IEMS Vol. 7, No. 2, pp. 143-149, September 2008.

Proposal of Approximation Analysis Method for GI/G/1 Queueing System

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Selected paper from APIEM 2006

Abstract. There have been some approximation analysis methods for a GI/G/1 queueing system. As one of them, an approximation technique for the steady-state probability in the GI/G/1 queueing system based on the iteration numerical calculation has been proposed. As another one, an approximation formula of the average queue length in the GI/G/1 queueing system by using the diffusion approximation or the heuristics extended diffusion approximation has been developed. In this article, an approximation technique in order to analyze the GI/G/1 queueing system is considered and then the formulae of both the steady-state probability and the average queue length in the GI/G/1 queueing system are proposed. Through some numerical examples by the proposed technique, the existing approximation methods, and the Monte Carlo simulation, the effectiveness of the proposed approximation technique is verified.

Keywords: Average queue length, GI/G/1 queueing system, Mixed Erlang distribution, Steady-state probability

1. INTRODUCTION

A GI/G/1 queueing system has been known as one of the most common and flexible queueing systems. Then, it is interesting subjects to develop a practical analysis method. However, the development of the practical analysis method is not easy because the finite information about interarrival and service time distributions in the GI/G/1 queueing system is assumed. Therefore, some approximation analysis methods for the GI/G/1 queueing system have been proposed (See Shanthikumar and Buzacott (1980)).

For instance, Heyman (1975) and Kobayashi (1974) have developed an approximation formula of the average queue length by using the diffusion approximation based on the first two moments of both of arrivals and service times. Then, using numerical experiments, Kramer and Lagenbach-Belz (1976) have heuristically extended Heyman's (1975) approximation formula, and further the equivalent approximation for the average queue length in the GI/G/1 queueing system have been obtained. Then, Shanthikumar and Buzacott (1980) have indicated that the approximation formula of the average queue length proposed by Kramer and Lagenbach-Belz (1976) have the superior accuracy. In addition, Kobayashi (1974) has proposed the method to obtain the approximate distribution of the number of customers in the

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system based on the approximation for the average queue length.

Moreover, Ott (1987) has investigated the approximation technique based on the iterative numerical calculation for obtaining the waiting time distribution of the APH/APH/1 queueing system. In addition, an approximation technique based on the iterative numerical calculation for obtaining the steady-state probability in the Ph/Ph/c queueing system has been proposed by Seelen (1986). Further, Takahashi and Takami (1976) have investigated the approximation technique based on the iterative numerical calculation for obtaining the steady-state probability in the GI/G/c queueing system. However, Ott (1987), Seelen (1986), and Takahashi and Takami (1976) have not obtained the explicit expressions for the average queue length and the steady-state probability.

In this article, we consider the approximation technique in order to analyze the GI/G/1 queueing system which is different from the above existing methods such as diffusion approximation, heuristics extended diffusion approximation and iterative numerical calculation. Then, the approximation technique for obtaining explicitly the average queue length and the steady-state probability distribution of the number of customers in the GI/G/1 queueing system are proposed. Through some numerical examples by the proposed technique, the existing approximation methods and the Monte Carlo simulation, the effectiveness of the proposed approximation analysis technique is verified.

2. NOTATIONS

In this section, we define and explain some symbols in this article.

E_F : Mean of distribution F,
V_F : Variance of distribution F,
C_F : Coefficient of variation of distribution F,
$\mathrm{Erl}(\alpha,\beta)$: Erlang distribution with phase α and mean $1/\beta$,
$\mathrm{Weib}(\xi,\theta)$: Weibull distribution with scale parameter ξ and shape parameter θ
$\mathrm{Log}(\eta,\delta)$: Log-normal distribution with location parameter η and scale parameter δ
$h(t;\alpha,\beta)$: Probability density function of $\mathrm{Erl}(\alpha,\beta)$
n : Number of customers in queueing system,
p_n : Steady-state probability in GI/G/1 queueing system,
L_E : Average queue length in GI/G/1 queueing system.

3. APPROXIMATION TECHNIQUE BY THE MIXED-ERLANG DISTRIBUTION FOR THE GENERAL DISTRIBUTION

In this section, we explain the approximation technique, where the mixture of two Erlang distributions of phase $k-1$ and phase k is substituted for the general distribution with known mean and variance.

Tijms (1994) has proposed an approximation technique using first two moments, where the following probability density function of the mixed Erlang distribution with phase $k-1$ and phase k is substituted for the general distribution G with mean E_{G} and variance V_{G}:

$$f(t)=pv^{k-1}\frac{t^{k-2}}{(k-2)!}e^{-vt}+(1-p)v^{k}\frac{t^{k-1}}{(k-1)!}e^{-vt},\ (t\geq 0), \tag{1}$$

where k is such that

$$\frac{1}{k}\leq\frac{V_{\mathrm{G}}}{(E_{\mathrm{G}})^2}\leq\frac{1}{k-1}, \tag{2}$$

and the parameters p and v are defined by

$$p=\frac{k\dfrac{V_{\mathrm{G}}}{(E_{\mathrm{G}})^2}-\sqrt{k\left(1+\dfrac{V_{\mathrm{G}}}{(E_{\mathrm{G}})^2}\right)-k^2\dfrac{V_{\mathrm{G}}}{(E_{\mathrm{G}})^2}}}{1+\dfrac{V_{\mathrm{G}}}{(E_{\mathrm{G}})^2}}, \tag{3}$$

$$v=\frac{k-p}{E_{\mathrm{G}}}. \tag{4}$$

In this approximation, the general distribution G is given the mixture distribution of the Erlang distribution with phase $k-1$, mean $(k-1)/v$ and the Erlang distribution with phase k, mean k/v.

However, we don't employ the above approximation presented in Equations (1)-(4), and propose another approximation in this article. The reason why we don't employ the approximation by Equations (1)-(4) is addressed in Section 4.

Instead of the approximation in Equations (1)-(4), we propose the following approximation. At first, the probability density function of $\mathrm{Erl}(\alpha,\beta)$ is represented as

$$h(t;\alpha,\beta)=(\alpha\beta)^{\alpha}\frac{t^{\alpha-1}}{(\alpha-1)!}e^{-\alpha\beta t},\ (t\geq 0).$$

Then, we have

$$E_{\mathrm{Erl}(\alpha,\beta)}=\frac{1}{\beta},$$

$$V_{\mathrm{Erl}(\alpha,\beta)}=\frac{1}{\alpha\beta^2},$$

$$\left(C_{\mathrm{Erl}(\alpha,\beta)}\right)^2=\frac{1}{\alpha}.$$

Under the notations mentioned in the previous section, we consider the approximation distribution with the following probability density function for the general distribution G instead of Equation (1):

$$h(t) = (1-p)h\left(t;\alpha-1,\beta\right) + ph\left(t;\alpha,\beta\right), (t \geq 0). \quad (5)$$

The Mixed Erlang distribution with the probability density function of Equation (5) is denoted by Mixed-Erl (α, β, p). Then, the mean $E_{\text{Mixed-Erl}(\alpha,\beta,p)}$ and variance $V_{\text{Mixed-Erl}(\alpha,\beta,p)}$ in the Mixed-Erl(α, β, p) are derived as

$$E_{\text{Mixed-Eri}(\alpha,\beta,p)} = \frac{1}{\beta},$$

$$V_{\text{Mixed-Eri}(\alpha,\beta,p)} = \frac{p}{(\alpha-1)\beta^2} + \frac{1-p}{\alpha\beta^2}.$$

Notice that the approximation of Equation (5) is different from the approximation of Equation (1) greatly at the point that the two Erlang distributions in the approximation of Equation (5) have the same mean, *i.e.* $E_{\text{Erl}(\alpha-1,\beta)} = E_{\text{Erl}(\alpha,\beta)} = E_{\text{G}}$. The approximation parameter β is provided as

$$\beta = \frac{1}{E_{\text{G}}} \quad (6)$$

Further, based on the first two moments, we obtain the approximation parameter α from the relation:

$$\left(C_{\text{Erl}(\alpha,\beta)}\right)^2 \leq \left(C_{\text{G}}\right)^2 \leq \left(C_{\text{Erl}(\alpha-1,\beta)}\right)^2.$$

This inequality leads to the following result for parameter α which is the same as Equation (2):

$$\frac{1}{\alpha} \leq \frac{V_{\text{G}}}{\left(E_{\text{G}}\right)^2} \leq \frac{1}{\alpha-1}. \quad (7)$$

Finally, by letting the variance of the Mixed-Erlang distribution correspond to the variance of the general distribution G, the mixture radio p is provided as follows:

$$p = (\alpha-1)\left\{\alpha(C_{\text{G}})^2 - 1\right\}. \quad (8)$$

Suppose that the mean and variance of the interarrival time distribution and the service time distribution are described by E_A, V_A, E_S, and V_S, respectively. Then, we have the coefficients of variation C_A and C_S in the interarrival and service time distributions, respectively.

In the approximation technique of Equations (5)-(8), we consider the interarrival time distribution as the mixed-Erlang distribution with the approximation parameters (ℓ, λ, r), where we have

$$\lambda = \frac{1}{E_A}, \quad (9)$$

$$\frac{1}{\ell} \leq C_A^2 \leq \frac{1}{\ell-1}, \quad (10)$$

$$r = (\ell-1)\left(\ell C_A^2 - 1\right). \quad (11)$$

Similarly, the service time distribution is specified as the Mixed-Erl(k, μ, q) with the following approximation parameters (k, μ, q):

$$\mu = \frac{1}{E_S}, \quad (12)$$

$$\frac{1}{k} \leq C_S^2 \leq \frac{1}{k-1}, \quad (13)$$

$$q = (k-1)\left(kC_S^2 - 1\right). \quad (14)$$

Remark that from $\ell \geq 1$ and $k \geq 1$, the squared coefficients of variation C_A^2 and C_S^2 should satisfy the following relations: $C_A^2 \leq 1$ and $C_S^2 \leq 1$. We consider only the situation where the coefficient of variation is equal to or less than 1. This case is useful in practical use.

4. PROPOSAL APPROXIMATION ANALYSIS FOR GI/G/1 QUEUEING SYSTEM BASED ON $\mathbf{Erl}(\ell,\lambda)/\mathbf{Erl}(k,\mu)/1$ QUEUEING SYSTEM ANALYSIS

In the previous section, we have explained the approximation technique for the interarrival time distribution and the service time distribution based on the Mixed-Erlang distribution. Then, we substitute the mixture of analysis of four Erlang interarrival/Erlang service /1 queueing systems presented in Table 1 for the analysis of the GI/G/1 queueing system.

Table 1. Queueing systems and mixture ratio.

Queueing Systems	Mixture Ratio
$\text{Erl}(\ell-1,\lambda)/\text{Erl}(k-1,\mu)/1$	rq
$\text{Erl}(\ell-1,\lambda)/\text{Erl}(k,\mu)/1$	$r(1-q)$
$\text{Erl}(\ell,\lambda)/\text{Erl}(k-1,\mu)/1$	$(1-r)q$
$\text{Erl}(\ell,\lambda)/\text{Erl}(k,\mu)/1$	$(1-r)(1-q)$

Further, we can analyze the $\text{Erl}(\ell,\lambda)/\text{Erl}(k,\mu)/1$ queueing system by adopting the following method presented by Kawamura (1980). Denote the present state in the $\text{Erl}(\ell,\lambda)/\text{Erl}(k,\mu)/1$ queueing system by (s, m, n), where s is the progress of the phase in the interarrival time distribution $\mathrm{Erl}(\ell,\lambda)$, m is the progress of the phase in the service time distribution $\mathrm{Erl}(k,\mu)$, and n is the number of customers in the $\mathrm{Erl}(\ell,\lambda)/\mathrm{Erl}(k,\mu)/1$ queueing system. Remark that if there is no customers in the queueing system, the state is denoted by $(s,*,0)$. And, let $p_{s,*,0}^{(\ell,k)}$ and $p_{s,m,n}^{(\ell,k)}$ be the steady-state probabilities of the states $(s,*,0)$ and (s,m,n). Define the following notations:

$$\phi=\frac{\ell\lambda}{k\mu},\ \rho=\lambda/\mu\,(<1).$$

Then, consider the variables u, v and w which satisfy the relations:

$$\begin{cases}\phi+uv-(1+\phi)v=0,\\ w=u^k=v^\ell.\end{cases} \tag{15}$$

Further, we transform the equation (15) into

$$v=\frac{\phi}{1+\phi-u},$$

and

$$\left(\frac{\phi}{1+\phi-u}\right)^\ell=u^k. \tag{16}$$

In this case, the equation (16) has $\ell+k$ different roots, including the root $u=1$, and among them exactly k roots have the absolute values less than 1.

Let $u_1,u_2,\cdots,u_k$ be the roots of equation (16) inside the unit circle and let $u_{k+1},u_{k+2},\cdots,u_{k+\ell}$ be the others and $u_{k+\ell}=1$. Then, it can be easily seen that $u_{k+1},u_{k+2},\cdots,u_{k+\ell-1}$ can not have the absolute value 1. Therefore, we get in succession:

$$0<|u_i|<1,\quad 0<|v_i|<1,\quad 0<|w_i|<1,\quad i=1,2,\cdots,k;$$
$$|u_i|>1,\quad |v_i|>1,\quad |w_i|>1,\quad i=k+1,\cdots,k+\ell-1;$$
$$u_{\ell+k}=1,\quad v_{\ell+k}=1,\quad w_{\ell+k}=1.$$

Then, we can obtain the steady-state probabilities $p_{s,*,n}^{(\ell,k)}$ and $p_{s,m,n}^{(\ell,k)}$ as follows:

$$p_{s,*,0}^{(\ell,k)}=\frac{(-1)^{s-1}\sigma_{s-1}(1-\rho)}{\prod_{j=1}^{\ell-1}\left(1-v_{k+j}\right)},\ s=1,2,\cdots,\ell, \tag{17}$$

$$p_{s,m,n}^{(\ell,k)}=\frac{-B\phi(1-\rho)}{A\prod_{j=1}^{\ell-1}\left(1-v_{k+j}\right)}, \tag{18}$$

$$s=1,2,\cdots,\ell;\quad m=1,2,\cdots,k;\quad n=1,2,3,\cdots,$$

where

$$\sigma_{s-1}=\begin{cases}1, & s=1;\\ \mathbf{e}_{\ell-s+1}^T\left\{\prod_{j=1}^{s-1}\left(\mathbf{L}_{\ell-s+1}\mathbf{V}_j\right)\right\}\mathbf{1}_{\ell-s+1}, & \\ & s=2,3,\cdots,\ell-1;\\ \sigma_{\ell-1}=\prod_{j=1}^{\ell-1}v_{k+j},\ \ s=\ell, & \end{cases}$$

$$\mathbf{1}_{\ell-s+1}^T=\begin{bmatrix}1 & \cdots & 1\end{bmatrix},$$

$$\mathbf{V}_j=\begin{bmatrix}v_{k+j} & 0 & \cdots & 0\\ 0 & v_{k+j+1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & v_{k+j+\ell-s}\end{bmatrix},$$

$$\mathbf{L}_{\ell-s+1}=\begin{bmatrix}1 & 0 & \cdots & 0\\ 1 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1\end{bmatrix},$$

$$\prod_{j=1}^{s-1}\left(\mathbf{L}_{\ell-s+1}\mathbf{V}_j\right)=\mathbf{L}_{\ell-s+1}\mathbf{V}_{s-1}\mathbf{L}_{\ell-s+1}\mathbf{V}_{s-2}\cdots\mathbf{L}_{\ell-s+1}\mathbf{V}_1,$$

$$\mathbf{e}_{\ell-s+1}^T=\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix},$$

$$\mathbf{A}=\begin{bmatrix}1 & 1 & \cdots & 1\\ u_1 & u_2 & \cdots & u_k\\ \vdots & \vdots & \ddots & \vdots\\ u_1^{k-1} & u_2^{k-1} & \cdots & u_k^{k-1}\end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix}1 & \cdots & 1 & 1\\ u_1 & \cdots & u_k & 1+\phi\\ \vdots & \ddots & \vdots & \vdots\\ u_1^{k-1} & \cdots & u_k^{k-1} & (1+\phi)^{k-1}\\ U_1^{(s,m,n)} & \cdots & U_k^{(s,m,n)} & 0\end{bmatrix},$$

$$U_j^{(s,m,n)}=u_j^{m-1}v_j^{s-1}w_j^{n-1},\quad j=1,2,\cdots,k.$$

We have particularly

$$p_{1,*,0}^{(\ell,k)}=\frac{1-\rho}{\prod_{j=1}^{\ell-1}\left(1-v_{k+j}\right)}.$$

Therefore, it can be easily verified that

$$p_0^{(\ell,k)}=\sum_{s=1}^{\ell}p_{s,*,0}^{(\ell,k)}=1-\rho, \tag{19}$$

where $p_0^{(\ell,k)}$ means the steady-state probability that there is no customer in the $\mathrm{Erl}(\ell,\lambda)/\mathrm{Erl}(k,\mu)/1$ queueing system. Further, the average queue length $L_E^{(\ell,k)}$ in the $\mathrm{Erl}(\ell,\lambda)/\mathrm{Erl}(k,\mu)/1$ queueing system is derived as follows:

$$L_E^{(\ell,k)}=\frac{1}{1-\rho}\left\{\frac{k+1}{2k}\rho^2-\frac{\ell-1}{2\ell}+\frac{s_0-(1-\rho)}{\ell}\right\}+\rho. \tag{20}$$

where

$$s_0 = \sum_{s=1}^{\ell} s p_{s,*,0}^{(\ell,k)}.$$

As mentioned the above, we consider the mixture of analysis of four Erlang interarrival/Erlang service/1 queueing systems presented in Table 1 to be the approximation of analysis of the GI/G/1 queueing system. Therefore, we can get the steady-state probability p_n and the average queue length L_E as follows:

$$p_0 = 1-\rho, \tag{21}$$

$$\begin{aligned} p_n &= rq\sum_{m=1}^{k-1}\sum_{s=1}^{\ell-1} p_{s,m,n}^{(\ell-1,k-1)} + r(1-q)\sum_{m=1}^{k}\sum_{s=1}^{\ell-1} p_{s,m,n}^{(\ell-1,k)} \\ &\quad +(1-r)q\sum_{m=1}^{k-1}\sum_{s=1}^{\ell} p_{s,m,n}^{(\ell,k-1)} \\ &\quad +(1-r)(1-q)\sum_{m=1}^{k}\sum_{s=1}^{\ell} p_{s,m,n}^{(\ell,k)}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} L_E &= rqL_E^{(\ell-1,k-1)} + r(1-q)L_E^{(\ell-1,k)} \\ &\quad +(1-r)qL_E^{(\ell,k-1)} + (1-r)(1-q)L_E^{(\ell,k)}. \end{aligned} \tag{23}$$

In the approximation analysis technique proposed in this section, the traffic intensities for respective Erlang interarrival/Erlang service/1 queueing systems in Table 1 have to be less than 1. If the traffic intensity for the original GI/G/1 queueing system is less than 1, we can always apply the approximation analysis technique proposed in this section because of the relation of $E_{\text{Erl}(\alpha-1,\beta)} = E_{\text{Erl}(\alpha,\beta)} = E_{\text{G}}$ by using the approximation of Equations (5)-(8). However, in the approximation presented of Equations (1)-(4), the traffic intensities for respective Erlang interarrival/Erlang service/1 queueing systems in Table 1 aren't always less than 1. It has been known that for any queueing systems with the traffic intensity exceeding 1, we can never obtain the steady-state in the queueing system. This is the reason why we don't employ the approximation technique of Equations (1)-(4).

5. NUMERICAL VERIFICATION

In this section, through some numerical results, we verify the validity of the proposed approximation analysis technique for the GI/G/1 queueing system. For comparison, we consider some existing approximation methods and the Monte Carlo simulation to obtain the steady-state probability and the average queue length. At first, we have to describe exactly the interarrival time distribution and the service time distribution in order to implement the Monte Carlo simulation. As an example, assume the interarrival time distribution to be the Weibull distribution $\text{Weib}(\xi,\theta)$ with the following probability density function $f_{\text{Weib}(\xi,\theta)}(t)$:

$$f_{\text{Weib}(\xi,\theta)}(t) = \left(\frac{\theta t^{\theta-1}}{\xi}\right)\exp\left(-\frac{t^{\theta}}{\xi}\right),$$

where let ξ and θ denote the scale parameter and the shape parameter in the Weibull distribution, respectively. In this case, the mean $E_{\text{Weib}(\xi,\theta)}$ and variance $V_{\text{Weib}(\xi,\theta)}$ in the Weibull distribution are given as

$$E_{\text{Weib}(\xi,\theta)} = \xi^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\theta}+1\right),$$

$$V_{\text{Weib}(\xi,\theta)} = \xi^{\frac{2}{\theta}}\left\{\Gamma\left(\frac{2}{\theta}+1\right) - \Gamma^2\left(\frac{1}{\theta}+1\right)\right\}.$$

Moreover, the service time distribution is assumed to be the log-normal distribution $\text{Log}(\eta,\delta)$ with the probability density function $f_{\text{Log}}(t)$:

$$f_{\text{Log}}(t) = \frac{1}{\sqrt{2\pi}\delta t}\exp\left\{-\frac{(\log t-\eta)^2}{2\delta^2}\right\},$$

where η and δ represent the parameters about location and scale. Further, we have the mean $E_{\text{Log}(\eta,\delta)}$ and variance $V_{\text{Log}(\eta,\delta)}$ in the log-normal distribution as

$$E_{\text{Log}(\eta,\delta)} = \exp\left(\eta + \frac{\delta^2}{2}\right),$$

$$V_{\text{Log}(\eta,\delta)} = \exp\left(2\eta+\delta^2\right)\left\{\exp\left(\delta^2\right)-1\right\}.$$

Therefore, on applying the proposed approximation analysis technique, we can interpret the means, variances and coefficients of variation in the interarrival time distribution and service time distribution as follows:

$$E_A = E_{\text{Weib}(\xi,\theta)}, \quad E_S = E_{\text{Log}(\eta,\delta)},$$

$$V_A = V_{\text{Weib}(\xi,\theta)}, \quad V_S = V_{\text{Log}(\eta,\delta)},$$

$$C_A^2 = V_A / E_A^2, \quad C_S^2 = V_S / E_S^2.$$

Furthermore, for another comparison, the approximation formula of the average queue length proposed by Kramer and Lagenbach-Belz (1976) should be employyed, because it has the superior accuracy. In addition, when the approximate queue length are obtained by using the approximation formula proposed by Kramer and Lagenbach-Belz, we can apply the derivation for the steady-state probability using diffusion approximation with a reflecting boundary (Kobayashi 1974).

Table 2. The approximation parameters (k, μ, q) for the Lognormal distribution as the service time distribution.

η	δ	E_S	V_S	$1/C_S^2$	k	μ	q
-2.428	0.5	0.1	0.00284	3.521	4	10.0	0.592

Table 3. The approximation parameters (ℓ, λ, p) for the Weibull distribution as the interarrival time distribution.

ξ	θ	E_A	V_A	$1/C_A^2$	ℓ	λ	r
1.3063	2.2	1.000	0.23024	4.34	5	1.0	0.395
0.2843	2.2	0.500	0.05756	4.34	5	2.0	0.395
0.1165	2.2	0.333	0.02558	4.34	5	3.0	0.395
0.0619	2.2	0.250	0.01439	4.34	5	4.0	0.395
0.0379	2.2	0.200	0.00921	4.34	5	5.0	0.395
0.0254	2.2	0.167	0.00640	4.34	5	6.0	0.395
0.0181	2.2	0.143	0.00470	4.34	5	7.0	0.395
0.0135	2.2	0.125	0.00360	4.34	5	8.0	0.395
0.0104	2.2	0.111	0.00284	4.34	5	9.0	0.395

Then, the approximation formula for the average queue length proposed by Kramer and Lagenbach-Belz (1976) is indicated as

$$L_K = \frac{\rho^2}{1-\rho} \cdot \frac{C_A^2 + C_S^2}{2} \Phi + \rho, \tag{24}$$

where

$$\Phi = \exp\left\{ -\frac{2(1-\rho)\left(1-C_A^2\right)^2}{3\rho\left(C_A^2 + C_S^2\right)} \right\}, \qquad C_A^2 \le 1.$$

Further, by substituting the result L_K into the approximation of the steady-state probability presented by Kobayashi (1974), we have

$$p_n = \begin{cases} 1-\rho, & n = 0; \\ \rho\left(1-\hat{\rho}\right)\hat{\rho}^{n-1}, & n \ge 1, \end{cases}$$

where

$$\hat{\rho} = \frac{L_K - \rho}{L_K}.$$

Table 2 shows the approximation parameters (k, μ, q) in the case that the Log-normal distribution is substituted for the general service time distribution. Similarly, the approximation parameters (ℓ, λ, r) in the Weibull distribution for the general interarrival time distribution are shown in Table 3. Further, Table 4 illustrates the average queue length based on the Monte Carlo simulation, the proposed analysis technique and the method of Kramer and Lagenbach-Belz (1976) under the data set in Tables 2 and 3. Further, in Table 5, the steady-state probabilities by the Monte Carlo simulation, the proposed technique and the approximation by Kobayashi (1974) are shown under the same data set. From Table 4, we find that the proposed analysis technique has the more effectiveness than the approximation of Kramer and Lagenbach-Belz. And, it is obvious that the steady-state probability based on the proposed technique has enough accuracy through Table 5 in the practical usage.

Table 4. The comparisons of the average queue length

ρ	L_S	L_E	L_K
0.1	0.100	0.100	0.100
0.2	0.203	0.202	0.201
0.3	0.313	0.308	0.306
0.4	0.435	0.426	0.422
0.5	0.578	0.566	0.560
0.6	0.764	0.746	0.739
0.7	1.033	1.009	1.002
0.8	1.514	1.486	1.479
0.9	2.865	2.818	2.812

6. CONCLUSION

In this article, the interarrival time and the service time of GI/G/1 queueing system are given as the Mixed-Erlang distribution, respectively. Then, we have proposed the approximation technique for the average queueing length and the steady-state probability based on the analysis results of four Erlang interarrival/Erlang service/1 queueing systems. As the result, we have presented the approximation formulae for the average queueing length and the steady-state probability. The effectiveness of the proposed approximation technique has been verified. In this article, we have proposed the approximation technique for the distribution, provided that the squared coefficient of variation is less than 1. The analysis technique in the case that the squared coefficient of variation is greater than 1.0 would be left as the future subject.

Table 5. The steady-state probability ($\rho = 0.8$) .

p_n	simulation	Proposed method	Kobayashi
p_0	0.199	0.200	0.200
p_1	0.407	0.398	0.433
p_2	0.218	0.220	0.199
p_3	0.097	0.095	0.091
p_4	0.043	0.041	0.042
p_5	0.020	0.018	0.019

ACKNOWLEDGMENT

This research was partially supported by the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-aid for Scientific Research (C), No.20510138, 2008-2010.

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