In this paper we establish a unique common fixed point theorem for a sequence of self-maps and an other self-map through weak compatibility satisfying a new generalized contractive condition in a generalized menger space, which generalizes and improves the results of [2], [8] and [10].

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2. Preliminaries

Definition 1 ([9]). A mapping $F : R \to R^+$ is called a *distribution* if it is non-decreasing left continuous with

$$\inf\{F(t) : t \in R\} = 0 \quad \text{and} \quad \sup\{F(t) : t \in R\} = 1.$$

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2. A *probabilistic metric space* (*PM-space*) is an ordered pair (X, F), where X is an abstract set of elements and $F : X \times X \to L$, defined by $(p, q) \longmapsto F_{p,q}$, where L is the set of all distribution functions i.e. $L = \{F_{p,q} | p, q \in X\}$, if the functions $F_{p,q}$ satisfy:

(a) $F_{p,q}(x) = 1$, for all $x > 0$, if and only if $p = q$;
(b) $F_{p,q}(0) = 0$;
(c) $F_{p,q} = F_{q,p}$;
(d) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$.

Definition 3 ([9]). A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if

(e) $t(a, 1) = a, t(0, 0) = 0;,$
(f) $t(a, b) = t(b, a);$
(g) $t(c, d) \geq t(a, b)$ for $c \geq a, d \geq b;$
(h) $t(t(a, b), c) = t(a, t(b, c)),$

for all $a, b, c, d \in [0, 1]$.

Definition 4 ([1]). A *generalized Menger space* is a triplet (X, F, t) where (X, F) is PM-space and t is a t-norm such that for all $p, q, r \in X$ and for all $x, y \geq 0$,

$$F_{p,r}(x + y) \geq t(F_{p,q}(x), F_{q,r}(y)).$$

If in a generalized Menger space $\lim_{t\to\infty} F_{x,y}(t) = 1$, then it is said to be a Menger space.

Definition 5. Let (X, F, t) be a Menger space with *sup* $_{0<x<1}$ $t(x, x) = 1$. A sequence $\{p_n\}$ in X is said to *converge to a point* p in X (written as $p_n \to p$) if for every $\epsilon > 0$ and $\lambda > 0, \exists$ an integer $M(\epsilon, \lambda)$ such that $F_{p_n,p}(\epsilon) > 1-\lambda, \forall n \geq M(\epsilon, \lambda)$. Further, the sequence is said to be a *Cauchy sequence* if for each $\epsilon > 0$ and $\lambda > 0, \exists$ an integer $M(\epsilon, \lambda)$ such that $F_{p_n,p_m}(\epsilon) > 1 - \lambda, \forall n, m \geq M(\epsilon, \lambda)$. A Menger space (X, F, t) is said to be *complete* if every Cauchy sequence in it converges to a point of it.

A complete metric space can be treated as a complete Menger space in the following way:

Proposition 6 ([7]). *If (X, d) is a metric space then the metric d induces a map from $X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, for all $p, q \in X$ and $x \in R$. Further, if the t-norm $t : [0, 1 \times [0, 1] \to [0, 1]$ is defined by $t(a, b) = min\{a, b\}$, then (X, F, t) is a Menger space. It is complete if (X, d) is complete.*

The space (X, F, t) so obtained is called *induced Menger space.*

In the following T_M will denote the minimum t-norm.

Definition 7 ([5]). Self mappings A and S of a Menger space (X, F, t) are said to be *weak compatible* if they commute at their coincidence points, i.e., $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.

In the following example, self maps A and S are weak compatible but they are non-commuting.

Example 8. Let (X, d) be a metric space, where $X = [0, 2]$ and (X, F, t) be the induced Menger space with $F_{p,q}(\epsilon) = H(\epsilon - d(p, q)), \forall p, q \in X$ and $\epsilon > 0$. Let I be the identity map on X. Define self maps A and S as follows;

$$A(x) = \begin{cases} 2 - x, & x \in [0, 1), \\ 2, & x \in [1, 2], \end{cases} \qquad S(x) = \begin{cases} x, & x \in [0, 1), \\ 2, & , x \in [1, 2]. \end{cases}$$

Now $AS(1/2) = 3/2$ and $SA(1/2) = 2$. Hence $AS(1/2) \neq SA(1/2)$ Thus (A, S) is non-commuting. Also the set of coincident points of A and S is $[1, 2]$. Now for any $x \in [1, 2], Ax = Sx = 2$ and $AS(x) = A(2) = 2 = S(2) = SA(x)$. Thus maps A and S are weak compatible though they are non-commuting.

3. MAIN RESULTS

Theorem 9. *Let $\{A_n\}$ be a sequence of self-maps and S be a self-map of generalized complete Menger space (X, F, T_M) satisfying:*
(3.11) *$A_n(X) \subseteq S(X)$, for all n;*
(3.12) *pairs (A_n, S) are weak compatible, for all n.*
(3.13) *$S(X)$ is complete;*
(3.14) *there exists $k \in [0, 1)$ such that for each pair (A_i, A_j) , for all $x, y \in X$, and for all $t > 0$,*

$$F^2_{A_ix,A_jy}(kt) \geq \min \left\{ \begin{array}{l} F^2_{A_ix,Sx}(t), F^2_{Sy,A_jy}(t), F^2_{Sx,Sy}(t), \\ F_{A_jy,Sx}(2t)F_{A_ix,Sy}(t), F_{A_jy,Sx}(2t)F_{A_ix,Sx}(t), \\ F_{A_jy,Sx}(2t)F_{Sx,Sy}(t), F_{A_jy,Sx}(2t)F_{A_jy,Sy}(t) \end{array} \right\}.$$

Then for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = A_nx_{n-1}$, for all n, is convergent and its limit is the unique common fixed point for all A_n and S.

Proof. : Define sequences $\{x_n\}$ and $\{y_n\}$ in X by $A_nx_{n-1} = Sx_n = y_n$, for $n = 0, 1, 2...$ First we prove that $\{y_n\}$ is a Cauchy sequence in X. Putting

$x = x_{n-1}, y = x_n$ for the pair (A_n, A_{n+1}) in (3.14) we have,

$$\begin{aligned}
&F^2_{y_n,\, y_{n+1}}(kt)\\
&\quad = F_{A_n x_{n-1}, A_{n+1} x_n}(kt)\\
&\quad \geq \min\left\{\begin{array}{l} F^2_{A_n x_{n-1}, Sx_{n-1}}(t), F^2_{Sx_n, A_{n+1}x_n}(t), F^2_{Sx_{n-1}, Sx_n}(t),\\ F_{A_{n+1}x_n, Sx_{n-1}}(2t) F_{A_n x_{n-1}, Sx_n}(t), F_{A_{n+1}x_n, Sx_{n-1}}(2t) F_{A_n x_{n-1}, Sx_{n-1}}(t),\\ F_{A_{n+1}x_n, Sx_{n-1}}(2t) F_{Sx_{n-1}, Sx_n}(t), F_{A_{n+1}x_n, Sx_{n-1}}(2t) F_{A_{n+1}x_n, Sx_n}(t) \end{array}\right\}\\
&\quad = \min\left\{\begin{array}{l} F^2_{y_{n-1}, y_n}(t), F^2_{y_n, y_{n+1}}(t), F^2_{y_{n-1}, y_n}(t),\\ F_{y_{n+1}, y_{n-1}}(2t) F_{y_n, y_n}(t), F_{y_{n+1}, y_{n-1}}(2t) F_{y_n, y_{n+1}}(t),\\ F_{y_{n+1}, y_{n-1}}(2t) F_{y_n, y_{n-1}}(t), F_{y_{n+1}, y_{n-1}}(2t) F_{y_n, y_{n+1}}(t) \end{array}\right\}\\
&\quad \geq \min\left\{F^2_{y_{n-1}, y_n}(t), F^2_{y_n, y_{n+1}}(t), F^2_{y_{n-1}, y_n}(t)\right\}.
\end{aligned}$$

As

$$F_{y_{n+1}, y_{n-1}}(2t) \geq \min\{F_{y_{n-1}, y_n}(t), F^2_{y_n,\, y_{n+1}}(t) \ \geq \min\{F^2_{y_{n-1}, y_n}(t), F^2_{y_n, y_{n+1}}(t)\ \},$$

$$F_{y_{n+1}, y_{n-1}}(2t) F_{y_n, y_{n+1}}(t) \geq \min\{F^2_{y_n, y_{n+1}(t)}, F_{y_{n-1}, y_{n-1}}(t), F_{y_n, y_{n+1}}(t)\}$$

and

$$F_{y_{n+1}, y_{n-1}}(2t) F_{y_n, y_{n-1}}(t) \geq \min\{F^2_{y_{n-1}, y_n}(t), F_{y_{n-1}, y_n}(t), F_{y_n, y_{n+1}}(t)\}.$$

Thus

$$\begin{aligned}
F^2_{y_n, y_{n+1}}(kt) &\geq \min\{F^2_{y_{n-1}, y_n}(t), F^2_{y_n, y_{n+1}}(t)\}\\
&\geq \min\{F^2_{y_{n-1}, y_n}(t), F^2_{y_n, y_{n+1}}(t/k)\},\\
&\ \vdots\\
&\geq \min\{F^2_{y_{n-1}, y_n}(t), F^2_{y_n, y_{n+1}}(t/k^m)\}
\end{aligned}$$

Taking limit as $m \to \infty$, we get

$$F^2_{y_n, y_{n+1}}(kt) \geq F^2_{y_{n-1}, y_n}(t), \text{ for all } t > 0.$$

Hence

$$F^2_{y_n, y_{n+1}}(t) = 1, \forall t > 0. \tag{1}$$

Again

$$F_{y_n, y_{n+2}}(t) \geq T\{Fy_n, y_{n+1}(t/2), Fy_{n+1}, y_{n+2}(t/2)\}$$

implies

$$\lim_{n\to\infty} F_{y_n, y_{n+2}} = 1, \text{ in view of (1)}.$$

Also

$$F_{y_n, y_{n+3}}(t) \geq T\{Fy_n, y_{n+2}(t/2), Fy_{n+2}, y_{n+3}(t/2)\}$$

implies

$$\lim_{n\to\infty} F_{y_n, y_{n+3}} = 1.$$

Proceeding successively we have,

$$\lim_{n\to\infty} F_{y_n, y_{n+p}} = 1.$$

Thus $\{y_n\}$ is a Cauchy sequence in $S(X)$. Hence $\{y_n\} = \{Sx_n\} \to u \in S(X)$. As $S(X)$ is complete, there exists $v \in X$ such that

$$u = Sv. \tag{2}$$

Step I: For $m \in N$, taking $x = x_{n-1}$ and $y = v$ in (3.14) for the pair (A_n, A_m) and using (2) we get,

$$\begin{aligned} &F^2_{A_n x_{n-1}, A_m v}(kt) \\ &\quad \geq \min\left\{\begin{array}{l} F^2_{A_n x_{n-1}, Sx_{n-1}}(t), F^2_{Sv, A_m v}(t), F^2_{Sx_{n-1}, Sv}(t), \\ F_{A_m v, Sx_{n-1}}(2t) F_{A_m v, Sx_{n-1}}(t), F_{A_m v, Sx_{n-1}}(2t) F_{Sx_{n-1}, Sv}(t), \\ F_{A_m v, Sx_{n-1}}(2t) F_{A_m v, Sv}(t) \end{array}\right\} \end{aligned}$$

i.e.

$$F^2_{y_n, A_m v}(kt) \geq \min\left\{\begin{array}{l} F^2_{y_{n-1}, y_n}(t), F^2_{u, A_m v}(t), F^2_{y_{n-1}, u}(t), \\ F_{A_m v, y_{n-1}}(2t) F_{u, y_n}(t), F_{A_m v, y_{n-1}}(2t) F_{y_{n-1}, u}(t), \\ F_{A_m v, y_{n-1}}(2t) F_{u, y_{n-1}}(t), F_{A_m v, y_{n-1}}(2t) F_{A_m v, u}(t) \end{array}\right\}$$

Taking limit as $n \to \infty$, we get,

$$\begin{aligned} &F^2_{u, A_m v}(kt) \\ &\quad \geq \min\left\{\begin{array}{l} F^2_{u,u}(t), F^2_{u, A_m v}(t), F^2_{u,u}(t), F_{A_m v, u}(2t) F_{u,u}(t), \\ F_{A_m v, u}(2t) F_{u,u}(t), F_{A_m v, u}(2t) F_{u,u}(t), F_{A_m v, u}(2t) F_{A_m v, u}(t) \end{array}\right\} \\ &\quad = \min\{1, F^2_{u, A_m v}(t), 1, F_{A_m v, u}(2t), F_{A_m v, u}(2t), F_{A_m v, u}(2t)(t), \\ &\qquad F_{A_m v, u}(2t) F_{A_m v, u}(t)\} \\ &\quad \geq F^2_{u, A_m v}(t); \quad \forall t > 0 \end{aligned}$$

implies

$$F^2_{u, A_m v}(kt) \geq Fu, A_m v(t), \forall t > 0,$$

which gives $A_m v = u$. Thus, $A_m v = u = Sv$. As (A_m, S) is weak compatible we have,

$$A_m u = Su, \forall m. \tag{3}$$

Step II: Again putting $x = x_{n-1}$ and $y = u$ in (3.14) for the pair (A_n, A_m) and using (3) we get

$$\begin{aligned} &F^2_{A_n x_{n-1}, A_m u}(kt) \\ &\quad \geq \min\left\{\begin{array}{l} F^2_{y_{n-1}, Sx_{n-1}}(t), F^2_{Su, A_m u}(t), F^2_{Sx_{n-1}, Su}(t), \\ F_{A_m u, Sx_{n-1}}(2t) F_{A_n x_{n-1}, Su}(t), F_{A_m u, Sx_{n-1}}(2t) F_{A_n x_{n-1}, Sx_{n-1}}(t), \\ F_{A_m u, Sx_{n-1}}(2t) F_{Sx_{n-1}, Su}(t), F_{A_m u, Sx_{n-1}}(2t) F_{A_m u, Su}(t) \end{array}\right\}, \end{aligned}$$

and

$$F^2_{y_n, Su}(kt) \geq \min\left\{\begin{array}{l} F^2_{y_n, y_{n-1}}(t), F^2_{Su, Su}(t), F^2_{y_{n-1}, Su}(t), \\ F_{Su, y_{n-1}}(2t) F_{y_n, Su}(t), F_{Su, y_{n-1}}(2t) F_{y_{n-1}, y_n}(t), \\ F_{Su, y_{n-1}}(2t) F_{yn-1, Su}(t), F_{Su, y_{n-1}}(2t) F_{Su, Su}(t) \end{array}\right\}.$$

Taking limit as $n \to \infty$ we get

$$\begin{aligned}
F^2_{u,Su}(kt) &\geq \min\left\{\begin{array}{l} 1, 1, F^2_{u,Su}(t), F_{Su,u}(2t)F_{u,Su}(t), \\ F_{Su,u}(2t), F_{Su,u}(2t)F_{u,Su}(t), F_{Su,u}(2t) \end{array}\right\} \\
&= \min\{F^2_{u,Su}(t), F_{Su,u}(2t)F_{u,Su}(t), F_{Su,u}(2t)\} \\
&= F^2_{u,Su}(t),
\end{aligned}$$

which implies $F_{u,Su}(kt) \geq F_{Su,u}(t)$, for all $t > 0$, which gives $Su = u$. Thus, $A_m u = Su = u$ and we get that u is a common fixed point of the sequence of self-maps $\{A_n\}$ and S.

Uniqueness: Let z be another common fixed point of the sequence of self-maps $\{A_n\}$ and S. Then, $z = A_m z = Sz$, for all m. Putting $x = z$ and $y = u$ in (3.14) for the pair (A_1, A_2) we get,

$$\begin{aligned}
F^2_{A_1z,A_2u}(kt) &\geq \min\left\{\begin{array}{l} F^2_{A_1z,Sz}(t), F^2_{Su,A_2u}(t), F^2_{Sz,Su}(t), \\ F_{A_2u,Sz}(2t)F_{A_1z,Su}(t), F_{A_2u,Sz}(2t)F_{A_1z,Sz}(t), \\ F_{A_2u,Sz}(2t)F_{Sz,Su}(t), F_{A_2u,Sz}(2t)F_{A_2u,Su}(t) \end{array}\right\} \\
&= \min\{1, 1, F^2_{z,u}(t), F_{u,z}(2t)F_{z,u}(t), F_{u,z}(2t)\} \\
&= F^2_{z,u}(t)
\end{aligned}$$

i. e.

$$F_{z,u}(kt) \geq F_{z,u}(t), \text{ for all } t > 0,$$

which gives

$$F_{z,u}(t) = 1, \text{ for all } t > 0.$$

Hence $u = z$. Therefore u is the unique common fixed point of the sequence of self-maps $\{A_n\}$ and S. □

Again we note that for $a, b \in [0, 1]$ we have either $ab \geq a^2$ or $ab \geq b^2$. Hence $ab \geq \min\{a^2, b^2\}$. Thus for $a, b, c \in [0, 1]$ we have

$$ab \geq \min\{a^2, b^2\}, \ bc \geq \min\{b^2, c^2\}, \text{ and } ac \geq \min\{a^2, c^2\}.$$

It gives

$$\min\{ab, bc, ca\} \geq \min\{a^2, b^2, c^2\}.$$

Therefore,

$$\begin{aligned}
&\min\left\{F_{A_ix,Sx}(t)F_{Sx,Sy}(t), F_{Sx,Sy}(t)F_{A_jy,Sy}(t), F_{A_jy,Sy}(t)F_{A_ix,Sx}(t)\right\} \\
&\qquad\qquad \geq \min\{F^2_{A_ix,Sx}(t), F^2_{A_jy,Sy}(t), F^2_{Sx,Sy}(t)\}.
\end{aligned}$$

Thus from Theorem 9, it follows that

Corollary 10. *Let $\{A_n\}$ be a sequence of self-maps and S be a self-map of a complete Menger space (X, F, T_M) satisfying* (3.11), (3.12), (3.13) *and*

(3.21) *for all* i, j *for all* $x \in X$ *and* $\forall t > 0$, *there exists* $k \in (0, 1)$ *such that*

$$F^2_{A_ix,A_jy}(kt)$$
$$\geq \min \left\{ \begin{array}{l} F_{A_ix,Sx}(t)F_{Sx,Sy}(t), F_{Sx,Sy}(t).F_{A_jy,Sy}(t), F_{A_jy,Sy}(t)F_{A_ix,Sx}(t), \\ F_{A_jy,Sx}(2t)F_{A_ix,Sy}(t), F_{A_jy,Sx}(2t)F_{A_ix,Sx}(t), \\ F_{A_jy,Sx}(2t)F_{Sx,Sy}(t), F_{A_jy,Sx}(2t)F_{A_jy,Sy}(t) \end{array} \right\}.$$

Then for any $x_0 \in X$, *the sequence* $\{x_n\}$ *defined by* $x_n = A_nx_{n-1}$, *for all* n, *is convergent and its limit is the unique common fixed point for all* A_n *and* S.

Taking S to be a surjective map in Corollary 10, we get

Corollary 11. *Let* $\{A_n\}$ *be a sequence of self-maps and* S *be a surjective self-map of a complete generalized Menger space* (X, F, T_M) *satisfying* (3.12) *and* (3.21). *Then for any* $x_0 \in X$, *the sequence* $\{x_n\}$ *defined by* $x_n = A_nx_{n-1}$, *for all* n, *is convergent and its limit is the unique common fixed point for all* A_n *and* S.

Taking S to be an identity map in Corollary 11, we get

Corollary 12. *Let* $\{A_n\}$ *be a sequence of self-maps of a complete generalized Menger space* (X, F, min) *satisfying*

(3.41) *for all* i, j, *for all* $x \in X$ *and* $\forall t > 0$, *there exists* $k \in (0, 1)$ *such that,*

$$F^2_{A_ix,A_jy}(kt)$$
$$\geq \min \left\{ \begin{array}{l} F_{A_ix,x}(t)F_{x,y}(t), F_{x,y}(t).F_{A_jy,y}(t), F_{A_jy,y}(t)F_{A_ix,x}(t), \\ F_{A_jy,x}(2t)F_{A_ix,y}(t), F_{A_jy,x}(2t)F_{A_ix,x}(t), \\ F_{A_jy,x}(2t)F_{x,y}(t), F_{A_jy,x}(2t)F_{A_jy,y}(t) \end{array} \right\}.$$

Then for any $x_0 \in X$, *the sequence* $\{x_n\}$ *defined by* $x_n = A_nx_{n-1}$, *for all* n, *is convergent and its limit is the unique common fixed point of all* A_n.

In [10] Vasuki proved the following result:

Theorem ([10])**.** *Let* $\{A_n\}$ *be a sequence of self-maps of a complete Menger space* (X, F, t) *into itself with* $t(x, y) = min\{x, y\}$, *for every* $x, y \in [0, 1]$. *If for any two maps* A_i *and* A_j *the following inequality*

$$F^2_{A_ix,A_jy}(kt) \geq \min \left\{ \begin{array}{l} F_{A_ix,x}(t)F_{x,y}(t), F_{x,y}(t).F_{A_jy,y}(t), \\ F_{A_jy,y}(t)F_{A_ix,x}(t), F_{A_jy,x}(2t)F_{A_ix,y}(t) \end{array} \right\}.$$

holds for all $x, y \in X$, *where* $0 \leq k < 1$, *then for any* $x_0 \in X$, *the sequence* $\{x_n\}$ *defined by* $x_n = A_nx_{n-1}$, *for all* n, *is convergent and its limit is the unique common fixed point for all* A_n.

Remark 13. The quoted result of [10] follows from Corollary 12. Moreover, the contractive condition of our corollary is more general. Thus all the results of this paper from Theorem 9 to Corollary 12 are the successive betterments of the result of [10].

Taking S to be a surjective self-map in Theorem 9, we obtain the following result:

Corollary 14. *Let $\{A_n\}$ be a sequence of self-maps and S be a surjective self-map of a complete generalized Menger space (X, F, T_M) satisfying* (3.14) *and* (3.21)*. Then for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = A_n x_{n-1}$, for all n, is convergent and its limit is the unique common fixed point of all A_n and S.*

In [2] Milovanovic-Arandelovic established the following result:

Theorem ([2])**.** *Let $\{T_n\}$ be a sequence of self-mappings of a complete Menger space (X, F, t) and $S : X \to X$ be a continuous mapping such that $T_n(X) \subseteq S(X)$ and S is commuting with each T_n. Let $t(r, s) = min\{r, s\}$, for every $r, s \in [0, 1]$. Suppose that there exists a constant $k \in [0, 1)$ such that for any two maps T_i and T_j and for every $x, y \in X$,*

$$F^2_{T_ix,T_jy}(kt) \geq \min \left\{ \begin{array}{l} F^2_{T_ix,Sx}(t), F^2_{Sy,T_jy}(t), F^2_{Sx,Sy}(t), \\ F_{T_jy,Sx}(2t) F_{T_ix,Sy}(t), F_{T_jy,Sx}(2t) F_{T_ix,Sx}(t) \end{array} \right\}$$

holds for all $t > 0$, then there exists a unique common fixed point for all T_i and S.

Remark 15. Corollary 14 supplements and generalizes the above result of [2]. It stresses that if S is a surjective map its continuity is not required to prove the result for even a more general contraction. In addition, In that case the commutativity of the pairs (T_i, S) also was reduced to their weak compatibility only.

Theorem 9 is an important alternate result for the quoted result of [2], through weak compatibility without assuming the continuity of the map S and commutativity of the pairs (A_n, S) and still having a more general contraction.

Taking S to be an identity map in Corollary 14, we get

Corollary 16. *Let $\{A_n\}$ be a sequence of self-maps of a generalized complete generalized Menger space (X, F, T_M) satisfying*

(3.81) *there exists $k \in [0, 1)$such that for all i, j, for all $x, y \in X$, for all $t > 0$,*

$$F^2_{A_ix,A_jy}(kt) \geq \min \left\{ \begin{array}{l} F^2_{A_ix,x}(t), F^2_{y,A_jy}(t), F^2_{x,y}(t), \\ F_{A_jy,x}(2t) F_{A_ix,y}(t), F_{A_jy,x}(2t) F_{A_ix,x}(t), \\ F_{A_jy,x}(2t) F_{x,y}(t), F_{A_jy,x}(2t) F_{A_jy,y}(t) \end{array} \right\}.$$

Then for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = A_n x_{n-1}$, for all n, is convergent and its limit is the unique common fixed point for all A_n.

The study of fixed point in theory of PM-space was started by V. M. Sehgal and A. T. Bharucha-Reid in [8]. The following definition and theorem appeared in their paper.

Definition ([8])**.** A mapping f of a PM-space (X, F) into itself is a *contraction* if there exist $0 < k < 1$ such that for each x and y in X,

$$F_{fx,fy}(kt) \geq F_{x,y}(t), \text{ for all } t > 0.$$

Theorem ([8])**.** *Let (X, F, t) be a complete Menger space where $t(a, b) = \min\{a, b\}$. If f is any contraction, there exists a unique $p \in X$ such that $f(p) = p$. Moreover, $\lim_{n\to\infty} f^n(q) = p$ for each q in X.*

We obtain the following more complete result with a more general contractive condition from our Corollary 16.

Theorem 17. *Let f be a self-map of a complete generalized Menger space (X, F, T_M) satisfying*

(3.91) *there exists $k \in [0, 1)$ such that for $x, y \in X, \forall t > 0$*

$$F^2_{fx,fy}(kt) \geq \min \left\{ \begin{array}{l} F^2_{fx,x}(t), F^2_{fy,y}(t), F^2_{x,y}(t), \\ F_{fy,x}(2t)F_{fx,y}(t), F_{fy,x}(2t)F_{fx,x}(t), \\ F_{fy,x}(2t)F_{x,y}(t), F_{fy,x}(2t)F_{fy,y}(t) \end{array} \right\}.$$

Then there exists a unique $p \in X$ such that $f(p) = p$. Moreover, $\lim_{n\to\infty} f^n(q) = p$, for all n, for each q in X.

Proof. The result follows from Corollary 16 by taking $A_n = f$, for all n, as $x_n = f^n(x_0)$ there. □

Remark 18. Restricting the contractive condition of Theorem 17 to the third factor only, the quoted result of [8] follows.

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