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$\pi G\alpha$-LOCALLY CLOSED SETS AND $\pi G\alpha$-LOCALLY CONTINUOUS FUNCTIONS

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Abstract. In this paper we introduce $\pi G\alpha$–LC sets, $\pi G\alpha$–LC^* sets and $\pi G\alpha$–LC^{**} sets and different notions of generalizations of continuous functions in topological space and discuss some of their properties. Further we prove pasting lemma for $\pi G\alpha$–LC^{**} continuous functions and $\pi G\alpha$–LC^{**} irresolute functions.

1. Introduction

Norman Levine [10] introduced the concept of generalized closed sets in 1970. The notion of a locally closed set in a topological space was implicitly introduced by Kuratowski and Sierpienski [9]. According to Bourbaki [3] a subset of a topological space X is locally closed in X if it is the intersection of an open set and a closed set in X. In 1989, Ganster and Reilly [7] continued the study of locally closed set and also introduced the concept of LC-continuous functions to find a decomposition of continuous functions. Balachandran et al. [2] introduced the concept of generalized locally closed sets and obtained seven more different notions of generalized continuity. Arockia Rani et al. [1] introduced regular generalized locally closed sets and obtained six more new classes of generalized continuity using the concept of regular generalized closed sets [12].

The purpose of this paper is to introduce three new classes of sets called $\pi G\alpha$–LC set, $\pi G\alpha$–LC^* sets, $\pi G\alpha$–LC^{**} sets, which contains the class of glc-sets and α–LC sets by using the notion of $\pi g\alpha$-open and $\pi g\alpha$-closed sets. Also we introduce some different classes of continuity and irresoluteness and study some of their properties.

2. Preliminaries

Throughout this paper $(X,\tau),(Y,\sigma)$ and (Z,η) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space $X, \mathrm{cl}(A), \mathrm{int}(A)$ denotes the closure of A and interior

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of A repectively. $P(X)$ denotes the power set of X. A subset A of X is regular open if $A = \text{int}\,\text{cl}(A)$ and regular closed if $A = \text{cl}\,\text{int}(A)$. Finite union of regular open sets is called π-open. Recall the following definitions which will be used in sequel.

Definition 2.1. A subset A of a space (X, τ) is called

(a) *g-closed* [10] if $\text{cl}(A) \subset G$ whenever $A \subset G$ and G is open.
(b) *$\pi g\alpha$-closed* if $\alpha\,\text{cl}(A) \subset U$ whenever $A \subset U$ and U is π-open.
(c) *α-closed* [11] if $\text{cl}(\text{int}(\text{cl}(A))) \subset A$.

Definition 2.2. A subset A of (X, τ) is called

(a) *locally closed set* [6] if $A = G \cap F$ where G is open and F is closed.
(b) *generalized locally closed set* [2] (briefly *glc*-set) if $A = G \cap F$ where G is g-open and F is g-closed.
(c) *glc^*-set* [2] if there exist a g-open set G and a closed set F such that $A = G \cap F$.
(d) *glc^{**}-set* [2] if there exist an open set G and a g-closed set F such that $A = G \cap F$.
(e) *α-locally closed set* [8] (briefly *αlc*-set) if $A = G \cap F$ where G is α-open and F is α-closed.
(f) *α-lc^* set* [8] if there exist a α-open set G and a closed set F such that $A = G \cap F$.
(g) *α-lc^{**} set* [8] if there exist an open set G and a α-closed set F such that $A = G \cap F$.

The collection of all locally closed sets (resp. glc sets, α-*lc* sets, glc^* sets, α-lc^* sets, glc^{**} sets, α-lc^{**} sets) of (X, τ) will be denoted by $LC(X, \tau)$ (resp. $GLC(X, \tau)$, α-$LC(X, \tau)$, $GLC^*(X, \tau)$, $GLC^{**}(X, \tau)$, α-$LC^{**}(X, \tau)$).

Definition 2.3. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(a) *LC-continuous* [6] if $f^{-1}(V) \in LC(X, \tau)$ for each open set V of (Y, σ).
(b) *LC-irresolute* [6] if $f^{-1}(V) \in LC(X, \tau)$ for each $V \in LC(Y, \sigma)$
(c) *Sub-LC-continuous* [6] if there is a sub-base B for (Y, σ) such that $f^{-1}(V) \in LC(X, \tau)$ for each $V \in B$.

Definition 2.4. A space (X, τ) is called

(a) a *submaximal space* [4] if every dense subset of X is open.
(b) a *door space* [5] if every subset of X is either open or closed in X.
(c) a $\pi g\alpha$-$T_{\frac{1}{2}}$ space if every $\pi g\alpha$-closed set is α-closed.

3. $\pi G\alpha$-Locally Closed Sets

In this section we define $\pi g\alpha$-locally closed sets a weaker form of locally closed sets and compare it with the existing weaker forms of sets.

Definition 3.1. A subset S of (X, τ) is called

(a) $\pi g\alpha$-*locally closed* (briefly $\pi g\alpha$-*lc* set) if $S = A \cap B$ where A is $\pi g\alpha$-open in X and B is $\pi g\alpha$-closed in X.
(b) $\pi g\alpha$-lc^* *set* if there exist a $\pi g\alpha$-open set A and a closed set B such that $S = A \cap B$.
(c) $\pi g\alpha$-lc^{**} set if there exist a open set A and a $\pi g\alpha$-closed set B such that $S = A \cap B$.

The collection of all $\pi g\alpha$-*lc* sets (resp. $\pi g\alpha$-lc^* sets, $\pi g\alpha$-lc^{**}) of (X, τ) will be denoted by $\pi G\alpha$-$LC(X, \tau)$, (resp. $\pi G\alpha$-$LC^*(X, \tau)$, $\pi G\alpha$-$LC^{**}(X, \tau)$).

Remark 3.2. It is well known fact every closed (resp. open) set is locally closed. Every $\pi g\alpha$-open set (resp. $\pi g\alpha$-closed set) is $\pi g\alpha$-LC.

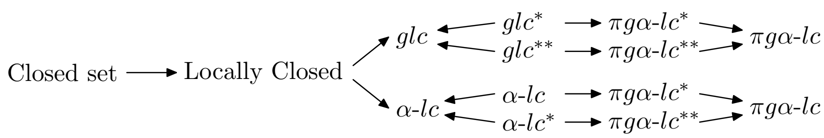
Remark 3.3. Every locally closed set is $\pi g\alpha$-locally closed but not conversely.

Example 3.4. Let $X = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}\}$. Then

$$LC(X) = \{\phi, X, \{a\}, \{b, c, d\}\}, \quad \pi G\alpha\text{–}LC(X) = P(X).$$

This show that $\pi g\alpha$-locally closed set need not to be locally closed.

Remark 3.5. From the definition above and definition 3.1 we have the following implications.



In the remark above the relationship cannot be reversible as the following example illustrates.

Example 3.6. (a) $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{b\}, \{c, d\}, \{b, c, d\}\}$
(i) $\{a, b, d\} \in \pi G\alpha$–$LC(X, \tau)$ but $\{a, b, d\} \notin \pi G\alpha$–$LC^*(X, \tau)$.
(ii) $\{a, b, c\} \in \pi G\alpha$–$LC^{**}(X, \tau)$ but $\{a, b, c\} \notin \pi G\alpha$–$LC^*(X, \tau)$.

(b) $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{c, d\}, \{d\}, \{a, d\}, \{a, c, d\}\}$
(i) $\{a, b, d\} \in \pi G\alpha$–$LC^*(X, \tau)$ but $\{a, b, d\} \notin g$–$LC^*(X, \tau)$.
(ii) $\{a, b, d\} \in g$–$LC(X, \tau)$ but $\{a, b, d\} \notin g$–$LC^*(X, \tau)$.

(c) $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$
(i) $\{a, b\} \in \pi G\alpha$–$LC(X, \tau)$ but $\{a, b\} \notin \alpha$–$LC(X, \tau)$.
(ii) $\{c\} \in \pi g\alpha$–$LC^*(X, \tau)$ but $\{c\} \notin \alpha$–$LC^*(X, \tau)$.
(iii) $\{c\} \in \pi G\alpha$–$LC^{**}(X, \tau)$ but $\{c\} \notin \alpha$–$LC^{**}(X, \tau)$.

Remark 3.7. If $A \in LC(X, \tau)$ then $A \in \pi G\alpha$–$LC^*(X, \tau)$ and $\pi G\alpha$–$LC^{**}(X, \tau)$. The converse is not true as seen in the following example. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$ then $LC(X, \tau) = \{\phi, X, \{a, b\}, \{c\}\}$, $\{a\} \notin LC(X, \tau)$ but $\{a\} \in \pi G\alpha$–$LC^*(X, \tau)$ and $\pi G\alpha$–$LC^{**}(X, \tau)$

Definition 3.8. A space is a $\pi g\alpha$-space if every $\pi g\alpha$-open set is open in X.

Proposition 3.9. *Let (X,τ) be a $\pi g\alpha$-space. Then*
(i) $\pi G\alpha\text{–}LC(X,\tau) = LC(X,\tau)$
(ii) $\pi G\alpha\text{–}LC(X,\tau) \subset GLC(X,\tau)$
(iii) $\pi G\alpha\text{–}LC(X,\tau) \subset \alpha LC(X,\tau)$

Proof. Obvious. □

Proposition 3.10. *If $\pi G\alpha O(X,\tau) = GO(X,\tau)$ then*

$$\pi G\alpha\text{–}LC(X,\tau) = GLC(X,\tau).$$

Proof. Follows from definition □

Proposition 3.11. *If X is a $\pi g\alpha\text{–}T_{\frac{1}{2}}$ space then*

$$\pi G\alpha\text{–}LC(X,\tau) = \alpha\text{–}LC(X,\tau).$$

Proof. Follows from definition. □

The converse of the above proposition need not hold.

Example 3.12. X=$\{a,b,c,d\}$, $\tau = \{\phi, X, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. Then

$$\pi G\alpha\text{–}LC(X,\tau) = \alpha\text{–}LC(X,\tau) = GLC(X,\tau) = P(X).$$

But

$$\begin{aligned} GO(X) &= \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\} \\ &\neq \pi G\alpha O(X,\tau), \\ \alpha O(X) &= \{\phi, X, \{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\} \\ &\neq \pi G\alpha O(X,\tau). \end{aligned}$$

Proposition 3.13. *If X is a $\pi g\alpha$-space, then*

$$\pi G\alpha\text{–}LC(X,\tau) = \pi G\alpha\text{–}LC^*(X,\tau) = \pi G\alpha\text{–}LC^{**}(X,\tau).$$

Proof. Straight forward. □

The hypothesis in proposition 3.13 can be weakened as follows.

Proposition 3.14. *If $\pi G\alpha O(X,\tau) \subset LC(X,\tau)$ and suppose that collection of all $\pi g\alpha$-closed sets ($\pi g\alpha$-open) sets are closed under finite intersection then $\pi G\alpha\text{–}LC(X) = \pi G\alpha\text{–}LC^*(X) = \pi G\alpha\text{–}LC^{**}(X)$*

Proof. Let $A \in \pi G\alpha\text{–}LC(X)$. Then $A = P \cap Q$ where P is $\pi g\alpha$-open and Q is $\pi g\alpha$-closed. Since $\pi G\alpha O(X,\tau) \subset LC(X,\tau)$ implies $\pi G\alpha C(X,\tau) \subset LC(X,\tau)$, we have Q is locally closed. Let $Q = M \cap N$ where M is open and N is closed. Hence $A = (P \cap M) \cap N$ where $(P \cap M)$ is $\pi g\alpha$-open and N is closed. Hence $A \in \pi G\alpha\text{–}LC^*(X)$. For any space X, $\pi G\alpha - LC^*(X) \subset \pi G\alpha\text{–}LC(X)$. Thus $\pi G\alpha\text{–}LC(X) = \pi G\alpha\text{–}LC^*(X)$.Let $B \in \pi G\alpha\text{–}LC(X)$. Then $B = P \cap Q$ where P is $\pi g\alpha$-open and Q is $\pi g\alpha$-closed. Since $\pi G\alpha O(X,\tau) \subset LC(X,\tau)$ implies P is locally closed, we have $P = M \cap N$ where M is open and N is closed. Hence $A = M \cap (N \cap Q)$ where M is open and $N \cap Q$ is $\pi g\alpha$-closed. For any space X, $\pi G\alpha\text{–}LC^{**}(X) \subset \pi G\alpha\text{–}LC(X)$. Thus $\pi G\alpha\text{–}LC(X) = \pi G\alpha\text{–}LC^{**}(X)$. □

Now, we obtain a characterization for $\pi G\alpha$–$LC^*(X)$ sets as follows.

Theorem 3.15. *For a subset S of (X,τ) the following are equivalent.*

(i) $S \in \pi G\alpha$–$LC^*(X,\tau)$
(ii) $S = P \cap \text{cl}(S)$ *for some $\pi g\alpha$-open set P*
(iii) $\text{cl}(S) - S$ *is $\pi g\alpha$-closed*
(iv) $S \cup (X - \text{cl}(S))$ *is $\pi g\alpha$-open.*

Proof. 1$\Rightarrow$ 2: Let $S \in \pi G\alpha$–$LC^*(X,\tau)$. Then there exist a $\pi g\alpha$-open set P and a closed set F in (X,τ) such that $S = P \cap F$. Since $S \subset P$ and $S \subset \text{cl}(S)$, we have $S \subset P \cap \text{cl}(S)$. Conversely, since $\text{cl}(S) \subset F$, $P \cap \text{cl}(S) \subset P \cap F = S$. Hence $S = P \cap \text{cl}(S)$.

2$\Rightarrow$1: Since P is $\pi g\alpha$ -open and $\text{cl}(S)$ is closed, $S = p\cap \text{cl}(S) \in \pi G\alpha$–$LC^*(X,\tau)$.

3$\Rightarrow$ 4: Let $F = cl(S) - S$. Then F is $\pi g\alpha$-closed, by assumption. $X - F = X \cap (\text{cl}(S) - S)^c = S \cup (X - \text{cl}(S))$. Since $X - F$ is $\pi g\alpha$-open, we have that $S \cup (X - cl(S))$ is $\pi g\alpha$-open.

4$\Rightarrow$ 3: Let $U = S \cup (X - \text{cl}(S))$. Then U is $\pi g\alpha$-open. This implies $X - U = X - (S \cup (X - \text{cl}(S))) = (X - S) \cap \text{cl}(S) = \text{cl}(S) - S$ is $\pi g\alpha$-closed.

2$\Rightarrow$4: Let $S = P \cap \text{cl}(S)$ for some $\pi g\alpha$-open set P. $S \cup (X - \text{cl}(S)) = P \cap (\text{cl}(S) \cup X - \text{cl}(S)) = P \cap X = P$ which is $\pi g\alpha$-open.

4$\Rightarrow$ 2: Let $U = S \cup (X - \text{cl}(S))$. Then U is $\pi g\alpha$-open. Now $U \cap \text{cl}(S) = (S \cup (X - \text{cl}(S))) \cap \text{cl}(S) = (S \cap \text{cl}(S)) \cup (X - \text{cl}(S) \cap \text{cl}(S)) = S \cup \phi = S$ for some $\pi g\alpha$-open set U. □

Remark 3.16. It is not true that $S \in \pi G\alpha$–$LC^*(X,\tau)$ iff $S \subset \text{int}(S \cup (X - cl(S)))$. Let $S = \{b, c\}$ be a subset of the topological space (X,τ) given in example 3.6 (a). Then $S \notin int(S \cup (X - \text{cl}(S))$, but $S \in \pi G\alpha$–$LC^*(X,\tau)$.

Definition 3.17. A topological space (X,τ) is called $\pi g\alpha$-submaximal, if every dense subset in it is $\pi g\alpha$-open.

Definition 3.18. Let (X,τ) be a topological space.If X is a submaximal, then it is $\pi g\alpha$-submaximal.

Proof. Follows from definition. □

Converse of the above is not true as seen in the following example.

Example 3.19. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Let $A = \{a, b\}$. A is dense in X, such that A is $\pi g\alpha$-open but not open.

Theorem 3.20. *A topological space (X,τ) is $\pi g\alpha$-submaximal if and only if $\pi g\alpha$–$LC^*(X,\tau) = P(X)$.*

Proof. Necessity: Let $S \in P(X)$. Let $U = S \cup (X - \text{cl}(S))$. Then $\text{cl}(U) = X$. U is dense in X and X is $\pi g\alpha$-submaximal implies U is $\pi g\alpha$-open. By theorem 3.15, $S \in \pi G\alpha$–$LC^*(X,\tau)$. *Sufficiency:* Let S be a dense subset of (X,τ). Then $S \cup (X - \text{cl}(S)) = S \cup \phi = S$. Now $S \in P(X) \Rightarrow S \in \pi G\alpha$–$LC^{**}(X,\tau)$. By theorem 3.15 $S \cup (X - \text{cl}(S)) = S$ is $\pi g\alpha$-open. Hence (X,τ) is $\pi g\alpha$-submaximal. □

Theorem 3.21. *For a subset S of (X,τ) if $S \in \pi G\alpha LC^{**}(X,\tau)$ then there exist a open set P such that $S = P \cap \mathrm{cl}(S)$ where $\mathrm{cl}(S)$ is the $\pi g\alpha$-closure of S.*

Proof. Let $S \in \pi G\alpha\text{–}LC^{**}(X,\tau)$. Then there exist an open set P and $\pi g\alpha$-closed set F of (X,τ) such that $S = P \cap F$. Since $S \subset P$ and $S \subset \mathrm{cl}(S)$, we have $S \subset P \cap \mathrm{cl}(S)$. Since $\mathrm{cl}(S) \subset F$, we have $P \cap \mathrm{cl}(S) \subset P \cap F \subset S$. Thus $S = P \cap \mathrm{cl}(S)$. □

4. Properties of $\pi g\alpha\text{–}LC$ sets

Theorem 4.1. *Let A and B be any two subsets of (X,τ). Suppose that the collection of $\pi g\alpha$-closed sets of (X,τ) is closed under finite intersections, then the following are true.*

(a) *If $A \in \pi G\alpha\text{–}LC(X,\tau)$ and B is $\pi g\alpha$-open or $\pi g\alpha$-closed then $A \cap B \in \pi G\alpha\text{–}LC(X,\tau)$.*

(b) *If $A \in \pi G\alpha\text{–}LC^*(X,\tau)$, $B \in \pi G\alpha\text{–}LC^*(X,\tau)$ then*

$$A \cap B \in \pi G\alpha\text{–}LC^*(X,\tau).$$

Proof. (a) $A \in \pi G\alpha\text{–}LC(X,\tau)$ implies $A \cap B = (G \cap F) \cap B$ for some $\pi g\alpha$-open set G and $\pi g\alpha$-closed set F. If B is $\pi g\alpha$-open then $A \cap B = (G \cap B) \cap F \in \pi G\alpha\text{–}LC(X,\tau)$. If B is $\pi g\alpha$-closed, then $A \cap B = G \cap (F \cap B) \in \pi G\alpha\text{–}LC(X,\tau)$, since $F \cap B$ is $\pi g\alpha$-closed.

(b) $A, B \in \pi G\alpha\text{–}LC^*(X,\tau)$ then by theorem 3.15, there exist $\pi g\alpha$-open sets P and Q such that $A = P \cap \mathrm{cl}(A)$ and $B = Q \cap cl(B)$. $P \cap Q$ is also $\pi g\alpha$-open. Then $A \cap B = (P \cap Q) \cap (\mathrm{cl}(A) \cap \mathrm{cl}(B)) \in \pi G\alpha\text{–}LC^*(X,\tau)$. □

Proposition 4.2. *Let A and B be any two subsets of (X,τ). Suppose that the collection of all $\pi g\alpha$-closed sets of (X,τ) is closed under finite intersection. If $A \in \pi G\alpha\text{–}LC^{**}(X,\tau)$ and B is closed or open, then $A \cap B \in \pi G\alpha\text{–}LC^{**}(X,\tau)$.*

Proof. If $A \in \pi G\alpha\text{–}LC^{**}(X,\tau)$, then there exist an open set G and a $\pi g\alpha$-closed set F of (X,τ) such that $A \cap B = (G \cap F) \cap B$. If B is open, then $A \cap B = (G \cap B) \cap F \in \pi g\alpha\text{–}LC^{**}(X,\tau)$. If B is closed, then $A \cap B = G \cap (F \cap B) \in \pi g\alpha\text{–}LC^{**}(X,\tau)$. □

Proposition 4.3. *Let A and Z be any two subsets of (X,τ) and let $A \subset Z$. Suppose that the collection of all $\pi g\alpha$ -open sets of (X,τ) is closed under finite intersection. If Z is $\pi g\alpha$ -open in (X,τ) and regular closed and if $A \in \pi G\alpha\text{–}LC^*(Z,\tau/Z)$ then $A \in \pi G\alpha\text{–}LC^*(X,\tau)$.*

Proof. If $A \in \pi G\alpha\text{–}LC^*(Z,\tau/Z)$, there is a $\pi g\alpha$ -open set G in $(Z,\tau/Z)$ such that $A = G \cap cl_z(A)$ where $cl_Z(A) = Z \cap cl(A)$. Since G and Z are $\pi g\alpha$ -open, $G \cap Z$ is also $\pi g\alpha$ -open. This implies that $A = (G \cap Z) \cap cl(A) \in \pi G\alpha\text{–}LC^*(X,\tau)$. □

Remark 4.4. The following examples shows that one of the assumptions in the above theorem (i.e) Z is $\pi g\alpha$-open in (X,τ) cannot be removed.

Example 4.5. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{b\}, \{c, d\}, \{b, c, d\}\}$. Let $Z = A = \{a, b, d\}$. Then Z is not $\pi g\alpha$-open in X. $\tau/Z = \{\phi, Z, \{b\}, \{d\}, \{b, d\}\}$. $A \in \pi g\alpha\text{–}LC^*(Z, \tau/Z)$ but $A \notin \pi g\alpha\text{–}LC^*(X, \tau)$.

Lemma 4.6. *Let Z be regular open and $\pi g\alpha$-closed in (X, τ) and $F \subset Z$. If F is $\pi g\alpha$-closed in $(Z, \tau/Z)$ then F is $\pi g\alpha$-closed in (X, τ).*

Proof. Straight forward. □

Theorem 4.7. *Suppose that the collection of all $\pi g\alpha$-closed sets of (X, τ) is closed under finite intersection. If Z is $\pi g\alpha$-closed, regular open in (X, τ) and $A \in \pi G\alpha\text{–}LC^*(Z, \tau/Z)$ then $A \in \pi G\alpha\text{–}LC(X, \tau)$.*

Proof. Let $A \in \pi G\alpha\text{–}LC^*(Z, \tau/Z)$. Then $A = G \cap F$ for some $\pi g\alpha$-open set G in $(Z, \tau/Z)$, and some closed set F in $(Z, \tau/Z)$. F is closed in Z, Z is $\pi g\alpha$-closed and regular open in X implies F is $\pi g\alpha$-closed in (X, τ). Hence $A = G \cap F \in \pi G\alpha\text{–}LC(X, \tau)$. □

Proposition 4.8. *If Z is closed and open (X, τ) and $A \in \pi G\alpha\text{–}LC(Z, \tau/Z)$ then $A \in \pi G\alpha\text{–}LC(X, \tau)$.*

Proof. Let $A \in \pi G\alpha\text{–}LC(Z, \tau/Z)$. Then $A = G \cap F$ where G is $\pi g\alpha$-open in Z and F is $\pi g\alpha$-closed in Z. Since Z is closed and open in (X, τ) by lemma 4.6, G and F are $\pi g\alpha$-open and $\pi g\alpha$-closed respectively in (X, τ). Therefore $A \in \pi G\alpha\text{–}LC(X, \tau)$. □

Theorem 4.9. *If Z is $\pi g\alpha$-closed, regular open in (X, τ) and*

$$A \in \pi G\alpha\text{–}LC^{**}(Z, \tau/Z)$$

then $A \in \pi G\alpha\text{–}LC^{**}(X, \tau)$.

Proof. Let $A \in \pi G\alpha\text{–}LC^{**}(Z, \tau/Z)$. Then $A = G \cap F$ where G is open in Z and F is $\pi g\alpha$-closed in Z. Since Z is $\pi g\alpha$-closed in (X, τ) and regular open G and F are open sets and $\pi g\alpha$-closed sets respectively in (X, τ). Then $A \in \pi G\alpha\text{–}LC^{**}(Z, \tau/Z)$. □

Definition 4.10. Let $A, B \subset X$. Then A and B are said to be *separated* if $A \cap \text{cl}(B) = \phi$ and $B \cap \text{cl}(A) = \phi$. [1]

Proposition 4.11. *Suppose the collection of all $\pi g\alpha$-open sets of (X, τ) are closed under finite unions. Let $A, B \in \pi G\alpha\text{–}LC^*(X, \tau)$. If A and B are separated in (X, τ) then $A \cup B \in \pi G\alpha\text{–}LC^*(X, \tau)$.*

Proof. Since $A, B \in \pi G\alpha\text{–}LC^*(X, \tau)$ by theorem 3.14 there exist $\pi g\alpha$-open sets P and Q of (X, τ) such that $A = P \cap \text{cl}(A)$ and $B = Q \cap \text{cl}(B)$. Put $U = P \cap (X - \text{cl}(B))$ and $V = Q \cap (X - \text{cl}(A))$. Then U and V are $\pi g\alpha$-open subsets of (X, τ). Then $A = U \cap \text{cl}(A)$ and $B = V \cap \text{cl}(B)$ and $U \cap \text{cl}(B) = \phi$, $V \cap \text{cl}(A) = \phi$, hold. Consequently $A \cup B = (U \cup V) \cap (\text{cl}(A \cup B))$, showing that $A \cup B \in \pi G\alpha\text{–}LC^*(X, \tau)$. □

Remark 4.12. The following example shows that one of assumption of proposition 4.11 (i.e. A and B are separated) cannot be removed.

In example 3.6 (a), $\{a\} \in \pi G\alpha\text{–}LC^*(X,\tau), \{b,d\} \in \pi G\alpha\text{–}LC^*(X,\tau)$. However $\{a\}$ and $\{b,d\}$ are not separated and $\{a,b,d\} \notin \pi G\alpha\text{–}LC^*(X,\tau)$.

Theorem 4.13. *Let $\{Z_i : i \in \wedge\}$ be a finite $\pi g\alpha$-closed cover of (X,τ) and let A be a subset of (X,τ). If $A \cap Z_i \in \pi G\alpha\text{–}LC^{**}(Z_i, \tau/Z_i)$ for each $i \in \wedge$, then $A \in \pi G\alpha\text{–}LC^{**}(X,\tau)$.*

Proof. For each $i \in \wedge$, there exist an open set $U_i \in \tau/Z_i$ and $\pi g\alpha$-closed set F_i of $(Z_i, \tau/Z_i)$, such that $A \cap Z_i = (U_i \cap F_i) \cap Z_i = U_i \cap (F_i \cap Z_i)$. Then $A = \cup\{A \cap Z_i : i \in \wedge\} = \cup\{U_i : i \in \wedge\} \cap [\cup\{Z_i \cap F_i : i \in \wedge\}]$ and hence $A \in \pi G\alpha\text{–}LC^{**}(X,\tau)$. □

Theorem 4.14. *Let (X,τ) and (Y,σ) be any two topological spaces. Then*

(i) *If $A \in \pi G\alpha\text{–}LC(X,\tau)$ and $B \in \pi G\alpha\text{–}LC(Y,\sigma)$, then*
$$A \times B \in \pi G\alpha\text{–}LC(X \times Y, \tau \times \sigma).$$
(ii) *If $A \in \pi G\alpha\text{–}LC^*(X,\tau)$ and $B \in \pi G\alpha\text{–}LC^*(Y,\sigma)$, then $A \times B \in \pi G\alpha\text{–}LC^*(X \times Y, \tau \times \sigma)$.*
(iii) *If $A \in \pi G\alpha\text{–}LC^{**}(X,\tau)$ and $B \in \pi G\alpha\text{–}LC^{**}(Y,\sigma)$, then $A \times B \in \pi G\alpha\text{–}LC^{**}(X \times Y, \tau \times \sigma)$.*

Proof. Let $A \in \pi G\alpha\text{–}LC(X,\tau)$ and $B \in \pi G\alpha\text{–}LC(Y,\sigma)$. Then there exist $\pi g\alpha$-open sets V and V^1 of (X,τ) and $\pi g\alpha$-closed sets W and W^1 of (Y,σ) respectively such that $A = V \cap W$ and $B = V^1 \cap W^1$. Then $A \times B = (V \cap W) \times (V^1 \cap W^1) = (V \times V^1) \cap (W \times W^1)$ holds and hence $A \times B \in \pi G\alpha\text{–}LC(X \times Y, \tau \times \sigma)$.

Proofs of (ii) and (iii) are similar to (i). □

5. $\pi G\alpha$–LC Continuity and $\pi g\alpha$–LC Irresoluteness

In this section we use $\pi G\alpha\text{–}LC$ sets, $\pi G\alpha\text{–}LC^*$ sets, $\pi G\alpha\text{–}LC^{**}$ sets to generalize $\pi G\alpha\text{–}LC$ continuous functions, $\pi G\alpha\text{–}LC$ irresolute functions.

Definition 5.1. a) A function $f : (X,\tau) \to (Y,\sigma)$ is called *$\pi G\alpha$–LC continuous* (resp. *$\pi G\alpha$–LC^* continuous*, *$\pi G\alpha$–LC^{**} continuous*) if $f^{-1}(V) \in \pi G\alpha\text{–}LC(X,\tau)$ (resp. $f^{-1}(V) \in \pi G\alpha\text{–}LC^*(X,\tau)$, $f^{-1}(V) \in \pi G\alpha\text{–}LC^{**}(X,\tau)$) for every $V \in \sigma$.
b) A function $f : (X,\tau) \to (Y,\sigma)$ is called *$\pi G\alpha$–LC irresolute* (resp. *$\pi G\alpha$–LC^* irresolute*, *$\pi G\alpha$–LC^{**} irresolute*) if $f^{-1}(V) \in \pi G\alpha\text{–}LC(X,\tau)$ (resp. $f^{-1}(V) \in \pi G\alpha\text{–}LC^*(X,\tau)$, $f^{-1}(V) \in \pi G\alpha\text{–}LC^{**}(X,\tau)$) for every $V \in \pi G\alpha\text{–}LC(Y,\sigma)$ (resp. $V \in \pi G\alpha\text{–}LC^*(Y,\sigma)$, $V \in \pi G\alpha\text{–}LC^{**}(Y,\sigma)$)

Proposition 5.2. *If f is $\pi G\alpha$–LC irresolute then it is $\pi G\alpha$–LC continuous.*

Proof. Follows from definition 5.1. □

Proposition 5.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function.*

(i) *If f is LC-continuous, then f is $\pi G\alpha$–LC^* continuous and $\pi G\alpha$–LC^{**} continuous.*
(ii) *If f is $\pi G\alpha$–LC^* continuous then f is $\pi G\alpha$–LC continuous.*
(iii) *If f is $\pi G\alpha$–LC^{**} continuous then f is $\pi G\alpha$–LC continuous.*
(iv) *If f is $\pi G\alpha$–LC^* irresolute then f is $\pi G\alpha$–LC^* continuous.*
(v) *If f is $\pi G\alpha$–LC^{**} irresolute then f is $\pi G\alpha$–LC^{**} continuous.*

Proof. Straight forward. □

Converse of the above need not be true in general as can be seen in the following examples.

Example 5.4. (1) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}\}$, $\sigma = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. f is $\pi G\alpha$–LC^* continuous and $\pi G\alpha$–LC^{**} continuous but not LC-continuous.

(2) Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{b\}, \{c, d\}, \{b, c, d\}\}$ and $\sigma = \{\phi, Y, \{c\}, \{a, b, d\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then f is $\pi G\alpha$–LC continuous but not $\pi G\alpha$–LC^* continuous since $\{a, b, d\} \in (Y, \sigma)$ but $\{a, b, d\} \notin \pi G\alpha$–$LC^*(X, \tau)$.

(3) Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{b\}, \{c, d\}, \{b, c, d\}\}$, $\sigma = \{\phi, X, \{a, c, d\}, \{b\}\}$ and $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then f is $\pi G\alpha$–LC^* continuous but not $\pi G\alpha$–LC^*-irresolute since $\{a, b, d\} \in \pi G\alpha$–$LC^*(Y, \sigma)$ but $\{a, b, d\} \notin \pi G\alpha$–$LC^*(X, \tau)$.

Proposition 5.5. *Any map defined on a door space is $\pi G\alpha$–LC irresolute.*

Proof. Let (X, τ) be door space and (Y, σ) be any space. Define a map $f : (X, \tau) \to (Y, \sigma)$. Let $A \in \pi G\alpha$–$LC(Y, \sigma)$. Then $f^{-1}(A)$ is either open or closed in (X, τ). In both cases $f^{-1}(A) \in \pi G\alpha$–$LC(X, \tau)$. Hence f is $\pi G\alpha$–LC irresolute. □

Theorem 5.6. *A topological space (X, τ) is $\pi g\alpha$-submaximal iff every function having (X, τ) as it domain is $\pi G\alpha$–LC^* continuous.*

Proof. Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a function. By Theorem 3.20, we have that $f^{-1}(V) \in P(X) = \pi G\alpha$–$LC^*(X, \tau)$ for each open set V of (Y, σ). Therefore f is $\pi G\alpha$–LC^* continuous. Conversely, let every map having (X, τ) as domain be $\pi G\alpha$–LC^* continuous. Let $Y = \{0, 1\}$ be the Sierpinski space with topology $\sigma = \{Y, \phi, \{0\}\}$. Let V be a subset of (X, τ) and $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(x) = 0$ for every $x \in V$ and $f(x) = 1$ for every $x \notin V$. By assumption, f is $\pi G\alpha$–LC^* continuous and hence $f^{-1}\{0\}) = V \in \pi G\alpha$–$LC^*(X, \tau)$. Therefore we have $P(X) = \pi G\alpha$–$LC^*(X, \tau)$ and by theorem 3.20, (X, τ) is $\pi g\alpha$ -submaximal. □

Proposition 5.7. *If $f : (X, \tau) \to (Y, \sigma)$ is $\pi G\alpha$–LC^{**} continuous and subset B is open in (X, τ), then the restriction of f to B say $f/B : (B, \tau/B) \to (Y, \sigma)$ is $\pi G\alpha$–LC^{**} continuous.*

Proof. Let V be an open set of (Y,σ). Then $f^{-1}(V) = G \cap F$ for some open set G and $\pi g\alpha$-closed set F of (X,τ). Now $G \cap B \in \tau/B$ and F is a $\pi g\alpha$-closed subset of $(B,\tau/B)$. But $(f/B)^{-1}(V) = (G\cap B)\cap F$. Hence $(f/B)^{-1}(V) \in \pi G\alpha\text{–}LC^{**}(B,\tau/B)$. This implies that f/B is $\pi G\alpha\text{–}LC^{**}$ continuous. □

We recall the definition of the combination of two functions: Let $X = A\cup B$ and $f : A \to Y$ and $h : B \to Y$ be two functions. We say that f and h are *compatible* if $f/A\cap B = h/A\cap B$. If $f : A \to Y$ and $h : B \to Y$ are compatible then the function $f\nabla h : X \to Y$ defined as

$$(f\nabla h)(x) = \begin{cases} f(x) & \text{for every } x \in A, \\ h(x) & \text{for every } x \in B, \end{cases}$$

is called the combination of f and h.

Pasting lemma for $\pi G\alpha\text{–}LC^{**}$continuous (resp. $\pi G\alpha\text{–}LC^{**}$-irresolute) functions.

Theorem 5.8. *Let $X = A\cup B$, where A and B are $\pi g\alpha$-closed subsets of (X,τ) and $f : (A,\tau/A) \to (Y,\sigma)$ and $h = (B,\tau/B) \to (Y,\sigma)$ be compatible functions*

(a) *If f and h are $\pi G\alpha\text{–}LC^{**}$ continuous, then $f\nabla h : (X,\tau) \to (Y,\sigma)$ is $\pi G\alpha\text{–}LC^{**}$ continuous.*
(b) *If f and h are $\pi G\alpha\text{–}LC^{**}$ irresolute, then $f\nabla h : (X,\tau) \to (Y,\sigma)$ is $\pi G\alpha - LC^{**}$ irresolute.*

Proof. a) Let $V \in \sigma$. Then $(f\nabla h)^{-1}(V)\cap A = f^{-1}(V)$ and $(f\nabla h)^{-1}(V)\cap B = h^{-1}(V)$. By assumption $(f\nabla h)^{-1}(V)\cap A \in \pi G\alpha - LC^{**}(A,\tau/A)$ and $(f\nabla h)^{-1}(V)\cap B \in \pi G\alpha - LC^{**}(B,\tau/B)$. Therefore by Theorem 4.13,

$$(f\nabla h)^{-1}(V) \in \pi G\alpha\text{–}LC^{**}(X,\tau)$$

and hence $f\nabla h$ is $\pi G\alpha\text{–}LC^{**}$-continuous.
b) Proof is similar to (a) □

Next we have the theorem concerning the composition of functions.

Theorem 5.9. *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ be the functions. Then*

(a) *$g\circ f$ is $\pi G\alpha\text{–}LC$ irresolute (resp.$\pi G\alpha\text{–}LC^{*}$ irresolute, $\pi G\alpha\text{–}LC^{**}$ irresolute) if f and g are $\pi G\alpha\text{–}LC$ irresolute (resp. $\pi G\alpha\text{–}LC^{*}$ irresolute, $\pi G\alpha\text{–}LC{**}$ irresolute.)*
(b) *$g\circ f$ is $\pi G\alpha\text{–}LC$ continuous if f is $\pi G\alpha\text{–}LC$ irresolute and g is $\pi G\alpha\text{–}LC$ continuous.*
(c) *$g\circ f$ is $\pi G\alpha\text{–}LC^{*}$ continuous if f is $\pi G\alpha\text{–}LC^{*}$ continuous and g is continuous.*
(d) *$g\circ f$ is $\pi G\alpha\text{–}LC$ continuous if f is $\pi G\alpha\text{–}LC$ continuous and g is continuous.*

(e) *$g \circ f$ is $\pi G\alpha$–LC^* continuous if f is a $\pi G\alpha$–LC^* irresolute and g is $\pi G\alpha$–LC^* continuous.*
(f) *$g \circ f$ is $\pi G\alpha$–LC^{**} continuous if f is $\pi G\alpha$–LC^{**} irresolute and g is $\pi G\alpha$–LC^{**} continuous.*

Proof. Follows from definition 5.1 and 5.2. □

6. Sub $\pi G\alpha$–LC^*-continuity

Definition 6.1. A function $f : (X,\tau) \to (Y,\sigma)$ is called *sub-$\pi G\alpha$–LC^*-continuous* if there exist a basis B for (Y,σ) such that $f^{-1}(U) \in \pi G\alpha$–$LC^*(X,\tau)$ for each $U \in B$.

Proposition 6.2. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function:*
(a) *f is sub-$\pi G\alpha$–LC^*-continuous iff there is a sub-basis of C of (Y,σ) such that $f^{-1}(U) \in \pi G\alpha$–$LC^*(X,\tau)$ for each $U \in C$.*
(b) *If f is sub-LC-continuous then f is sub-$\pi G\alpha$–LC^*-continuous.*

Proof. (a) By assumption, there exist a basis B for (Y,σ) such that $f^{-1}(U) \in \pi G\alpha$–$LC^*(X,\tau)$ for each $U \in B$. Since B is also a sub-basis for (Y,σ), the proof is obvious. Conversely, for a sub-basis C, let $C_\delta = \{A \subset Y :$ A is an intersection of finitely many sets belonging to $C\}$. Then C_δ is a basis for (Y,σ). For $U \in C_\delta, U = \cap\{F_i : F_i \in C_i, i \in \wedge\}$ where $\wedge$ is a finite set. By assumption and Proposition 4.1 (b) we have $f^{-1}(U) = \cap\{f^{-1}(F_i) : i \in \wedge\} \in \pi G\alpha$–$LC^{**}(X,\tau)$.
(b) obtained from (a) and Definition 2.3 (c). □

Converse of Proposition 6.2 is not true as can be seen in the the following example.

Example 6.3. Let $X = Y = \{a,b,c\}$ and $\tau = \{X,\phi,\{a\},\{a,c\}\}$ and σ be the topology induced by a base B of Y. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity function. If $B = \{Y,\{c\}\}$ then f is sub-$\pi G\alpha$–LC^*-continuous but not sub $LC-$continuous since $f^{-1}(\{c\}) = \{c\} \in \pi G\alpha$–$LC^*(X,\tau)$ but $f^{-1}(\{c\}) = \{c\} \notin LC(X,\tau)$.

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